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ON THE FORMATION OF CLASSICAL AND NON-CLASSICAL SHOCKS IN A TUBE LINED WITH HELMOLTZ RESONATORS

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Abstract

The coefficient of nonlinearity $\Gamma = 1 + B/(2A)$ is of essential importance for the acoustic wave propagation in single-phase gaseous media in thermodynamic equilibrium. In contrast to the well-known perfect gases, which are characterized by values of $\Gamma > 1$, real gases featuring very large values of specific heats and commonly referred to as BZT fluids have the distinguishing property that Γ is found to become negative over a finite range of temperatures and pressures. The existence of thermodynamic states with $\Gamma < 0$ in the dense gas regime leads to the occurrence of phenomena which have no equivalent in ideal gas dynamics. For example, if the equilibrium state of the BZT fluid is chosen to be close to the transition line $\Gamma = 0$, the propagation of planar nonlinear sound waves is governed by a Burgers equation extended with a cubic nonlinearity term, which may result in the simultaneous generation of compression and rarefaction shocks. The analysis presented here focuses on the properties of acoustic waves transmitted through a BZT fluid contained in a rigid tube which is connected to an array of Helmholtz resonators in its axial direction. Such a system gives rise to dispersion as well and, thus, the identification of physically acceptable discontinuous solutions in the limit of vanishing dissipation and dispersion has to be approached by a special regularization principle: In contrast to the classical, non-dispersive case where the admissibility of a discontinuity is ensured by the existence of a viscosity dominated inner shock structure, the shocks are now generated as limits of diffusive-dispersive traveling waves. The thus obtained shock admissibility criteria crucially depend on the precise ratio of dispersion to dissipation in the system. This may lead to wave solutions violating the well-known Oleinik entropy criterion since their discontinuities emanate rather than absorb waves. Such shocks are termed ”non-classical”.

INTRODUCTION

It is well known that in many physical problems the evolution of nonlinear waves can be described by the so-called Korteweg-de Vries-Burgers (KdVB) equation

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = \beta \frac{\partial^2 u}{\partial x^2} + \gamma \frac{\partial^3 u}{\partial x^3}, \quad f(u) = \Gamma \frac{u^2}{2}, \quad \beta > 0, \tag{1}$$

if a balance between nonlinearity, dissipation, and dispersion exists. However, if the nonlinear effects characterized by the parameter Γ are dominant, i.e. in the the limit $\beta \to 0$ and $\gamma \to 0$, the evolution equation reduces to the inviscid Burgers equation, with its formal solutions, in general, featuring regions of multivaluedness that have to be removed by the insertion of discontinuities according to the wave speed ordering principle

$$v_{wb} > v_s > v_{wa}, \quad v_w = \frac{df}{du}, \quad v_s = \frac{[f]}{[u]}. \tag{2}$$

Here v_w and v_s denote, respectively, the wave speed and the shock speed, while the subscripts b and a refer to states before (left of) and after (right of) the shock and $[A]$ denotes the jump of the quantity A, i.e. $[A] = A_a - A_b$. The relationship (2), which is now commonly referred to as the Lax (entropy) criterion [7], in general is considered to be a necessary and sufficient condition for admissible discontinuities such that they are stable and do not violate the natural requirement that acceptable weak solutions must depend continuously on the initial and boundary values. Each jump can thus be regarded as representing a dissipative-dispersive profile whose streamwise extent is so small compared to the characteristic wavelength that it collapses into a single point.

Furthermore, previous investigations dealing with nonlinear wave propagation involving dissipation and dispersion have also demonstrated that in some cases the quadratic flux function $f(u)$ appearing in the KdVB equation (1) has to be extended with an additional cubic term:

$$f(u) = \Gamma \frac{u^2}{2} + \epsilon \Lambda \frac{u^3}{6}, \quad \epsilon \ll 1. \tag{3}$$

Examples include acoustic waves in fluid filled viscoelastic tubes [3], internal waves in a two-layer film flow [2], dust-acoustic waves in plasmas [8], and kinematic waves in suspensions of particles in fluids [6]. The appearance of the additional cubic term in f is a direct consequence of the fact that in these problems the nonlinearity parameter Γ may change sign depending on the particular conditions imposed on the unperturbed state. Obviously, the asymptotic analysis which has led to Eq. (1) assumes $\Gamma = O(1)$ and breaks down in the vicinity of the transition point where the nonlinear parameter vanishes. Within a neighborhood where $\Gamma = O(\epsilon)$, $0 < \epsilon \ll 1$, small terms neglected so far become of the same order as the already included quadratic correction term and a separate analysis is required. Introducing the new (slow) time scale $T = \epsilon t$ and the parameters $\hat{\Gamma} = \Gamma/\epsilon$, $\hat{\beta} = \beta/\epsilon$, and $\hat{\gamma} = \gamma/\epsilon$ then yields the modified Korteweg-de Vries-Burgers (mKdVB) equation

$$\frac{\partial u}{\partial T} + \frac{\partial \hat{f}}{\partial x} = \hat{\beta} \frac{\partial^2 u}{\partial x^2} + \hat{\gamma} \frac{\partial^3 u}{\partial x^3}, \quad \hat{f}(u) = \hat{\Gamma} \frac{u^2}{2} + \Lambda \frac{u^3}{6}, \quad \hat{\beta} > 0. \tag{4}$$

In the limit of small dissipative as well as dispersive effects, i.e. $\hat{\Gamma} = O(1)$, $\Lambda = O(1)$, $\hat{\beta} \ll 1$, and $|\hat{\gamma}| \ll 1$, Eq. (4) reduces to a scalar hyperbolic conservation law where the flux $\hat{f}$ is not convex, i.e. $\hat{f}''$ changes sign. The problem is then said to be characterized by the presence of mixed nonlinearity.

In [6] and [10] it has been shown that if the condition $|\tilde{\gamma}| \ll \tilde{\beta}^2 \ll 1$ holds, the problem of finding the appropriate shock admissibility criteria simplifies to the selection of a traveling wave solution via the so-called vanishing viscosity approach. As pointed out in [4], such a solution can be equivalently selected by using the Oleinik [9] entropy criterion, which for the (non-convex) flux function $\hat{f}(u)$ from Eq. (3) reduces to

$$v_{wb} \geq v_s \geq v_{wa}. \tag{5}$$

In this limiting case, the jump criterion turns out to be equivalent to the Lax wave speed ordering principle (2) in the generalized sense that for either the left-hand state or the right-hand state of the shock the equality sign has to be included. Discontinuities with $v_s = v_{wa}$ or $v_s = v_{wb}$ are called *sonic shocks* and represent the jumps of maximum possible strength, if the state before or, respectively, the state after is fixed.

In the general case where (small) dissipation is balanced by (small) dispersion, however, this concept fails and the insertion of discontinuities in order to obtain physically acceptable weak solutions has to be approached in the following way, see [6], [10]: The key idea is to construct a traveling wave solution based on a dissipative-dispersive regularization which gives the jump conditions in the limit of small dissipation and dispersion. Despite the plausibility of the geometric inequalities of Eq. (5), shocks may exist that emanate rather than absorb waves and, thus, violate the Oleinik criterion. Such shocks are termed ”non-classical”.

The consequence of the above discussion is that the possibility of $\Gamma < 0$ in combination with weak dispersive effects obviously provides new physical phenomena in gas dynamics and acoustics and, thus, is also of great importance for the case considered here, namely, the acoustic waves transmitted through a circular tube which is *a)* filled with a gas having an acoustic nonlinearity parameter $\Gamma = 1 + B/(2A)$ that can change sign and *b)* whose wall is lined with an array of Helmholtz resonators, see Fig. 1.

PROBLEM FORMULATION

The propagation of nonlinear acoustic waves in a duct with an array of Helmholtz resonators has been extensively investigated in the past by Sugimoto and co-workers, see e.g. [11], [12]. According to [11], the system of equations governing the sound wave evolution for the open tube case shown in Fig. 1 can be written as:

$$\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial f}{\partial \phi} = -\delta_R \frac{\partial^{1/2} u}{\partial \phi^{1/2}} + \beta \frac{\partial^2 u}{\partial \phi^2} - K\frac{\partial q}{\partial \phi}, \quad f(u) = \Gamma \frac{u^2}{2}, \quad 0 < \beta \ll \delta_R \ll 1, \\ &\frac{\partial^2 q}{\partial \phi^2} + \delta_r \frac{\partial^{3/2} q}{\partial \phi^{3/2}} + \Omega q = \Omega u, \quad 0 < \delta_r \ll 1, \quad K > 0, \quad \Omega > 0. \end{aligned} \tag{6}$$

Here, the quantities u and q are proportional to the pressures p and p_c in the tube and the cavities, respectively, while δ_R and δ_r measure the ratios of the boundary layer thickness to

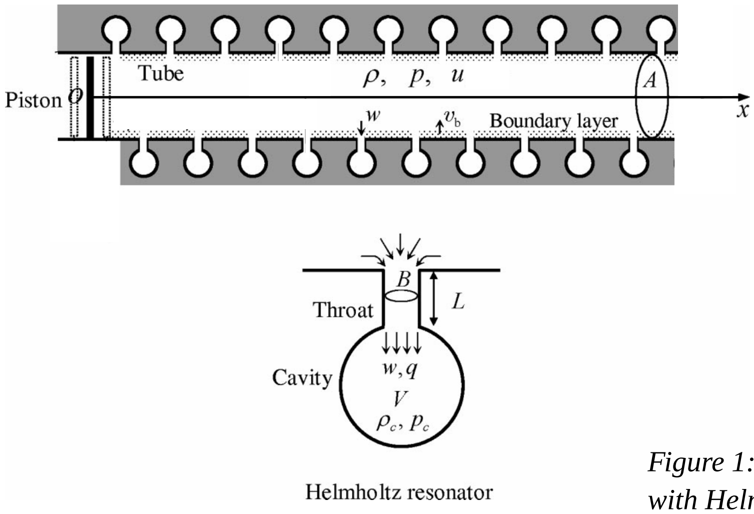

Figure 1: Illustration of an open tube with Helmholtz resonators.

the radius of the tunnel and that of the throat. In addition, the coupling parameter K and the tuning parameter Ω control the dispersive effects of the array of resonators, and the term proportional to β describes the diffusivity of sound on the core region. Furthermore, ϕ and x denote the retarded time measured in a frame moving with the linear sound speed and the far-field space variable, respectively.

For perfect gases with constant specific heats, $\Gamma = (\kappa + 1)/2 > 1$, where κ represents the ratio of the specific heats, and the fluid motion is said to exhibit positive nonlinearity. In this case, only *compression shocks* can form and propagate in the gas. The possibility that also *rarefaction shocks* can form in real fluids, i.e. that Γ might become negative, seems to have been recognized first by Bethe [1] and independently by Zel'dovich [14]. A class of fluids involving gases with high specific heats can be identified for which the curvature of the isentropes in the pressure-density state space is reversed near the coexistence curve in the vicinity of the critical point and, consequently, Γ changes sign. More sophisticated studies based on the Martin-Hou equation of state are due to Thompson and co-workers, see e.g. [13], who gave specific examples of such fluids, which include hydrocarbons and fluorocarbons of moderate complexity. In recognition of these investigations, fluids having the distinguishing property that the fundamental derivative can change sign are commonly referred to as BZT fluids. Furthermore, Kluwick [5] demonstrated that if the equilibrium state of the BZT fluid is chosen to be in the neighborhood to the transition line where $\Gamma = 0$, and thus $\Gamma = O(\epsilon)$, $0 < \epsilon \ll 1$, the propagation of planar, weakly nonlinear acoustic waves in rigid tubes is governed by a modified Burgers equation where $f(u)$ has to be extended with an additional cubic nonlinearity term. As a consequence, $f(u)$ in Eq. (6) has to be replaced with $\epsilon\hat{f}(u)$, with $\hat{f}(u)$ given by Eq. (4).

Introducing the new scales and parameters

$$X = \phi - xK, \quad T = \epsilon x, \quad \hat{\beta} = \frac{\beta}{\epsilon}, \quad \hat{\gamma} = \frac{K}{\Omega\epsilon}, \quad \hat{K} = \frac{K}{\epsilon}, \quad \hat{\delta}_R = \frac{\delta_R}{\epsilon} \tag{7}$$

and assuming that the viscous *and* the dispersive effects are small, the system (6) can be recast into (c.f. Eq. (4))

$$\frac{\partial u}{\partial T} + \frac{\partial \hat{f}}{\partial X} = -\hat{\delta}_R \frac{\partial^{1/2} u}{\partial X^{1/2}} + \hat{\beta}\frac{\partial^2 u}{\partial X^2} + \hat{\gamma}\left(\frac{\partial^3 q}{\partial X^3} + \delta_r \frac{\partial^{5/2} q}{\partial X^{5/2}}\right), \quad \hat{f}(u) = \hat{\Gamma}\frac{u^2}{2} + \Lambda\frac{u^3}{6},$$
$$q = u - \frac{\hat{\gamma}}{\hat{K}}\left(\frac{\partial^2 q}{\partial X^2} + \delta_r \frac{\partial^{3/2} q}{\partial X^{3/2}}\right), \quad \hat{\beta} \ll \hat{\delta}_R \ll 1, \quad \hat{\gamma} \ll 1. \tag{8}$$

In analyzing the shock admissibility conditions of the wave solutions traveling in the system (8), we shall concentrate on the case where the inner shock structures are governed by the diffusivity in the core region and the dispersion due to the array of resonators. Inspection of Eqs. (8) then shows that within the spatial ranges $\Delta X = O(\hat{\beta})$ and $\Delta X = O(\hat{\gamma}^{1/2})$ consumed by the inner shock profiles the boundary layer effects are negligibly small. In the following, therefore, the terms proportional to $\hat{\delta}_R$ and δ_r will be omitted.

SHOCK ADMISSIBILTY

For convenience the parameters $\hat{\Gamma}$ and Λ appearing in Eqs. (8) are eliminated by introducing the suitably scaled quantities

$$U = \frac{\Lambda}{\hat{\Gamma}}u, \quad Q = \frac{\Lambda}{\hat{\Gamma}}q, \quad \tilde{X} = \frac{\Lambda}{\hat{\Gamma}^2}X, \quad \tilde{\beta} = \frac{\Lambda^2}{\hat{\Gamma}^4}\hat{\beta}, \quad \tilde{\gamma} = \frac{\Lambda^3}{\hat{\Gamma}^6}\hat{\gamma}, \quad \tilde{K} = \frac{\Lambda}{\hat{\Gamma}^2}\hat{K}. \tag{9}$$

Neglecting boundary layer effects, the system (8) then assumes the following form:

$$\frac{\partial U}{\partial T} + \frac{\partial F}{\partial \tilde{X}} = \tilde{\beta}\frac{\partial^2 U}{\partial \tilde{X}^2} + \tilde{\gamma}\frac{\partial^3 Q}{\partial \tilde{X}^3}, \quad F(U) = \frac{U^2}{2} + \frac{U^3}{6},$$
$$Q = U - \frac{\tilde{\gamma}}{\tilde{K}}\frac{\partial^2 Q}{\partial \tilde{X}^2}, \quad \tilde{\beta} \ll 1, \quad \tilde{\gamma} \ll 1. \tag{10}$$

In order to decide which jump discontinuities are admissible for weak solutions of Eqs. (10) in the limit $\tilde{\gamma} \sim \tilde{\beta}^2 \ll 1$, we study traveling wave solutions of the form

$$U = U(\xi), \quad Q = Q(\xi), \quad \xi = \tilde{X} - V_s T,$$
$$(U, Q) \to U_b \quad \text{for} \quad \xi \to -\infty, \quad (U, Q) \to U_a \quad \text{for} \quad \xi \to \infty. \tag{11}$$

By using the scaled quantities

$$G = \frac{2}{[U]}\left(\frac{U_b + U_a}{2} - U\right), \quad H = \frac{2}{[U]}\left(\frac{U_b + U_a}{2} - Q\right), \quad \eta = \frac{\sqrt{6}|[U]|}{12\sqrt{\tilde{\gamma}}}\xi \tag{12}$$

in place of U, Q and ξ, the shock structure problem reads

$$(G^2 - 1)(B + G) = dG' + H'', \quad H = G - \frac{1}{k}H'', \quad (G, H) \to \pm 1 \quad \text{for} \quad \eta \to \mp\infty, \tag{13}$$

where " $'$ " denotes differentiation with respect to η and

$$B = -\frac{6}{[U]}\left(1 + \frac{U_b + U_a}{2}\right), \quad k = \frac{24\tilde{K}}{[U]^2}, \quad d = \frac{2\sqrt{6}}{|[U]|}\sigma, \quad \text{with} \quad \sigma = \frac{\tilde{\beta}}{\sqrt{\tilde{\gamma}}}. \tag{14}$$

Obviously, if $d \gg 1$, i.e. $\tilde{\gamma} \ll \tilde{\beta}^2 \ll 1$, dissipation dominates over dispersion and, as pointed out in the Introduction, then only jumps satisfying the Oleinik criterion (5) can occur (classical shocks). However, in the more general case of $d = O(1)$, the limit to small dissipation and dispersion has to be be carried out by letting $\tilde{\beta}$ and $\tilde{\gamma}$ tend to zero with the parameter $\sigma = O(1)$ kept fixed. As before, in this limit ξ tends to zero almost everywhere and, in original variables, the shock layer collapses into a jump discontinuity.

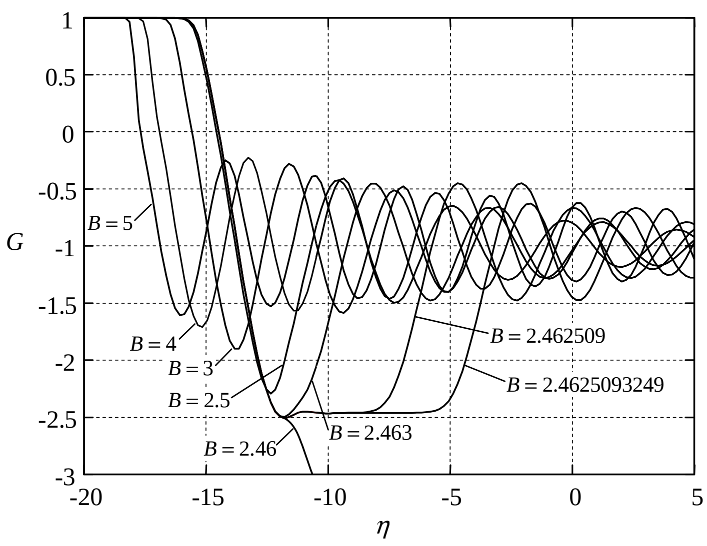

Figure 2: Typical results for $B \to B_{cr}$: The shock layer splits into two sublayers. Inside the first G decreases from 1 to a value $-B_{cr} < -1$ and increases to -1 inside the second sublayer to satisfy the boundary condition for $\eta \to \infty$. Both sublayers are separated by a pronounced plateau region where $G = -B_{cr}$ whose length tends to infinity as $B \to B_{cr}$; $k = 5$, $d = 0.5$, $B_{cr} \approx 2.46251$

A representative numerical solution to the shock layer problem (13) is plotted in Fig. 2. It clearly shows that shock layer solutions can be obtained for values $B \geq B_{cr}(d,k) > 1$ only. Interestingly, as in the case of the mKdVB equation (4) discussed in [6] and [10], the solution for the critical value $B = B_{cr}(d,k)$ splits into two sublayers separated by a pronounced plateau region where $G = -B_{cr}$.

The possibility that the shock layer problem (13) possesses a solution for the modified boundary conditions

$$(G,\, H) \to 1 \quad \text{for} \quad \eta \to -\infty, \quad (G,\, H) \to -B \quad \text{for} \quad \eta \to \infty, \tag{15}$$

with the eigenvalue $B = B_{cr}(d,k)$, enables the calculation of a state $(U_A,\, F(U_A))$ after the first shock layer which, in the original variables T and $\tilde{X}$, corresponds to the right-hand state of an admissible jump having a left-hand state $(U_b,\, F(U_b))$, see Fig. 3: Eq. (14) shows that for fixed values of U_b, σ and $\tilde{K}$, the shock strength $[U]_{cr}$ of a jump from $(U_b,\, F(U_b))$ to a critical state $(U_{a,cr},\, F(U_{a,cr}))$ is implicitly given by the relation

$$[U] = [U]_{cr}: \quad B_{cr}\left(\frac{2\sqrt{6}}{|[U]|}\sigma, \frac{24\tilde{K}}{[U]^2}\right)[U] + 3[U] = -6 - 6U_b, \quad B_{cr} > 1. \tag{16}$$

This critical state has the distinguishing property that its Rayleigh line, i.e. the chord connecting states before and after jumps, is at the same time a Rayleigh line connecting the states

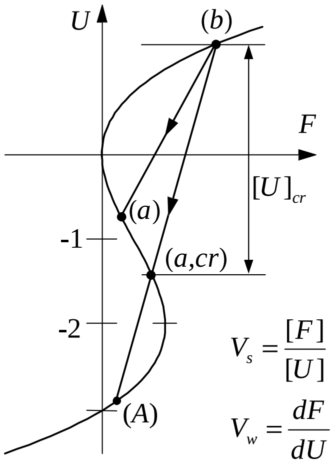

Figure 3: Rayleigh lines of classical and non-classical shocks for $U_b > U_a$: If U_a is continuously reduced one eventually reaches the limiting case of a classical non-sonic shock with $U_a = U_{a,cr}$. For the same state U_b it is, however, possible also to construct a non-classical shock with right-hand state U_A determined by Eq. (17). Similar considerations can be performed for jumps having $U_b < U_a$. For e.g. $U_b = 2.1865$, $\tilde{K} = 2.5520$, $\sigma = 0.3572$, Eqs. (16) and (17) lead to a solution $[U]_{cr} = -3.5$, $U_{a,cr} = -1.3135$, $B_{cr} = 2.46251$ (as in Fig. 2) and $U_A = -3.8729$.

$(U_b,\, F(U_b))$ and $(U_A,\, F(U_A))$. In accordance to that, solving Eq. (12) for $U = U_A$, with $G = -B_{cr}$ and $[U] = [U]_{cr}$, leads to

$$U_A = -2U_b - 3 - [U]_{cr}. \tag{17}$$

A shock from $(U_b,\, F(U_b))$ to $(U_A,\, F(U_A))$ is stronger than a classical sonic shock with $V_s = V_{wa}$, which would correspond to a value $B_{cr}(d,k) = 1$, i.e. $V_{wa} = V_{wA}$. Furthermore, as shown in Fig. 3, the wave and shock speeds satisfy the ordering relationship $V_{wb} > V_s < V_{wA}$. Obviously, such a shock violates the Oleinik admissibility criterion (5), although it is clearly admissible owing to the existence of a shock layer solution.

Asymptotic results

If $k \gg 1$, i.e. $\tilde{K} \sim \hat{K} \gg 1$ and, since $\tilde{\gamma} \sim \hat{\gamma} \ll 1$, necessarily also $\Omega \gg 1$, the shock layer problem (13) can be solved analytically by linearization at the point $k \to \infty$. This yields the expansion

$$B_{cr}(d,k) = 3 - \sqrt{2}d + \frac{1}{k}\frac{\sqrt{2}d}{10}\left(24 - 12\sqrt{2}d + d^2\right) + O(k^{-2}). \tag{18}$$

Here, the leading order term coincides with the solution for the mKdVB equation (4) given in [6] and [10].

In contrast to that, for the case of dispersion dominating over dissipation, i.e. $d \ll 1$ and, therefore, $\tilde{\beta}^2 \ll \tilde{\gamma} \ll 1$, the asymptotic expansion with respect to d results in

$$B_{cr}(d,k) = 3 - d\frac{\sqrt{k}\sqrt{k-8}}{16}\left[\sqrt{2} - \frac{\sqrt{6}(k-8)(3+\sqrt{3})(1+\sqrt{3})}{18\sqrt{k+4}(2+\sqrt{3})}\operatorname{arsinh}\left(\frac{2}{\sqrt{k}}\right)\right.$$
$$\left. - \frac{(k-32)}{6\sqrt{k-8}}\operatorname{arsin}\left(2\sqrt{\frac{2}{k}}\right)\right] + O(d^2). \tag{19}$$

It should be noted that this expression is valid for $k \geq 8$ only. If $d \ll 1$ and $k < 8$, no shock layer solution with $G = -B_{cr}$ for $\eta \to \infty$ exists.

CONCLUSIONS

For values of $d \sim \delta = O(1)$ or smaller, the region for admissible classical shocks is smaller than the region of admissibility following from the standard theory of nonlinear waves, reflecting the fact that shock layer solutions can be obtained for values $B \geq B_{cr}(d,k) > 1$ only. In this sense, the admissibility criterion derived from the existence of a dissipative-dispersive shock structure is more restrictive than the Oleinik criterion. In addition, however, there exists a family of admissible non-classical shocks. Discontinuities of this latter type represent isolated solutions of the shock layer problem, that is to say, a continuous transition of a classical shock into a non-classical shock or vice versa is not possible in general. As a consequence, non-classical shocks may be generated in two ways only: First, by imposing appropriate boundary or initial conditions or, secondly, in an interior point by suitably increasing the strength of the sonic jumps, for example through the interaction with a wave fan.

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