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INSTANTANEOUS HARMONIC CONTROL FOR MULTIVARIABLE SYSTEMS: CONVERGENCE ANALYSIS AND EXPERIMENTAL VALIDATION

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Abstract

This paper derives convergence conditions for harmonic control algorithms when applied in instantaneous update form. Although similar results have previously been obtained for specific controller forms, for example in the context of a Multiple Error LMS algorithm, an alternative approach is used here that is applicable to a wide class of algorithms. The derivations are carried out for a square multivariable LTI control path and the algorithm includes the use of a general operator and a relaxation gain. The inclusion of the latter enables a simple stability condition to be stated. The convergence results are demonstrated using a multivariable naval vibration isolation test rig. The experimental results validate the theoretical findings and show that the algorithm provides a 40 dB reduction for a single-tone disturbance.

INTRODUCTION

Harmonic control is a well-known approach for actively reducing the effect of periodic disturbances in acoustics and mechanical vibration. In its standard form [1], the control approach is implemented iteratively using a steady-state approach where, following each corrective control action, the algorithm waits for transients to die out before executing the next update.

The disadvantage from using the steady-state approach is that the controller applies pure feedforward control between updates, where the time taken before the next corrective action can, depending on the system dynamics, be several milliseconds. Furthermore, long time periods between updates leads to a slow convergence rate for the algorithm.

In order to avoid these problems it is quite common, in industrial practice, to not wait for a steady-state before applying a corrective action but to implement the algorithm at each sampling point. In this paper this approach is referred to as "instantaneous harmonic control". The higher harmonic control approach, developed initially for helicopter vibration control was designed to operate in this way and it has been shown in [2] that this is equivalent to a classical feedback compensator with infinite gain at the disturbance frequency, thereby enabling standard stability analysis. It was also pointed out in [4] that this result was similar to that obtained in [5] for the case of a multiple error LMS algorithm operating on a sinusoidal reference. Both of these, however, are very specific implementations of instantaneous algorithms. In this paper a convergence proof is derived for a very general class of multiple channel harmonic algorithms which is validated using a multiple degree of freedom vibration isolation mount.

PRELIMINARIES

The starting point is the following discrete-time plant model

$$y(t) = G_c(q)u(t) + G_d(q)d(t) = y_c(t) + y_d(t) \tag{1}$$

where $G_c(q)$ is the control channel, $G_d(q)$ is the disturbance channel, $u(t)$ is the control input and $d(t)$ is the disturbance. The symbol q^{-1} stands for the standard left-shift operator. In the sequel it is assumed that $d(t)$ contains only a single tone, *i.e.* $d(t)=A\cos(\omega_0 t+\varphi)$ for some A, ω_0 and φ (the case of multiple harmonic control is considered in [6]). The vibration control design problem is to find a feedback controller that drives to output $y(t)$ to zero under the assumption of a one-tone disturbance. From now on it is assumed that $G_c(q)$ is a multivariable and square system and it is both controllable and observable.

In harmonic control the idea is to use a steady-state approach whereby the plant (1) is represented in the frequency domain by

$$y(e^{j\omega_o}) = G_c(e^{j\omega_o})u(e^{j\omega_o}) + G_d(e^{j\omega_o})d(e^{j\omega_o}) \tag{2}$$

When $G_d(e^{j\omega_o})$ is known and $d(e^{j\omega_o})$ can be measured, then an obvious feedforward solution for $u(e^{j\omega_o})$ results from solving equation (2) for $y(e^{j\omega_o}) = 0$. However in most situations this is not the case and a number of alternative iterative feedback solutions have been proposed [[1],[7]]; these have the generic form

$$u(k) = \alpha u(k-1) - \beta C y(k) \tag{3}$$

Where k represents an iteration or update index (following steady state), $\beta>0$ is a scalar learning or convergence gain, α is a relaxation or leakage gain ($0<\alpha\leq 1$)and C is a complex matrix, typically chosen to be

$$C = \begin{cases} G_c(e^{j\omega_0})^H \\ G_c(e^{j\omega_0})^{-1} \end{cases} \tag{4}$$

Note that for ease of exposition it has been assumed that α is a scalar but it can also represent the situation where the cost function

$$J = \left\| y(e^{j\omega_o}) \right\|_Q + \left\| u(e^{j\omega_o}) \right\|_R \tag{5}$$

is minimized. For this case, under the assumption that $R = rI$, then $\alpha = 1 - \beta r$ and $C = G_c(e^{j\omega_0})^H Q$. A number of convergence results exist for the steady state approach [1]; convergence analysis of the instantaneous implementation of (3) is presented in the following section.

INSTANTANEOUS IMPLEMENTATION

The first step in the instantaneous algorithm is to approximate the Fourier coefficient with its instantaneous value using the equation

$$\hat{y}(t) = y(t) e^{-j\omega_0 t} \tag{6}$$

where $e^{-j\omega_0 t}$ is a complex reference signal. Based on this instantaneous estimate, the control signal in the *frequency* domain is updated using the formula

$$\hat{u}(t) = \alpha \hat{u}(t-1) - \beta C \hat{y}(t) \tag{7}$$

which is equivalent to equation (3) but now the update interval is the sample rate *T*.

Finally, the control signal is transformed back to the time domain using an "instantaneous" inverse Fourier formula

$$u(t) = \underbrace{\hat{u}(t) e^{j\varpi_0 t}}_{u_1(t)} + \underbrace{\hat{u}(t)^* e^{-j\varpi_0 t}}_{u_2(t)} = 2\,\mathrm{Re}\{u_1(t)\} \tag{8}$$

Combining the equations for $u_1(t)$ and $u(t)$ results in

$$\begin{aligned} u_1(t) &= \hat{u}_1(t)e^{j\omega_o t} = e^{j\omega_o t}(\hat{u}_1(t-1) - \beta C\hat{y}(t)) \\ &= \alpha e^{j\omega_o T}u_1(t-1) - e^{j\omega_o t}\beta C e^{-j\omega_o t}y(t) \\ &= \alpha e^{j\omega_o T}u_1(t-1) - \beta C y(t) \\ &= \alpha e^{j\omega_o T}u_1(t-1) + \beta C(-G_c(q)u_1(t) - G_c(q)u_2(t) - G_d(q)d(t)) \end{aligned} \tag{9}$$

where $\hat{u}_1(t) := \hat{u}(t)$.

In a similar manner, the difference equation for $u_2(t)$ becomes

$$\begin{aligned} u_2(t) &= \alpha e^{-j\omega_o T}u_2(t-1) - \beta C^* y(t) \\ &= \alpha e^{-j\omega_o T}u_2(t-1) + \beta C^*(-G_c(q)u_1(t) - G_c(q)u_2(t) - G_d d(t)) \end{aligned} \tag{10}$$

Combing these two interconnected systems gives the following multivariable dynamic system (where the elements in the partition are also matrices)

$$\begin{bmatrix} I + \beta C G_c(q) - \alpha e^{j\omega_o T}q^{-1}I & \beta C G_c(q) \\ \beta C^* G_c(q) & I + \beta C^* G_c(q) - \alpha e^{-j\omega_o T}q^{-1}I \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} -\beta C G_d(q) \\ -\beta C^* G_d(q) \end{bmatrix} d(t) \tag{11}$$

Because $d(t)$ is a bounded function, it is a well-know result that the system (11) is BIBO stable, if the solutions of (in terms of q)

$$\det \begin{bmatrix} I + \beta C G_c(q) - \alpha e^{j\omega_o T}q^{-1}I & \beta C G_c(q) \\ \beta C^* G_c(q) & I + \beta C^* G_c(q) - \alpha e^{-j\omega_o T}q^{-1}I \end{bmatrix} = 0 \tag{12}$$

are inside the unit circle. Based on these derivations we get the following main result

Proposition 1: The harmonic controller is BIBO stable if for a given β the solutions of the equation (12) as a function of q are inside the unit circle.

However, this result can be developed further in the following way: as a starting point, it is a well-known result that for a block-partitioned matrix

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - BC) \tag{13}$$

if C and D commute, i.e. $CD=DC$. This is obviously the case in (12), because

$$(\beta C^* G_c(q))(I+\beta C^* G_c(q)-\alpha e^{-j\omega_o T}q^{-1}I)$$
$$=\beta C^* G_c(q)+\beta C^* G_c(q)\beta C^* G_c(q)-\alpha\beta C^* G_c(q)e^{-j\omega_o T}q^{-1}I \quad (14)$$
$$=(I+\beta C^* G_c(q)-\alpha e^{-j\omega_o T}q^{-1}I)(\beta C^* G_c(q))$$

It follows that

$$\det\begin{bmatrix} I+\beta CG_c(q)-\alpha e^{j\omega_o T}q^{-1}I & \beta CG_c(q) \\ \beta C^* G_c(q) & I+\beta C^* G_c(q)-\alpha e^{-j\omega_o T}q^{-1}I \end{bmatrix}$$
$$=\det((I+\beta CG_c(q)-\alpha e^{j\omega_o T}q^{-1})(I+\beta C^* G_c(q)-\alpha e^{-j\omega_o T}q^{-1})-\beta^2 CG_c(q)C^* G_c(q)$$
$$=\det((1-\alpha e^{-j\omega_o T}q^{-1})\beta CG_c(q)+(1-\alpha e^{j\omega_o T}q^{-1})\beta C^* G_c(q) \quad (15)$$
$$+(1-\alpha e^{j\omega_o T}q^{-1})(1-\alpha e^{-j\omega_o T}q^{-1})+\beta^2 CG_c(q)C^* G_c(q)I-\beta^2 CG_c(q)C^* G_c(q)I)$$
$$=\det((1-\alpha e^{-j\omega_o T}q^{-1})\beta CG_c(q)+(1-\alpha e^{j\omega_o T}q^{-1})\beta C^* G_c(q)$$
$$+(1-\alpha e^{j\omega_o T}q^{-1})(1-\alpha e^{-j\omega_o T}q^{-1})I)=0$$

Dividing this equation with $(1-\alpha e^{j\omega_o T}q^{-1})(1-\alpha e^{-j\omega_o T}q^{-1})=1-2\alpha\cos(\omega_o T)q^{-1}+\alpha^2 q^{-2}$ results in

$$\det\left(I+\beta\frac{((1-\alpha e^{-j\omega_o T}q^{-1})CG_c(q)+(1-\alpha e^{j\omega_o T}q^{-1})C^* G_c(q))}{1-2\alpha\cos(\omega_o T)q^{-1}+\alpha^2 q^{-2}}\right)=0 \quad (16)$$

This is a standard multivariable characteristic equation for the plant $G_c(q)$ with a compensator[1]

$$K(q)=\beta\frac{((1-\alpha e^{-j\omega_o T}q^{-1})C+(1-\alpha e^{j\omega_o T}q^{-1})C^*)}{1-2\alpha\cos(\omega_o T)q^{-1}+\alpha^2 q^{-2}} \quad (17)$$

Note that with $\alpha=1$ and $C=G_c(e^{j\omega_o})$, equation (17) is entirely equivalent to that obtained in [5] for the multiple error LMS algorithm with a sinusoidal reference.

On the assumption that $G_c(q)$ is stable and $|\alpha|<1$, there cannot be any unstable pole-zero cancellation between $G_c(q)$ and $K(q)$ so internal stability in terms of β can be determined from equation (16) using any suitable method from multivariable control theory. Moreover under this assumption the following proposition can be developed from the small gain theorem [8].

Proposition 2: The instantaneous harmonic algorithm will be stable if the learning

[1] Note that the matrices appear in the order $K(q)G_c(q)$ as equation (11) leads to an evaluation of input sensitivity.

gain is selected according to

$$\beta < \inf_{\omega} \frac{\left|1 - 2\alpha\cos(\omega_o T)e^{-j\omega T} + \alpha^2 e^{-2j\omega T}\right|}{\bar{\sigma}\left((1-\alpha e^{-j(\omega+\omega_o)T})CG_c(e^{j\omega T}) + (1-\alpha e^{-j(\omega-\omega_o)T})C^* G_c(e^{j\omega T})\right)} \tag{18}$$

It is clear that with α close to unity, this result will be highly conservative and an alternative approach using, for example, the Generalized Nyquist Theorem may be preferred. The approach also limits the maximum loop gain to unity thereby negating the disturbance attenuation properties of the algorithm. These results will be illustrated further in the following section.

The condition for the stability of the closed-loop system is naturally important, however, it does not tell anything about performance (i.e. where does $u(t)$ and therefore $y(t)$ converge to?). In order to analyse performance, note that closed-loop stability enables the asymptotic properties to be investigated using steady-state analysis. As a result equation (11) can be re-written as[2]

$$\begin{bmatrix} I + \beta CG_c(e^{j\omega_o}) - I & \beta CG_c(e^{j\omega_o}) \\ \beta C^* G_c(e^{j\omega_o}) & I + \beta C^* G_c(e^{j\omega_o}) - e^{-2j\omega_o} I \end{bmatrix} \begin{bmatrix} u_1(e^{j\omega_o}) \\ u_2(e^{j\omega_o}) \end{bmatrix} = \begin{bmatrix} -\beta CG_d(e^{j\omega_o}) \\ -\beta C^* G_d(e^{j\omega_o}) \end{bmatrix} d(e^{j\omega_o}) \tag{19}$$

The first row of equation (19) is (it is assumed that both C and $G_c(e^{j\omega_o})$ are invertible)

$$u_1(e^{j\omega_o}) + u_2(e^{j\omega_o}) = -\beta^{-1} G_c(e^{j\omega_o})^{-1} C^{-1} \beta CG_d(e^{j\omega_o}) d(e^{j\omega_o}) \tag{20}$$

or equivalently

$$u(e^{j\omega_o}) = -G_c(e^{j\omega_o})^{-1} G_d(e^{j\omega_o}) d(e^{j\omega_o}) \tag{21}$$

which is the result that would have been obtained from the feedforward solution of equation (2). The main results are summarised in the following proposition:

Proposition 3: Assume that β has been chosen so that it satisfies the stability condition of Proposition 1, $\alpha = 1$ and $d(t)$ is a sinusoidal signal, i.e. $d(t)=A\cos(\omega_0 t+\varphi)$ for some A, ω_0 and φ. In this case the instantaneous harmonic control algorithm will drive $y(t)$ to zero asymptotically.

Note that if $d(t)$ is *not* a pure sinusoidal signal or $\alpha \neq 1$, the instantaneous harmonic control algorithm will still be BIBO stable if it satisfies the stability condition of Proposition 1 or 2, but it will not drive $y(t)$ to zero.

[2] Note due to space limitations only the case of $\alpha = 1$ is considered but the approach can be extended to other cases.

EXPERIMENTAL RESULTS

The results of the previous section are demonstrated using the approach to design a controller for the active vibration isolation mount shown in Figure 1. The system has been used by the authors in previous studies on repetitive control [9][10]. This was originally developed in association with BAE Systems Marine during the late 1980's. The main purpose of this mount is for testing active isolation schemes for large marine machinery rafts. The system consists of a central standard passive elastomeric Naval mount around which are located 6 Ling 30N electro-dynamic shakers. These apply forces in parallel to the passive mount and the "stinger" attachments are arranged in a hexapod or Stewart platform style such that control can be applied to all six degrees of freedom (three orthogonal translational forces and three orthogonal torques). Located on top of the mount is an additional inertial shaker used to inject disturbance forces. The peak of the transmissibility curve occurs at 60Hz (mount resonance) and so this frequency is used in the study for the harmonic disturbance.

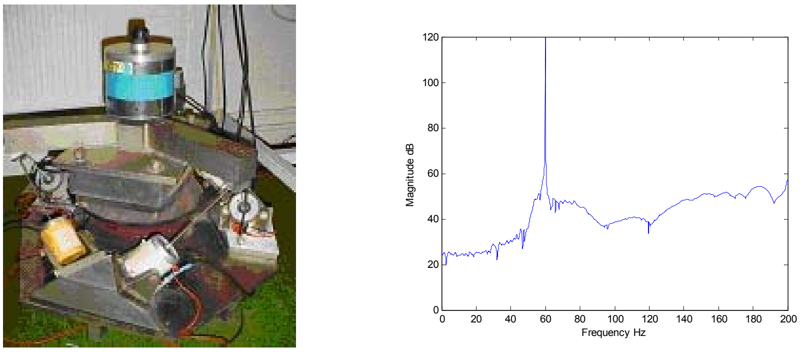

Figure 1 – Experimental active mount and magnitude of maximum eigenvalue

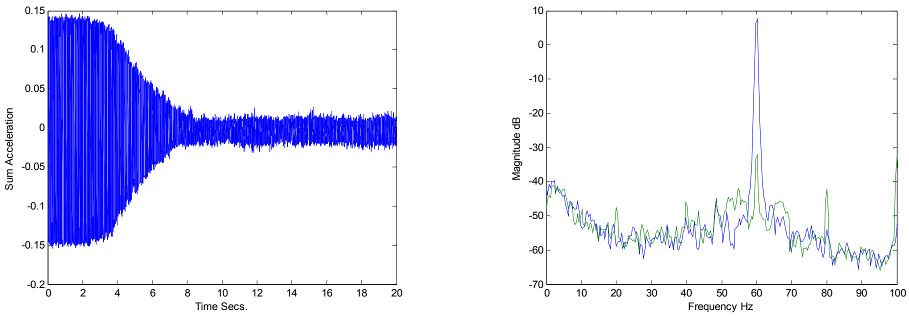

Figure 2 – Experimental system responses: Sum Acceleration

The right of Figure 1 shows the magnitude of the maximum eigenvalue of the loop transfer function as a function of frequency and with a relaxation gain close to unity. The figure is constructed from a calculation that utilises an experimentally measured frequency response function for the system and therefore is limited to a maximum frequency of 200Hz. In the region of 60Hz the phase remains within ±90° however a crossover occurs in the region of 200Hz and so a maximum limit is $\beta < 1\times10^{-3}$. To give some stability margin (in part due to the fact that the magnitude beyond 200Hz is unknown) a value of $\beta = 5\times10^{-5}$ is used. The resulting controller is, as predicted, stable and convergence is rapid (left of Figure 2). The right hand figure shows the power spectral density of the sum acceleration without control and following convergence. The conservatism of proposition 2 is demonstrated by the fact that this requires $\beta < 1\times10^{-7}$.

CONCLUSIONS

This paper has derived rigorous convergence conditions for an instantaneous implementation of a wide class of harmonic control algorithms. One form of the algorithm has been implemented on a multivariable vibration isolation test rig, and shown to be capable of producing a 40dB attenuation of a single-tone disturbance. This result can be considered as exceptionally good for such an application and also validates the theoretical findings. As some conservatism was used in the final gain selection, future work will concentrate on expanding the analysis to include robustness considerations.

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