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TORSIONAL VIBRATION ANALYSIS ON CIRCULAR AND ANNULAR PLATES

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Abstract

Circular and annular plates are widely used in machinery as basic components. Their torsional vibration behaviour is studied in the paper. An analytical model is developed and the natural frequencies and mode shapes are calculated for the torsional vibration under different boundary conditions. Results show that the torsional vibration behaviour of circular and annular plates is different compared with that of circular shafts. The torque moment in annular plates is transmitted via ring layers from the inner ring to outer ring and vice versa. In addition, the torsional vibration magnitude in terms of the angular displacement decreases with increasing radius. This can be explained by the fact that when the vibration energy is transmitted from the inner ring to outer ring, the circle becomes larger and the energy density becomes lower.

INTRODUCTION

Vibration of circular and annular plates has extensively been studied from the aspects of structure dynamics. Study on the torsional vibration behaviour usually employs a three-dimensional model and is often combined with flexural and axial vibration of the plates. Based on the Ritz method, Leissa and So [1, 2] calculated the free vibration frequencies of solid circular cylinders using a Fourier series in the circumferential direction and algebraic polynomials in the radial and axial direction as the admissible displacement functions. Using similar displacement functions, the Ritz method was applied in a 3-D analysis to obtain natural frequencies for thick circular and annular plates [3] and for the linearly tapered, annular plates [4]. In [1-4] the torsional vibration behaviour was not specifically be investigated, but was included as the axisymmetric vibration modes, although exact solutions were given in [1] for the torsional modes of the cylinders. Recently, Lu et al [5] studied torsional vibration of bellows using an equivalent thin-walled pipe model，but the torsional vibration behaviour of the annular plate components was not considered.

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Torsional vibration of thin circular and annular plates is different from that of slender rods, although they both are axisymmetric. In thin annular plates the torque moment is transmitted via ring layers from the inner ring to outer ring or from the outer ring to inner ring, whereas in slender rods it is transmitted through cross-sections from one end to another. Solutions for the thin circular and annular plates are less difficult to achieve because of the simplicity of the displacements for the torsional modes. Of the polar coordinates (r, θ), the circumferential one (θ) may be uncoupled in the solution [1], and the resulting analysis is only one-dimensional. This permits analytical solutions for the torsional modes of the thin circular and annular plates.

In this paper the governing equation for torsional vibration of thin circular and annular plates is derived. Then the torsional natural frequencies and mode shapes are calculated for four sets of different boundary conditions. The torsional vibration behaviour of thin circular and annular plates is analyzed to obtain the physical insight.

EQUATION OF MOTION

The torsion problem of thin circular and annular plates can be modelled as a plane stress problem. The element equilibrium equations for the plane stress or strain in the polar coordinates (r, θ) are given as [6]

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0, \quad \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + S = 0, \qquad \text{(1-a, b)}$$

where σ_r and σ_θ are the normal stress in the radial and circumferential direction respectively. $\tau_{r\theta}$ is the shear stress. R and S are the body force in the radial and circumferential direction respectively.

For the free torsional vibration of circular and annular plates both the normal stress and the shear stress may exist. The normal stress can be caused by the inertial force in the radial direction (represented by R in (1-a)), e.g. the centrifugal force, due to the circular motion of the plate. The shear stress can be caused by the inertial force in the circumferential direction (represented by S in (1-b)). For torsional vibration of circular and annular plates the circumferential wave number is equal to zero and the vibration modes are axisymmetric [1-4]. As a result, the shear, normal stress and strain are axisymmetric and vary only with the radial direction of the plates. The equilibrium equations (1-a) and (1-b) can therefore be simplified to, respectively,

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0, \quad \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + S = 0. \qquad \text{(2-a, b)}$$

Equations (2-a) and (2-b) now become *uncoupled* from each other, i.e. the inertial forces in the radial direction due to the circular motion of the plates cause only the normal stress and strain, and the inertial forces in the circumferential direction cause only the shear stress and strain. On the Base of the above analysis, the torque moment in a vibrating circular or annular plate can be assumed to be transmitted via ring layers from the inner ring to the outer ring and vice versa. The torsional vibration of each ring results from the shear stress in the circumferential direction. The shear strain in each ring layer can be considered to be of pure shearing strain, and their values are constant on each ring layer, although they vary with radius of the ring.

As the torque moment in a torsionally vibrating, circular or annular plate is transmitted via ring layers and the shear strain and stress on each ring is constant, an isolated ring in a circular or annular plate can be chosen as the basic element to derive the equation of motion for torsional vibration. A ring layer in an annular plate is schematically shown in Figure 1(a), where a and b are the inner and outer radius of the annular plate respectively. Considering an isolated ring from the annular plate at radius r, the torque moments applied to the inner and outer surfaces of the ring are denoted by T and $T + \partial T/\partial r\, dr$, respectively, as shown in Figure. 1(b). The torque moment T can be calculated according to the shear stress on the inner ring surface and is given by

$$T = 2\pi r^2 \delta \tau_{r\theta}, \tag{3}$$

where δ is the plate thickness and $\tau_{r\theta}$ is the shear stress.

According to Newton's second law, the equation of motion for torsional vibration of the ring layer can be written as

$$\rho I_p \frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial T}{\partial r} dr, \tag{4}$$

where φ is the angular displacement of the ring, ρ is the density of the plate, I_p is the rotational inertia moment of the ring about the plate centre O and can be calculated by

$$I_p = 2\pi r^3 \delta dr. \tag{5}$$

Because of the axisymmetric properties, each small sector of the ring is subjected to the same shear stress and strain. The shear stress and strain for an arbitrary element of the ring is schematically shown in Figure 1(c), where the stresses on the side surfaces of the element are not shown. The relationship between the shear stress $\tau_{r\theta}$ and strain $\gamma_{r\theta}$ is given by Hooke's law

$$\tau_{r\theta} = G\gamma_{r\theta}, \tag{6}$$

where G is the shear modulus.

Another relationship to be used here is between the shear strain $\gamma_{r\theta}$ and the angular displacement φ of the element of the ring. It can be derived from the geometry relationship in the polar coordinates (r, θ) between the shear strain and the translational displacements [6]:

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \tag{7}$$

where u denotes the radial displacement and v denotes the circumferential displacement and is given by

$$v = r\varphi. \tag{8}$$

As the shear strain $\gamma_{r\theta}$ is axisymmetric, equation (7) can be given in the form

$$\gamma_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r}. \tag{9}$$

Substituting equation (8) into equation (9) results in the relationship between the shear strain and angular displacement:

$$\gamma_{r\theta} = r\frac{\partial \varphi}{\partial r}. \tag{10}$$

Using equations (6) and (10) the torque moment T applied to the inner ring surface can be written as

$$T = 2\pi r^3 \delta G \frac{\partial \varphi}{\partial r}. \tag{11}$$

Its derivative with respective to radius r is derived as

$$\frac{\partial T}{\partial r} = 2\pi\delta G\left(3r^2 \frac{\partial \varphi}{\partial r} + r^3 \frac{\partial^2 \varphi}{\partial r^2}\right). \tag{12}$$

Finally, substituting equations (5) and (12) for I_p and $\partial T/\partial r$ in equation (4), respectively, the equation of motion for the torsional vibration of an annular plate can be written as

$$\frac{\partial^2 \varphi}{\partial t^2} = \frac{G}{\rho}\left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{3}{r}\frac{\partial \varphi}{\partial r}\right). \tag{13}$$

When the radius r is close to zero, $3/r$ reaches infinite and equation (13) becomes trivial. This is for the case of a circular plate.

NATURAL FREQUENCIES AND MODE SHAPES

When a circular or annular plate vibrates in one of its torsional modes, the solution to equation (13) may be taken in the form

$$\varphi = \Phi(r)e^{i\omega t}, \tag{14}$$

where ω is the angular frequency, $\Phi(r)$ represents a function of r that defines the natural mode shape of torsional vibration of the circular or annular plate. Substituting equation (14) into equation (13) yields

$$\frac{d^2\Phi}{dr^2} + \frac{3}{r}\frac{d\Phi}{dr} + \frac{\omega^2}{c^2}\Phi = 0, \tag{15}$$

where

$$c = \sqrt{G/\rho} \tag{16}$$

is the torsional wave propagating speed in the radial direction, which is the same as the torsional wave speed in a uniform rod propagating in the axial direction.

Introducing a non-dimensional variable $x = r/b$, and substituting x for r in equation (15) results in

$$\frac{d^2\Phi}{dx^2} + \frac{3}{x}\frac{d\Phi}{dx} + k^2\Phi = 0, \tag{17}$$

where

$$k = b\frac{\omega}{c} \tag{18}$$

is the total wave number within radius b.

Equation (17) is similar to a single degree of freedom vibration system. The second term in equation (17) works as a damping, with the damping coefficient $3/x$ varying inverse-proportionally with x. Thus the solution to Φ is expected to decay with increasing the non-dimensional distance x, although there is no analytical solution for equation (17).

Equation (17) is to be solved using a numerical method under the following four sets of different boundary conditions:

1. Inner ring free and outer ring free (free-free)
   $x = a/b$, $d\Phi/dx = 0$ and $x = 1$, $d\Phi/dx = 0$
2. Inner ring free and outer ring clamped (free-clamped)
   $x = a/b$, $d\Phi/dx = 0$ and $x = 1$, $\Phi = 0$
3. Inner ring clamped and outer ring free (clamped-free)
   $x = a/b$, $\Phi = 0$ and $x = 1$, $d\Phi/dx = 0$
4. Inner ring clamped and outer ring clamped (clamped-clamped)
   $x = a/b$, $\Phi = 0$ and $x = 1$, $\Phi = 0$

Equation (17) is solved using the fourth order Runge-Kutta method under the four sets of boundary conditions. Numerical integrations start from $x = a/b$, where the corresponding boundary conditions are applied as the initial values for Φ and $d\Phi/dx$. Different values for k are tried in the calculations in order to meet the corresponding boundary conditions at $x = 1$. Once the boundary conditions at $x = 1$ are satisfied by the value chosen for k, the solution is obtained. The natural frequencies of the torsional vibration can be determined via equation (18) according to k, whereas the mode shapes are plotted using the numerical solutions calculated for Φ at each x. For the free-free boundary condition there is a rigid body mode that can directly be derived from equation (17), with the natural frequency being zero (wave number $k = 0$) and the mode shape function Φ = constant.

RESULTS

The natural frequencies and mode shapes of torsional vibration are calculated under the four sets of boundary conditions. The results for the natural frequencies are listed in Table 1 in terms of wave number k for the first three vibration modes (except the rigid body mode). The ratios of the inner to outer radius of the annular plates in the calculations are chosen to be 0.01, 0.25, 0.50 and 0.80. The annular plate with ratio 0.01 approximately represents a circular plate. The outer radius of the annular plates is chosen to be $b = 100$ and the plate thickness is unity. The Young's modulus of the plate material is 2.1×10^{11} N/m^2 and the Poisson's ratio is 0.3. The material density is 7800 kg/m^3.

The natural frequencies can be seen from Table 1 to increase with the ratio of the inner to outer radius for all the boundary conditions considered. The reason for this is that the torsional stiffness of an annular plate increases with radius. For the plate with ratio 0.01, which is approximately regarded as a circular plate, the natural frequencies are almost the same under the free-free and clamped-free boundary conditions, and under the free-clamped and clamped-clamped boundary conditions as well. This is because the torsional stiffness at the inner radius is very small if the inner radius is small. Thus the clamped boundary condition at the small inner radius results in similar effects to those from the free boundary condition. When the inner radius of an annular plate increases and becomes larger, the two different boundary conditions at the inner radius cause different effects on the natural frequencies. This can be observed from Table 1.

The torsional vibration mode shapes calculated are plotted in Figures. 2 and 3 for the free-free and clamped-free boundary conditions respectively. The mode shapes are shown for the first three vibration modes. All the mode shapes show a decaying behaviour with increasing x in terms of the angular displacement magnitude. The decay rate becomes lower with increasing ratio a/b of the inner to outer radius. This decaying behaviour can be explained by the governing equation, equation (17), where the damping coefficient $3/x$ varies inverse-proportionally with x. Thus a smaller ratio a/b causes higher damping effects, whereas a lager ratio a/b results in smaller damping effects.

Comparing the mode shapes under the free-free and clamped-free boundary conditions of the annular plates with $a/b = 0.01$, the angular vibration magnitudes rise dramatically from $x = 0.01$ to about $x = 0.05$ under the clamped-free boundary conditions, see Figure. 3(a), and then they vary with x similarly to those under the free-free boundary conditions. This is because of the small torsional stiffness at the small inner radius for the annular plates with $a/b = 0.01$. Thus the natural frequencies under the two boundary conditions are almost the same, as analyzed before.

CONCLUSIONS

Torsional vibration of circular and annular plates has been studied. An analytical model has been developed using the axisymmetric property of the torsional vibration. The results show that the torsional vibration behaviour of circular and annular plates is different compared with that of slender rods. The torque moment in circular and annular plates is transmitted via ring layers from the inner ring to outer ring or from the outer ring to inner ring, whereas in slender rods it is transmitted from one end to another through the cross-sections. As the torsional stiffness of annular plates increases with radius, the vibration magnitude in terms of the angular displacement decreases with increasing radius. This can be explained by the fact that when the vibration energy is transmitted from the inner ring to outer ring, the circle becomes larger and the energy density becomes lower. It is similar to the situation of spherical wave propagation from its source.

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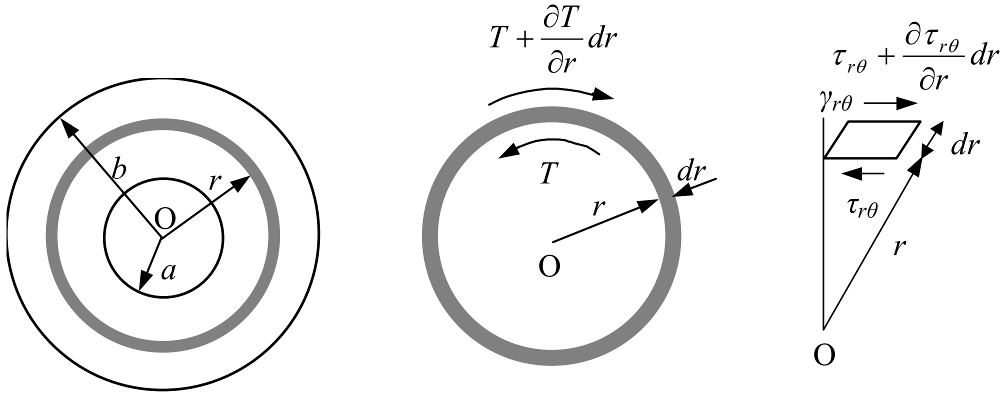
Table 1. Natural frequencies of torsional vibration of annular plates

Boundary conditions	$\frac{a}{b}$	$\omega_n = \frac{k_n}{b}\sqrt{\frac{G}{\rho}}$, $n = 1, 2, 3, \ldots$		
		k_1	k_2	k_3
free-free	0.01	5.14	11.62	14.80
	0.25	5.32	9.14	13.12
	0.50	6.81	12.86	19.05
	0.80	15.86	31.49	47.17
free-clamped	0.01	3.83	7.02	10.17
	0.25	3.90	7.45	11.29
	0.50	4.55	10.09	16.13
	0.80	8.95	23.97	39.52
clamped-free	0.01	5.14	11.65	14.84
	0.25	6.17	10.41	14.62
	0.50	9.18	15.56	21.89
	0.80	23.26	39.09	54.85
clamped-clamped	0.01	3.83	7.03	10.20
	0.25	4.45	8.54	12.68
	0.50	6.39	12.62	18.89
	0.80	15.74	31.43	47.13



(a) (b) (c)

Figure 1 – Torque moment, shear stress and strain in annular plate.

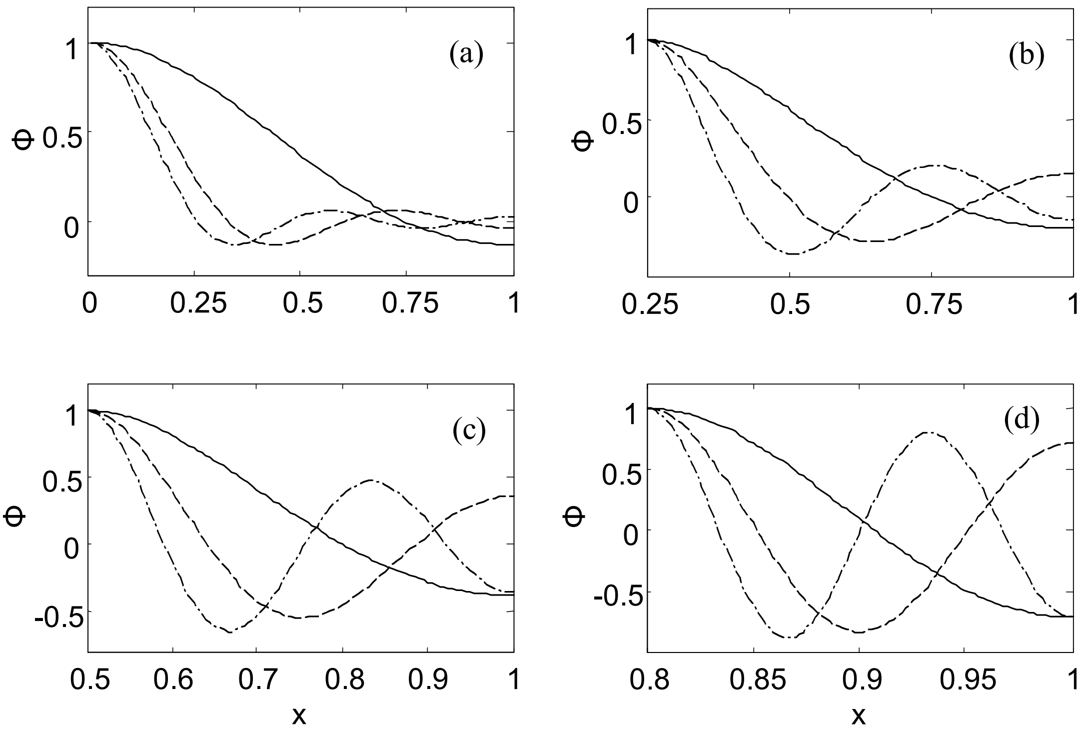


Figure 2 – Torsional vibration mode shapes of annular plates under free-free boundary conditions. (a) a/b = 0.01, (b) a/b = 0.25, (c) a/b = 0.5, (d) a/b = 0.8. — 1st order mode shape, – – 2nd order mode shape, – · – 3rd order mode shape.

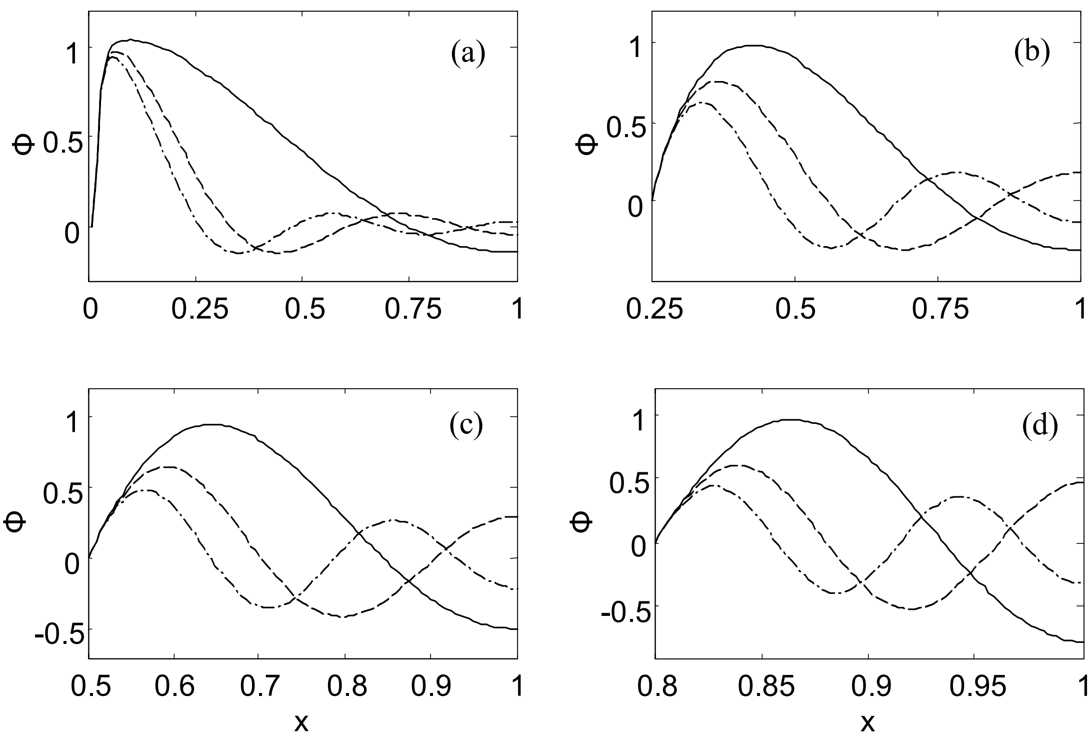


Figure 3 – Torsional vibration mode shapes of annular plates under clamped-free boundary conditions, key as for Figure 2.