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THREE-DIMENSIONAL VIBRATION OF A ROTATING RING EXCITRD BY ELECTRICAL FIELD

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Abstract

In this paper we study the vibration behavior of a micro-gyro, which consists of a ring and its supporting structure. This ring is made from $\{111\}$ silicon wafer. The geometric parameters of the system are designed so that the in-plane natural frequencies of the ring is tuned to be equal to the out-of-plane ones; therefore the output signal can be made of the same order as that of the input one. The relationships between the three components of the angular velocity and the output vibrating magnitudes of the ring are derived analytically by using perturbation technique.

INTRODUCTION

Traditional vibrating gyroscopes such as Delco's hemispherical resonator gyroscope [1] made of fused quartz is of size in centimeter. It is of high accuracy, but very cost. And it can only measure single-axis angular rate. Due to the rapid development of MEMS technology many gyros of size in micrometer were designed. Typical examples include the vibrating nickel ring gyroscope [2] and the silicon ring one of British Aerospace [3]. The former use capacitors as actuators and sensors, while the latter uses electromagnetic methods. There are also many other ring-type gyro patents [4,5] appeared recently with some modification in structures and fabrications. Juneau [6] showed that two-axis designs of ring-type gyros are possible.

Although there many papers and patents talking about ring micro-gyros, most of them gave only the conceptual designs and lack rigorous analysis.

Gallacher et al. [7] seems to be the first to study the design of three-axis gyros. However, their deign didn't include the mechanism of actuator, the effect of driving forces on the output amplitudes is unknown. Also the coupling terms are completely neglected in their work.

In this paper the three-axis gyros excited by electrostatic field of capacitors are investigated. The equations of vibration are derived by Hamilton' principle. The natural frequencies of the in-plane modes are tuned to those of the out-of-plane modes by adjusting the geometric parameters following the derived formula. Then we solve the complete equations of motion including the coupling terms by using perturbation technique. The nonlinear relationship between the three angular rates and the vibration amplitude of the ring are derived in closed form. The effect of the driving voltage and the gap of the capacitor surfaces on the output signal can be obtained through our analytical solution.

Frequency Analysis

The top view of the ring gyro is shown in Fig. 1. The pairs of capacitors (PD1, PD3), (PD2,PD4) are used as actuators. The diameter, width, and thickness are denoted by a, h, and b respectively. Let u, v, w be the radial, tangential, and out-of-plane displacement of the neutral line, U_r, U_θ, U_z are the corresponding displacements of any point on the ring as shown in Fig. 2. ϕ_i is the rotation angle about z-axis due to in-plane bending, ϕ_0 is the rotation angle about x-axis due to out-of-plane bending, and ϕ is the twist angle about the y-axis due to torsion.

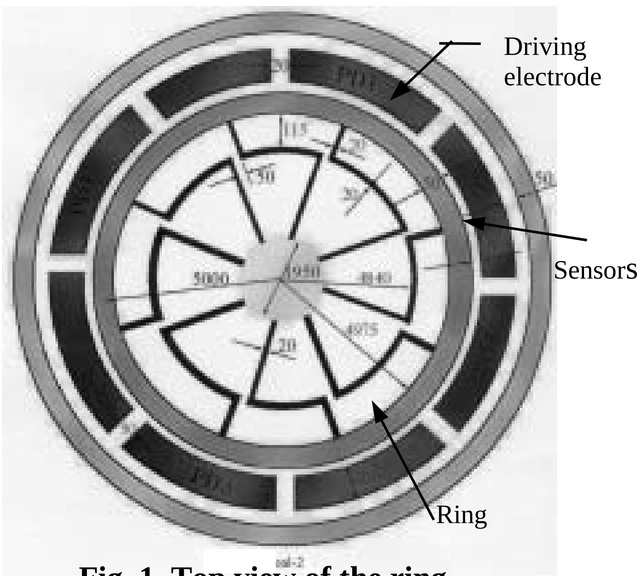

Fig. 1. Top view of the ring

We adopt the Euler's beam theory , the displacements [8] are given by

$$U_r = u(\theta,t) + z\phi(\theta,t)$$

$$U_\theta = v(\theta,t) + r\phi_i - z\phi_o = v(\theta,t) + \frac{r}{a}\left(v(\theta,t) - \frac{\partial u(\theta,t)}{\partial \theta} \right) - \frac{z}{a}\frac{\partial w(\theta,t)}{\partial \theta} \tag{1}$$

$$U_z = w(\theta,t) - r\phi(\theta,t);$$

$$\phi_i = \left(v - \partial u / \partial \theta \right) / a, \qquad \phi_0 = \partial w / a\partial \theta .$$

The strain-displacement relationships [9] are

$$\varepsilon_{\theta\theta} = \frac{v' + u}{a} + \frac{x}{a^2}(v' - u'') - \frac{z}{a}(\frac{w''}{a} - \phi),$$

$$\gamma_{r\theta} = \frac{z}{a}(\frac{w'}{a} + \phi'), \qquad \gamma_{\theta z} = -\frac{x}{a}(\frac{w'}{a} + \phi'). \tag{2}$$

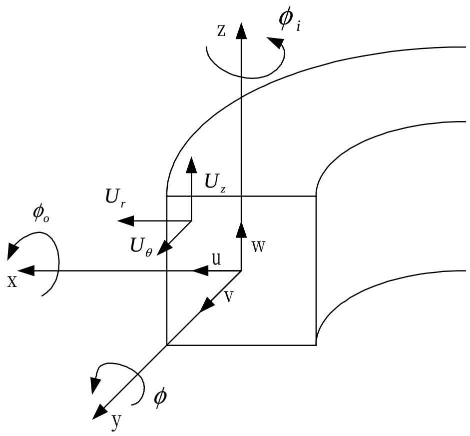

Fig. 2. Then displacements of a ring's cross-section.

Using Hamilton's principle we get the equations of motion as

$$\ddot{u} + \frac{E}{a^2\rho}(u + v') + \frac{h^2}{12a^2}(\ddot{v}' - \ddot{u}'') + \frac{h^2E}{12a^4\rho}(u'''' - v''') = 0,$$

$$\ddot{v} - \frac{12E}{12a^2\rho + h^2\rho}u' - \frac{h^2}{12a^2 + h^2}\ddot{u}' - \frac{E}{a^2\rho}v'' + \frac{h^2E}{12a^4\rho + a^2h^2\rho}u''' = 0,$$

$$\ddot{w} - \frac{G}{12a^4\rho}w'' - \frac{(b^2G + h^2G + b^2E)}{12a^3\rho}\phi'' - \frac{b^2}{12a^2}\ddot{w}'' + \frac{b^2E}{12a^4\rho}w'''' = 0 \tag{3}$$

$$\ddot{\phi} + \frac{b^2E}{a^2(b^2 + h^2)\rho}\phi - \frac{(b^2G + h^2G + b^2E)}{a^3(b^2 + h^2)\rho}w'' - \frac{G}{a^2\rho}\phi'' = 0.$$

We assume the solution in the form

$$u = Ae^{j(\omega_1 t + n\theta)},\ v = Be^{j(\omega_1 t + n\theta)},\ w = Ce^{j(\omega_2 t + m\theta)},\ \phi = De^{j(\omega_2 t + m\theta)}. \tag{4}$$

Substituting (4) into (3) gives

$$\omega_{11n}^2 = \frac{E(12a^2(n^2+1) + h^2(2n^2+1)(n^2-1) - \sqrt{(12a^2+h^2)[h^2(3n^2-1)^2 + 12a^2(n^2+1)^2]})}{2a^2[12a^2 + h^2(n^2+1)]\rho} \tag{5a}$$

$$\omega_{12n}^2 = \frac{E(12a^2(n^2+1) + h^2(2n^2+1)(n^2-1) + \sqrt{(12a^2+h^2)[h^2(3n^2-1)^2 + 12a^2(n^2+1)^2]})}{2a^2[12a^2 + h^2(n^2+1)]\rho}. \tag{5b}$$

where ω_{11n} and ω_{12n} are the natural frequencies of the in-plane n*th* vibration modes. The difference is that ω_{11n} is the frequency of inextensible mode, while ω_{12n} is that of the extensible mode. Similarly, the frequencies of the out-of-plane m*th* modes are

$$\omega_{21m} = \omega_{21m}\left(E,G,a,h,b,m\right), \quad \omega_{22m} = \omega_{21m}\left(E,G,a,h,b,m\right). \tag{6}$$

The amplitude ratios in (4) are

$$\frac{B}{A} = j\frac{(12a^2+h^2n^4)E - a^2(12a^2+h^2n^2)\rho\omega_1^2}{n(12a^2E+h^2n^2E-a^2h^2\rho\omega_1^2)}, \tag{7a}$$

$$\frac{D}{C} = \frac{-m^2[(b^2+h^2)G+b^2m^2E]+a^2(12a^2+b^2m^2)\rho\omega_2^2}{am^2[(b^2+h^2)G+b^2E]}. \tag{7b}$$

If we choose the lowest two modes and assume that $b,h \ll a$, equation (7) reduces to

$$\frac{B}{A} \approx j\frac{1}{n}, \qquad \frac{D}{C} \approx -\frac{m^2[(b^2+h^2)G+h^2E]}{a[(b^2+h^2)m^2G+h^2E]}. \tag{8}$$

The data of silicon wafer are $E = 165\text{GPa}$, $G = 67.6\text{GPa}$, $\rho = 2330\text{kg/m}^3$, $a = 4000\mu\text{m}$, $h = 100\mu\text{m}$, $b = 100\mu\text{m}$. In order to reconcile the in-plane and out-of-plane frequencies, that is, $\omega_{12n} = \omega_{22m}$ for $(n,m) = (2,3)$, we find that the geometric parameters are restricted to satisfy the equation

$$b = 0.34h. \tag{9}$$

and the radius has no effect on this reconciliation.

Analysis of Ring's gyroscopes

In this section we consider the effect of Corioslis force due to the angular velocity input. We are going to use the Lagrange's equation to derive equations of vibration. The eight supporting beam are included. Their strained energy is evaluated exactly, but their kinetic energy is approximated by considering their velocity as one half of the ring at the contact point. The sensing coefficients are derived, which are the key parameters that affect the performance of the gyros.

From equations (4) and (8) the displacements of the neutral line of the ring can be expressed as

$$u = X_1(t)Cos(n\theta) + X_2(t)Sin(n\theta), \; v = -\frac{1}{n}[X_1(t)Sin(n\theta) - X_2(t)Cos(n\theta)],$$
$$w = X_3(t)Cos(m\theta) + X_4(t)Sin(m\theta), \tag{10}$$

$$\phi = \frac{-m^2[(b^2+h^2)G+b^2m^2E]+a^2(12a^2+b^2m^2)\rho\omega_2^2}{am^2[(b^2+h^2)G+b^2E]}[X_3(t)Cos(m\theta)+X_4(t)Sin(m\theta)]$$

where $X_1(t)$, $X_2(t)$, $X_3(t)$, and $X_4(t)$ are generalized coordinates.

Strain energy of the suspensions

The structure of the suspensions of the ring is shown in Fig. 3.

The bending moments at different section of the suspension (shown in Fig. 4) are

$$M_1 = M + Qr, \; M_2 = M + QL_1 - Rs, \; M_3 = M + Q(L_1+t) - RL_2. \tag{11}$$

The total in-plane strain energy is

$$U_{si} = \frac{1}{2EI_x}\left(\int_0^{L_1} M_1^2 dr + \int_0^{L_2} M_2^2 ds + \int_0^{L_3} M_3^2 dt\right)$$

$$=\frac{1}{6EI_x}\left[\begin{array}{l}3L_1M^2+3L_1^2MQ+L_1^3Q^2+\frac{\left(M+L_1Q\right)^3}{R}-\frac{\left(M+L_1Q-L_2R\right)^3}{Q}\\ -\frac{\left(M+L_1Q-L_2R\right)^3}{R}+\frac{\left(M+\left(L_1+L_3\right)Q-L_2R\right)^3}{Q}\end{array}\right]. \quad (12)$$

The use of Castigliano theorem

$$u_s\left(\theta,t\right)=\frac{\partial U_{si}}{\partial R},\quad v_s\left(\theta,t\right)=\frac{\partial U_{si}}{\partial Q},\ \phi_{is}\left(\theta,t\right)=\frac{\partial U_{si}}{\partial M}, \quad (13)$$

gives the relationship between the forces and bending moments and the displacements as

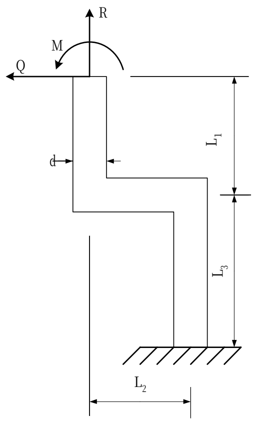

Fig. 3. The geometry of the suspension

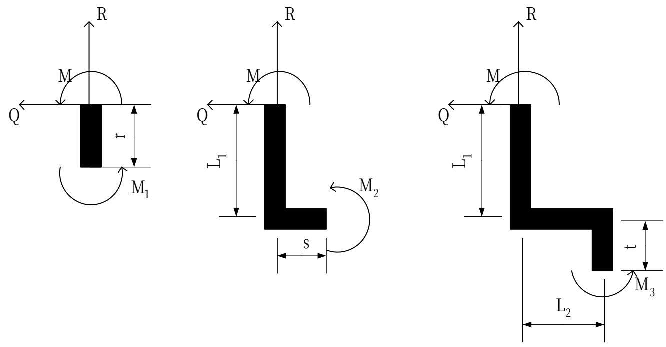

Fig. 4. Forces and bending moment diagram of the suspension

$$\begin{Bmatrix}N\\ M_d\\ M_g\end{Bmatrix}=\begin{bmatrix}K_{si11} & K_{si12} & K_{si13}\\ K_{si12} & K_{si22} & K_{si23}\\ K_{si13} & K_{si23} & K_{si33}\end{bmatrix}\begin{Bmatrix}w_s\left(\theta,t\right)\\ \phi_s\left(\theta,t\right)\\ \phi_{os}\left(\theta,t\right)\end{Bmatrix}, \quad (14)$$

where $K_{si12}=\frac{3bd^3E\left(2L_1L_3^2+L_2L_3^2+L_1^2\left(L_2+2L_3\right)\right)}{4L_2^2\left(L_1+L_2+L_3\right)\left(L_1^2-L_1L_3+L_3^2\right)\left(L_2L_3+L_1\left(L_2+3L_3\right)\right)}$, in general,

$K_{sikl}=K_{sikl}\left(E,G,a,h,b,L_1,L_2,L_3\right)$. Substituting (14) into (12) yields the total in-plane strain energy in terms of generalized coordinates as

$$V_{si}=k_{ni}(X_1^2(t)+X_2^2(t)), \quad (15a)$$

where

$$k_{ni}=4K_{si11}+K_{si22}+\frac{3\left(K_{si23}+K_{si32}\right)}{a}+\frac{9K_{si33}}{a^2}. \quad (15b)$$

Similarly, the total out-of-plane strain energy of the eight suspensions in terms of generalized coordinates is

$$V_{si}=k_{ni}(X_1^2(t)+X_2^2(t)). \quad (16)$$

Let $\mathbf{x}=\left(X_1(t),X_2(t),X_3(t),X_4(t)\right)^{\mathrm{T}}$ and we apply the Lagrange's equations to derive equations of vibration

$$\ddot{x}+\begin{bmatrix}0 & -\lambda_1\Omega_z & \lambda_2\Omega_y & -\lambda_2\Omega_x\\ \lambda_1\Omega_z & 0 & \lambda_2\Omega_x & \lambda_2\Omega_y\\ -\lambda_3\Omega_y & -\lambda_3\Omega_x & 0 & -\lambda_4\Omega_z\\ \lambda_3\Omega_x & -\lambda_3\Omega_y & \lambda_4\Omega_z & 0\end{bmatrix}\dot{x}+\begin{bmatrix}\xi_1\omega_1 & 0 & 0 & 0\\ 0 & \xi_1\omega_1 & 0 & 0\\ 0 & 0 & \xi_2\omega_2 & 0\\ 0 & 0 & 0 & \xi_2\omega_2\end{bmatrix}\dot{x}$$
$$+\begin{bmatrix}\omega_1^2 & 0 & 0 & 0\\ 0 & \omega_1^2 & 0 & 0\\ 0 & 0 & \omega_2^2 & 0\\ 0 & 0 & 0 & \omega_2^2\end{bmatrix}x=0 \tag{17}$$

where $\lambda_1,\cdots,\lambda_4$ are the important parameters of the gyros and are called sensing coefficients. The larger the λ's are, the more sensitive the gyro will be and the better the resolution is. With the non-dimensional quantities $\gamma=h/a$, $\eta=b/a$, $\kappa=V\!\!\!/\,/a^3$ where $V\!\!\!/$ is the volume of a single suspension, the explicit form of λ_1 and λ_2 is

$$\lambda_1=\frac{48(2\pi\gamma\eta+2\kappa)}{60\pi\gamma\eta+9\pi\gamma^3\eta+60\kappa},\quad \lambda_2=\frac{2}{5}\left(1+\frac{6\pi\gamma^3\eta(7E\eta^2+3G(\gamma^2+\eta^2))}{(E\eta^2+9G(\gamma^2+\eta^2))(\pi\gamma\eta(20+3\gamma^2)+20\kappa)}\right) \tag{18}$$

Electrostatic Actuators

The electrodes of the driving capacitor are modeled as parallel plates for approximately calculating the electrostatic force

$$dF=\frac{\epsilon_0 V_e^2}{2\Delta^2}dA, \tag{19}$$

where ϵ_0 is the dielectric constant, V_e applied voltage, Δ the gap of the parallel plates, A the area of the electrode plate. In terms of the radial displacement of the ring we can write $\Delta=\Delta_i-U_r$, where Δ_i is the initial gap in in-plane direction. Assuming $U_r\ll\Delta_i$ and expanding the term $1/\Delta^2$ up to first order we have

$$dF_i=\frac{\varepsilon_0 V_{ek}^2}{2\Delta_i^2}(1+\frac{2U_r}{\Delta_i})dA\,. \tag{20}$$

where Δ_{ek} is the k*th* electrode's potential. Using the Lagrange's equations we get the equations of vibration with electrostatic field as

$$\ddot{x}+\begin{bmatrix}0 & -\lambda_1\Omega_z & \lambda_2\Omega_y & -\lambda_2\Omega_x\\ \lambda_1\Omega_z & 0 & \lambda_2\Omega_x & \lambda_2\Omega_y\\ -\lambda_3\Omega_y & -\lambda_3\Omega_x & 0 & -\lambda_4\Omega_z\\ \lambda_3\Omega_x & -\lambda_3\Omega_y & \lambda_4\Omega_z & 0\end{bmatrix}\dot{x}+\begin{bmatrix}\xi_1\omega_1 & 0 & 0 & 0\\ 0 & \xi_1\omega_1 & 0 & 0\\ 0 & 0 & \xi_2\omega_2 & 0\\ 0 & 0 & 0 & \xi_2\omega_2\end{bmatrix}\dot{x}$$
$$+\begin{bmatrix}\omega_1^2 & 0 & 0 & 0\\ 0 & \omega_1^2 & 0 & 0\\ 0 & 0 & \omega_2^2 & 0\\ 0 & 0 & 0 & \omega_2^2\end{bmatrix}x+\begin{bmatrix}a_1 & 0 & 0 & 0\\ 0 & a_2 & 0 & 0\\ 0 & 0 & a_3 & 0\\ 0 & 0 & 0 & a_4\end{bmatrix}x=\begin{bmatrix}q_1\\ 0\\ 0\\ 0\end{bmatrix} \tag{21}$$

Consider the driving voltages to be

$$V_{e1}(t)=V_{e3}(t)=V_P Sin(\nu t)\,,\; V_{e2}(t)=V_{e4}(t)=V_P Cos(\nu t)\,. \tag{22}$$

Then one of the coefficients a_i, says a_1 has the explicit form as

$$a_1=-\frac{12a^2n^2\varepsilon_0 V_P^2[4\alpha+Sin(4\alpha)]}{\pi\rho h\Delta_i^3[h^2(n^2-1)^2+12a^2(n^2+1)]} \tag{23a}$$

and the forcing element q_1 is

$$q_1=-\frac{12a^2n^2\varepsilon_0 V_P^2 Sin(2\alpha)Cos(2\nu t)}{h\rho\Delta_i^2\pi[h^2(n^2-1)^2+12a^2(n^2+1)]}\,. \tag{23b}$$

Since the damping ratio ξ_i are small, also the ratio of the angular rate to the natural frequency is small, thus we can use the perturbation technique to obtain first-order approximate analytical solutions. Now we introduce

$$T=\omega_1 t\,,\; \xi_1=\varepsilon\bar{\xi}_1\,,\; \frac{a_1}{\omega_1^2}=\varepsilon\bar{a}_1,\; \frac{a_2}{\omega_1^2}=\varepsilon\bar{a}_2\; \frac{\Omega_x}{\omega_1}=\varepsilon\bar{\Omega}_x,\; \frac{\Omega_y}{\omega_1}=\varepsilon\bar{\Omega}_y,\; \frac{\Omega_z}{\omega_1}=\varepsilon\bar{\Omega}_z,$$

$$\frac{a_3}{\omega_1^2}=\varepsilon^2\bar{a}_3[6\alpha+Sin(6\alpha)Cos(2\nu t)],\; \frac{a_4}{\omega_1^2}=\varepsilon^2\bar{a}_4[-6\alpha+Sin(6\alpha)Cos(2\nu t)],$$

$$\omega_r=\frac{\omega_2}{\omega_1},\; \frac{q_1}{\omega_1^2}=\varepsilon^3\bar{q}_1,\; x(t)=X(T) \tag{24}$$

Thus equation (21) can be parameterized in ε as

$$\ddot{\mathbf{X}}+\varepsilon\begin{bmatrix}\bar{\xi}_1 & -\lambda_1\bar{\Omega}_z & \lambda_2\bar{\Omega}_y & -\lambda_2\bar{\Omega}_x\\ \lambda_1\bar{\Omega}_z & \bar{\xi}_1 & \lambda_2\bar{\Omega}_x & \lambda_2\bar{\Omega}_y\\ -\lambda_3\bar{\Omega}_y & -\lambda_3\bar{\Omega}_x & \bar{\xi}_2 & -\lambda_4\bar{\Omega}_z\\ \lambda_3\bar{\Omega}_x & -\lambda_3\bar{\Omega}_y & \lambda_4\bar{\Omega}_z & \bar{\xi}_2\end{bmatrix}\dot{\mathbf{X}}+\begin{bmatrix}1+\varepsilon\bar{a}_1 & 0 & 0 & 0\\ 0 & 1+\varepsilon\bar{a}_2 & 0 & 0\\ 0 & 0 & \omega_r^2+\varepsilon^2\bar{a}_3[6\alpha+Sin(6\alpha)Cos(2\nu t)] & 0\\ 0 & 0 & 0 & \omega_r^2+\varepsilon^2\bar{a}_4[-6\alpha+Sin(6\alpha)Cos(2\nu t)]\end{bmatrix}\mathbf{X}=\varepsilon^3\begin{bmatrix}\bar{q}_1\\0\\0\\0\end{bmatrix} \tag{25}$$

Assume the solution is in the form

$$X(T)=\begin{bmatrix}Q_{11}(T)\\Q_{21}(T)\\Q_{31}(T)\\Q_{41}(T)\end{bmatrix}+\varepsilon\begin{bmatrix}Q_{12}(T)\\Q_{22}(T)\\Q_{32}(T)\\Q_{42}(T)\end{bmatrix}+O(\varepsilon^2)\,. \tag{26}$$

The zero-order solution can be easily obtained as

$$\begin{gathered}Q_{11}(T)=A_{11}e^{jT}\quad Q_{21}(T)=A_{21}e^{jT}\\ Q_{31}(T)=A_{31}e^{j\omega_r T}\quad Q_{21}(T)=A_{21}e^{jT}\end{gathered}\,. \tag{27}$$

Substituting this into the second order governing equations and by eliminating the secular terms we can find solution for A_{11}, A_{21}, A_{31}, and A_{41}. Then the first order solutions $X_i(t)$ are obtained. We define the ratios of the output signal to the input one $X_1(t)$ as $\gamma_2=X_2(t)/X_1(t)$, $\gamma_3=X_3(t)/X_1(t)$, $\gamma_4=X_4(t)/X_1(t)$. (28)

The equations which relate the angular rates Ω_i to the displacements of the ring are obtained explicitly in the form

$$\Omega_x = \frac{\Omega_{xN}}{\Omega_{xD}}, \qquad \Omega_{xD} = \lambda_3[(1+\gamma_2^2)\lambda_1\lambda_3 - (\gamma_3^2+\gamma_4^2)\lambda_2\lambda_4]$$

$$\begin{aligned}\Omega_{xN} = &[\gamma_2^2\gamma_4\lambda_3\lambda_4\xi_1 + \gamma_4\xi_2((\gamma_3^2+\gamma_4^2)\lambda_2\lambda_4 - \lambda_1\lambda_3) + \gamma_2\gamma_3\lambda_3(\lambda_4\xi_1+\lambda_1\xi_2)]\omega \\ &- j\gamma_2\lambda_4\frac{a_2}{\omega}(\gamma_3+\gamma_2\gamma_4)\end{aligned} \quad . \qquad (29)$$

Equation (29) shows that the relation between Ω_i and γ_j are nonlinear. The coupling effect is shown in Fig. 7, which reveals that Ω_y affects the linear relation between γ_4 and Ω_x by shifting the curve to the left.

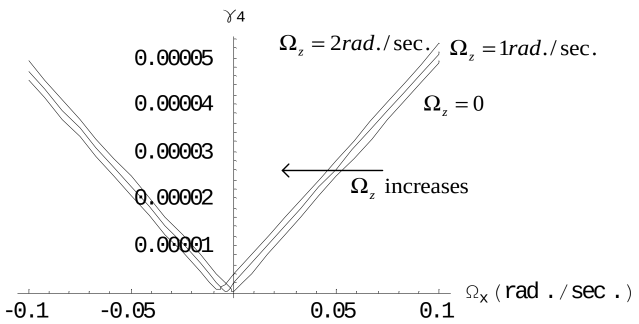

Fig. 7. The linear relation between Ω_x and γ_4 , and the Coupling effect.

Conclusions

In this paper the performance of the three-axis ring gyro is analyzed successfully. We have found the equation in which the in-plane frequency can be tuned o that of the out-of-planes one. Also we drive the exact closed-form solution for the relationship between the angular rates and the amplitudes of ring's vibration, in which the coupling effect is included.

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