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MOORED DOCK UNDER RANDOM WAVES: A STOCHASTIC PERTURBATION AND LINEARIZATION TECHNIQUE

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Abstract

Aim of this paper is to predict the motion of a dock moored by a couple of cables in irregular waves. Wave forces, of random nature, are determined by a modified Morison equation. The nonlinear effects of the loads and of the cables governing equations are taken into account. The perturbation and the statistical linearization technique is applied to the non linear reaction force of the cable as well to the hydrodynamic load.

INTRODUCTION

A linearization technique is developed to predict the horizontal motion of a floating moored dock loaded by hydrodynamic random forces. This structure is generally known as Catenary Anchor Leg Mooring (CALM). The dock is represented by a lumped mass, the mooring cables by equivalent nonlinear springs and the hydrodynamic forces is modelled by a modified Morison equation [1-4]. The model of the floating dock leads to a nonlinear random ordinary differential equation. Although the problem could be approached by numerical integration, because of the stochastic nature of the excitation, this would imply a large number of runs to produce results of statistical significance. An alternative solution is proposed and it is based on the linearization of the model. A perturbation and a statistical linearization techniques are presented [5-9].

The procedure allows the linearization, in a statistical sense, of the cable forces as well as of the hydrodynamic load and it can be easily generalized to be applied to different dock configurations. The results, compared with those obtained by a Monte-Carlo simulations, show, in terms of statistical moments of the dock response, a very satisfactory agreement.

NONLINEAR STATICS OF THE CABLE AND REDUCTION TO A CUBIC NONLINEARITY

Let s be the curvilinear abscissa, x and z Cartesian coordinates of the cable points (see figure 1), T the tension along the cable and W its weight per unit length. The static equilibrium equations of the cable read:

$$\partial\left(T\,\partial x/\partial s\right)/\partial s = W \quad , \quad \partial\left(T\,\partial z/\partial s\right)/\partial s = 0 \tag{1}$$

Inextensibility along the axial direction implies:

$$\left(\partial x/\partial s\right)^2 + \left(\partial z/\partial s\right)^2 = 1 \tag{2}$$

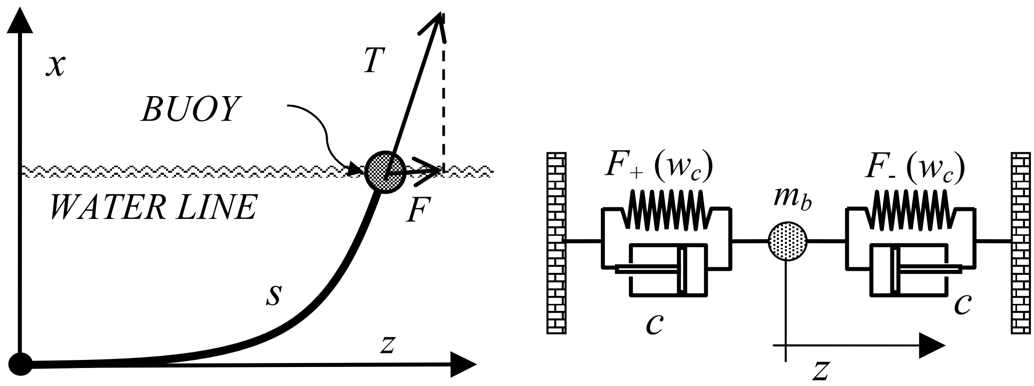

Figure 1 – Physical model

Substitution of (2) into (1) and consideration of void boundary conditions (figure 1) produce an equation which provides the force-displacement relationship at each cable point. The horizontal displacement of the cable end at the water plane is given by the following equation:

$$w_c = h\sqrt{2\frac{F_{\max}}{W\,h}+1} - h\sqrt{2\frac{F}{W\,h}+1} + \frac{F}{W}\cosh^{-1}\left(1+\frac{W\,h}{F}\right) \tag{3}$$

that provides the desired nonlinear dependence of the displacement of the cable end.
The total mooring actions on the dock are here approximated by a simpler cubic restoring force. In this configuration (figure 1) the total restoring force of a couple of cables (figure 2) is given by:

$$F_T\,(w_c) = F\,(w_{c0} + w_c) - F\,(w_{c0} - w_c) = F_+ - F_- \tag{4}$$

where w_c and w_{c0} are the relative cable end displacement and the horizontal distance between the anchor and the cable end at the water plane, respectively. Taylor series expansion of F_T up to the third order provides:

$$F_T\,(w_c) = \gamma_1\,w_c + \gamma_3\,w_c^3 \quad , \quad \gamma_1 = 2/w_c'\Big|_{F_0} \; , \; \gamma_3 = -\left[3\left(w_c''/w_c'\right)^2 - w_c'''/w_c'\right]/3(w_c')^3\Big|_{F_0} \tag{5}$$

RANDOM HYDRODYNAMIC LOAD AND THE DOCK EQUATION OF MOTION

The modified Morison equation provides the wave load q per unit length on a circular cylinder in terms of the fluid-structure relative velocity $(\dot{\xi}-\dot{w})$:

$$q(t)=C_I\,\ddot{\xi}+m_a\,\ddot{w}+C_D\left|\dot{\xi}-\dot{w}\right|(\dot{\xi}-\dot{w}) \tag{6}$$

$\xi(x,z,t)$ is the particle displacement within the water, C_I and C_D are the inertia and the drag coefficient, respectively, and m_a is the added mass. The fluid displacement, ξ, is the following random function:

$$\xi(x,z,t)=-\sum_{n=1}^{\infty}A_n(\omega_n,x)\sin\left[(\omega_n^2/g)z-\omega_n t+\varphi_n\right] \tag{7}$$

where ω_n is the wave frequency, g the gravity acceleration and φ_n a random phase (*i.e.* uniformly distributed over the interval [0, 2π]). The amplitude coefficients, A_n, are deterministic functions of the Pierson-Moskowitz spectrum [5].
In order to achieve a series expansion of ξ whit random coefficients, equation (7) can be modified as follows:

$$\xi(x,t)=\sum_{n=-\infty}^{\infty}B_n(\omega_n,\varphi_n,x)\chi_n(t) \tag{8}$$

where B_n are random complex coefficients and $\chi_n(t)=e^{j\omega_n t}$.
The equation of motion of the dock is:

$$\ddot{\zeta}+a_1\dot{\zeta}+(a_2+3a_4\xi^2)\varsigma+\left[a_3\left|\dot{\zeta}\right|\dot{\zeta}-3a_4\xi\zeta^2+a_4\zeta^3\right]=f(\xi) \tag{9}$$

where the following positions are assumed: $\zeta=\xi-w$ is the relative displacement, $m=m_b+m_a$, $a_1=2\tilde{c}/m$, $a_2=\gamma_1/m$, $a_3=C_D/m$, $a_4=\gamma_3/m$, $a_5=C_I/m$ and the new force relationship is:

$$f(\xi)=(1-a_5)\ddot{\xi}+a_1\dot{\xi}+a_4\xi^3+a_2\xi \tag{10}$$

LINEARIZATION OF THE PROBLEM

Equation (9) is a stochastic nonlinear ordinary differential equation, it is stochastic because it has random coefficients and a random right hand term. Its solution can be reached by three different procedures:

- a deterministic and stochastic perturbation technique coupled together,
- a statistical linearization method attended by a stochastic perturbation technique or, in a very particular case, by using the Ito's equation solution,

In this paper it is called deterministic perturbation technique the procedure which allows to write a new set of linear stochastic ordinary differential equations as a replacement of equation (9).

The deterministic and stochastic perturbation method

For sake of simplicity equation (9) is written as follows:

$$\ddot{\zeta}+a_1\dot{\zeta}+\beta(\xi)\zeta+\varepsilon g(\dot{\zeta},\zeta,\xi)=f(t) \tag{11}$$

where the new symbols have the following definitions:

$$\beta(\xi)=a_2+3a_4\xi^2 \quad , \qquad g(\dot{\zeta},\zeta,\xi)=a_3\left|\dot{\zeta}\right|\dot{\zeta}-3a_4\xi\zeta^2+a_4\zeta^3 \tag{12}$$

Function g is the small non linear term. Since ε is small, the solution is searched in the form of series power of ε:

$$\zeta(\xi,t,\varepsilon)=\zeta_0(\xi,t)+\varepsilon\zeta_1(\xi,t)+\varepsilon^2\zeta_2(\xi,t) \tag{13}$$

Function g can be expanded into a power series in ε about the solution $(\dot{\zeta}_0,\zeta_0)$. Therefore, by inserting equations (13) and the series expansion of g into (11) and by ordering a system of linear stochastic differential equation can be written as:

$$\begin{cases} \ddot{\zeta}_0+a_1\dot{\zeta}_0+a_2\zeta_0+3a_4\xi^2\zeta_0=(1-a_5)\ddot{\xi}+a_1\dot{\xi}+a_4\xi^3+a_2\xi \\ \ddot{\zeta}_1+a_1\dot{\zeta}_1+a_2\zeta_1+3a_4\xi^2\zeta_1=3a_4\xi\zeta_0^2-a_3\left|\dot{\zeta}_0\right|\dot{\zeta}_0-a_4\zeta_0^3 \\ \ddot{\zeta}_2+a_1\dot{\zeta}_2+a_2\zeta_2+3a_4\xi^2\zeta_2=6a_4\xi\zeta_0\zeta_1-2a_3\left|\dot{\zeta}_0\right|\dot{\zeta}_1-3a_4\zeta_0^2\zeta_1 \end{cases} \tag{14}$$

and they can be solved recursively.

Equation system (14) is a set of linear random differential equation. Let us remember equation (8), the solution of the system (14) can be expanded in terms of the coefficients B_n, which are random parameters, as:

$$\zeta_i=\zeta_{i0}(t)+B_n\,\zeta_{i1\,n}(t)+B_n B_m\,\zeta_{i2\,nm}(t) \qquad i=0,1,2 \tag{15}$$

By substituting equation (15) into equations (14) and ordering this new set of differential equations, the following systems can be written:

$$\begin{cases} \ddot{\zeta}_{00}+a_1\dot{\zeta}_{00}+a_2\zeta_{00}=0 \\ \ddot{\zeta}_{01\,n}+a_1\dot{\zeta}_{01\,n}+a_2\zeta_{01\,n}=(1-a_5)\ddot{\chi}_n+a_1\dot{\chi}_n+a_2\chi_n \\ \ddot{\zeta}_{02\,nm}+a_1\dot{\zeta}_{02\,nm}+a_2\zeta_{02\,nm}=-3a_4\chi_n\chi_m\zeta_{00} \end{cases} \tag{16}$$

$$\begin{cases} \ddot{\zeta}_{10}+a_1\dot{\zeta}_{10}+a_2\zeta_{10}=-a_3\left|\dot{\zeta}_{00}\right|\dot{\zeta}_{00}-a_4\dot{\zeta}_{00}^3 \\ \ddot{\zeta}_{11\,n}+a_1\dot{\zeta}_{11\,n}+a_2\zeta_{11\,n}=-2a_3\left|\dot{\zeta}_{00}\right|\dot{\zeta}_{01\,n}-3a_4(\dot{\zeta}_{00}^2\dot{\zeta}_{01\,n}+\zeta_{00}\chi_n) \\ \ddot{\zeta}_{12\,nm}+a_1\dot{\zeta}_{12\,nm}+a_2\zeta_{12\,nm}=3a_4[\zeta_{01\,n}\chi_n-\dot{\zeta}_{00}(\dot{\zeta}_{01\,n}\dot{\zeta}_{01\,m}+\dot{\zeta}_{00}\dot{\zeta}_{02\,nm})]- \\ \quad -a_3\left|\dot{\zeta}_{00}\right|\left(\dot{\zeta}_{01\,n}\dot{\zeta}_{01\,m}/\dot{\zeta}_{00}+2\dot{\zeta}_{02\,nm}\right) \end{cases} \tag{17}$$

$$\begin{cases} \ddot{\zeta}_{20}+a_1\dot{\zeta}_{20}+a_2\zeta_{20}=-2a_3\left|\dot{\zeta}_{00}\right|\dot{\zeta}_{01}-3a_4\zeta_{00}^2\zeta_{10} \\ \ddot{\zeta}_{21\,n}+a_1\dot{\zeta}_{21\,n}+a_2\zeta_{21\,n}=-2a_3\left|\dot{\zeta}_{00}\right|\left(\dot{\zeta}_{10}\dot{\zeta}_{01\,n}/\dot{\zeta}_{00}+\dot{\zeta}_{11\,n}\right)- \\ \quad -3a_4\zeta_{00}(\zeta_{00}\zeta_{11\,n}+2\zeta_{10}\zeta_{01\,n}+2\zeta_{10}\chi_n) \\ \ddot{\zeta}_{22\,nm}+a_1\dot{\zeta}_{22\,nm}+a_2\zeta_{22\,nm}=-a_3\left|\dot{\zeta}_{00}\right|(2\dot{\zeta}_{10}\dot{\zeta}_{02\,nm}+\dot{\zeta}_{01\,n}\dot{\zeta}_{11\,m}+ \\ \quad +\dot{\zeta}_{11\,n}\dot{\zeta}_{01\,m}+2\dot{\zeta}_{00}\dot{\zeta}_{12\,nm})/\dot{\zeta}_{00}-3a_4[\zeta_{00}^2\zeta_{12\,nm}+\zeta_{10}\zeta_{01\,n}\zeta_{01\,m}+ \\ \quad +2\zeta_{00}\zeta_{01\,n}\zeta_{11\,n}-2\chi_n(\zeta_{00}\zeta_{11\,n}+\zeta_{01\,n}\zeta_{10})] \end{cases} \tag{18}$$

This is a set of linear deterministic differential equations. If homogeneous initial condition are assumed the following set of differential equation is yielded:

$$\begin{cases} \ddot{\zeta}_{01\,n} + a_1\,\dot{\zeta}_{01\,n} + a_2\,\zeta_{01\,n} = (1-a_5)\,\ddot{\chi}_n + a_1\,\dot{\chi}_n + a_2\,\chi_n \\ \ddot{\zeta}_{12\,nm} + a_1\,\dot{\zeta}_{12\,nm} + a_2\,\zeta_{12\,nm} = 3\,a_4\,\zeta_{01\,n}\,\chi_m - a_3\,\dot{\zeta}_{01\,n}\,\dot{\zeta}_{01\,m} \end{cases} \tag{19}$$

Since the interest is focused on the steady state response and by remembering equation (8) the variance of the relative displacement can be calculated as follows:

$$E\left\{ \zeta^2 \right\} = E\left\{ B_n^2 \right\} \left| \frac{-\omega_n^2\,(1-a_5) + j\,\omega_n\,a_1 + a_2}{-\omega_n^2 + j\,\omega_n\,a_1 + a_2} \right|^2 + O\left(B_n^3 \right) \tag{20}$$

Statistical linearization and stochastic perturbation method

The basic idea of the statistical linearization lies in replacing a nonlinear system, i.e. $\mathcal{L}(x,\dot{x},t)+\mathcal{NL}(x,\dot{x},t)=f(t)$, by an "equivalent" linear one, i.e. $\mathcal{L}(y,\dot{y},t)+\mathcal{L}_{eq}(y,\dot{y},\mathscr{p},t)=f(t)$, so that the difference between the response of the two systems is minimal in some probabilistic sense [1, 2]. More precisely, the classical approach wants that the characteristic parameters of the equivalent system are determined by requiring that the mean-square difference between the nonlinear forces and their counterpart in the linear systems, $\mathscr{E}(x,\dot{x},\mathscr{p},t)=\mathcal{L}_{eq}(x,\dot{x},\mathscr{p},t)-\mathcal{NL}(x,\dot{x},t)$, is minimal. The unknown parameters $\mathscr{p}$ are calculated minimizing the mean-square of the equation error, i.e. by solving the following equation:

$$\partial E\left\{\mathscr{E}^2\right\}/\partial\,\mathscr{p} = 0 \tag{21}$$

The expectations in (21) should ideally be evaluated using the exact joint probability density function (pdf) of the nonlinear response $x(t)$ and of its derivative $\dot{x}(t)$: $p(x,t;\dot{x},t)$. Since x is the problem solution, its pdf is in general unknown. Therefore, a trial function is considered to calculate the statistical moment which are the coefficients of equation (21).
Therefore, the new equivalent equation is:

$$\ddot{\zeta} + (a_1 + c_{SLeq})\,\dot{\zeta} + (a_2 + 3\,a_4\,\xi^2 + k_{SLeq})\,\zeta = f(\xi) \tag{22}$$

The parameters c_{SLeq} and k_{SLeq} are calculated by the solution of the problem shown in equation (21)
By assuming that the solution process $\zeta(t)$ and its derivative $\dot{\zeta}(t)$ are Gaussian, stationary, independent and with zero mean, the following position holds (Kazakov):

$$\begin{aligned} &E\{\zeta^3\}=0\,,\; E\{\dot{\zeta}\,\zeta\}=E\{\dot{\zeta}\,\zeta^2\}=E\{\dot{\zeta}\,\zeta^3\}=0 \\ &E\{\left|\dot{\zeta}\right|\dot{\zeta}\,\zeta\}=0\,,\; E\{\left|\dot{\zeta}\right|\dot{\zeta}^2\}/E\{\dot{\zeta}^2\}=E\{(\left|\dot{\zeta}\right|\dot{\zeta})'\}=\sqrt{8/\pi}\;\sigma_{\dot{\zeta}} \end{aligned} \tag{23}$$

Therefore, the coefficients are:

$$c_{SLeq} = a_3\,\sqrt{8/\pi}\,\sigma_{\dot{\zeta}} \quad , \quad k_{SLeq} = 3\,a_4\,\sigma_{\zeta}^2 \tag{24}$$

Equation (22) is a random linear ordinary differential equation: $3a_4\xi^2$ is the random coefficient. A stochastic perturbation technique is developed by remembering the series expansion (8). As in equation (15), solution ζ is expanded in terms of the coefficients B_n (remember (15)).

Equation (15) is substituted into equation (22). By ordering, the following set of linear independent differential equation can be written.

$$\begin{cases} \ddot{\zeta}_0 + (a_1 + a_3\sqrt{8/\pi}\,\sigma_{\dot{\zeta}})\dot{\zeta}_0 + (a_2 + 3a_4\sigma_\zeta^2)\zeta_0 = 0 \\ \ddot{\zeta}_{1n} + (a_1 + a_3\sqrt{8/\pi}\,\sigma_{\dot{\zeta}})\dot{\zeta}_{1n} + (a_2 + 3a_4\sigma_\zeta^2)\zeta_{1n} = (1-a_5)\ddot{\chi}_n + a_1\dot{\chi}_n + a_2\chi_n \\ \ddot{\zeta}_{2nm} + (a_1 + a_3\sqrt{8/\pi}\,\sigma_{\dot{\zeta}})\dot{\zeta}_{2nm} + (a_2 + 3a_4\sigma_\zeta^2)\zeta_{2nm} = 0 \end{cases} \tag{25}$$

When void initial condition are considered, the zero and second order give void solutions $\zeta_0 = 0$ and $\zeta_{2nm} = 0$. The first order equations solutions, ζ_{1n} , are sufficient to obtain the relative displacement:

$$\zeta(t) = B_n\,\zeta_{1n}(t) \tag{26}$$

The statistical moments of the velocity and of the displacement are in general unknown. An approximated numerical procedure can be applied to evaluate these and to solve equation (25).

A rough approximation of equation (25) (the terms with the statistical moments are neglected) allows to calculate $\zeta_{1n}^{(0)}$.

Therefore, the approximated variance and the standard deviation are:

$$\begin{aligned} \sigma_{\dot{\zeta}}^{(0)} &= \sqrt{E\left\{B_n^2\right\}\left|\frac{j\omega_n\left(-\omega_n^2(1-a_5) + j\omega_n a_1 + a_2\right)}{-\omega_n^2 + j\omega_n a_1 + a_2}\right|^2} \\ \sigma_\zeta^{2\,(0)} &= E\left\{B_n^2\right\}\left|\frac{-\omega_n^2(1-a_5) + j\omega_n a_1 + a_2}{-\omega_n^2 + j\omega_n a_1 + a_2}\right|^2 \end{aligned} \tag{27}$$

These relationships can be used to start an iterative procedure in which the statistical moments calculated at the previous step allow to achieve the solution at the new iteration. Therefore, the statistical moments are:

$$\begin{aligned} \sigma_{\dot{\zeta}}^{(i)} &= \sqrt{E\left\{B_n^2\right\}\left|\frac{j\omega_n\left(-\omega_n^2(1-a_5) + j\omega_n a_1 + a_2\right)}{-\omega_n^2 + j\omega_n(a_1 + a_3\sqrt{8/\pi}\,\sigma_{\dot{\zeta}}^{(i-1)}) + (a_2 + 3a_4\sigma_\zeta^{2\,(i-1)})}\right|^2} \\ \sigma_\zeta^{2\,(i)} &= E\left\{B_n^2\right\}\left|\frac{-\omega_n^2(1-a_5) + j\omega_n a_1 + a_2}{-\omega_n^2 + j\omega_n(a_1 + a_3\sqrt{8/\pi}\,\sigma_{\dot{\zeta}}^{(i-1)}) + (a_2 + 3a_4\sigma_\zeta^{2\,(i-1)})}\right|^2 \end{aligned} \tag{28}$$

This calculus can be repeated for $i = 1, 2, 3, \cdots$ until the convergence is reached.

NUMERICAL RESULTS

A numerical test is developed in the following. The buoy has a diameter of 10m, a mass of 12000Kg. The sea depth under the dock is 50m. Two steel cables anchor the buoy, their suspended length is 150m. Four different wind velocities are considered to study the response of the system: 5, 8, 10, 15m/s.
By a numerical integration, the solution of equation (9) is calculated. Figure 2 shows the phase space of displacement and velocity and the respective pdf. The more the velocity increases, the more the pdf are not Gaussian (see figure 2).

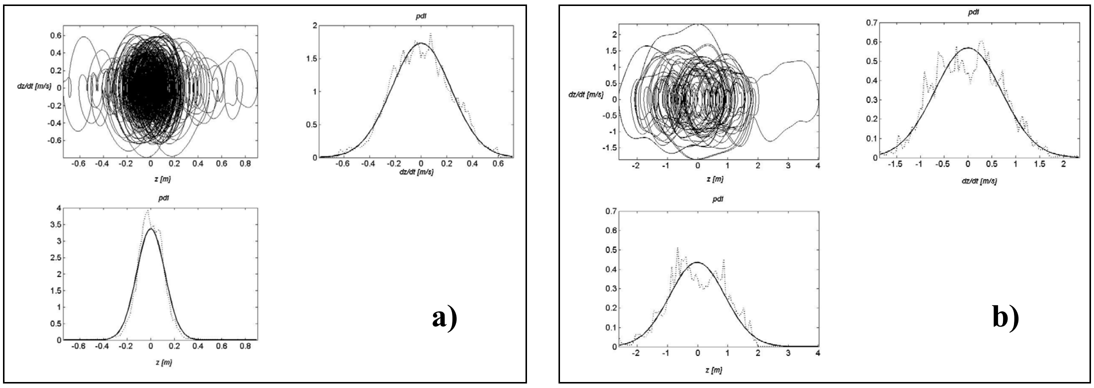

Figure 2 – Phase space and pdf of velocity and displacement: a) V=5m/s, b) V=15m/s

In table 1 the autocorrelation of ζ, R_ζ, calculated in $\tau=0$ (remember that the phenomenon is stationary), is compared with the variance calculated by equations (20) and (28).

Table 1 – Comparison between the numerical results for different wind velocities

Wind velocity [m/s]	*P.M. frequency*	$R_\zeta(0)$	σ^2_{zz} *pert.*	σ^2_{zz} *lin. stat.*
5	0.275	0.0098	0.01	0.01
8	0.17	0.066	0.074	0.074
10	0.135	0.156	0.202	0.201
15	0.09	0.813	1.751	1.739

Figure 3 shows the power spectral density of ζ. The first peak corresponds to the natural frequency of the rough linearization, $\omega_n=\sqrt{a_2}$, the second to the maximum of the Pierson Moskowitz spectrum.

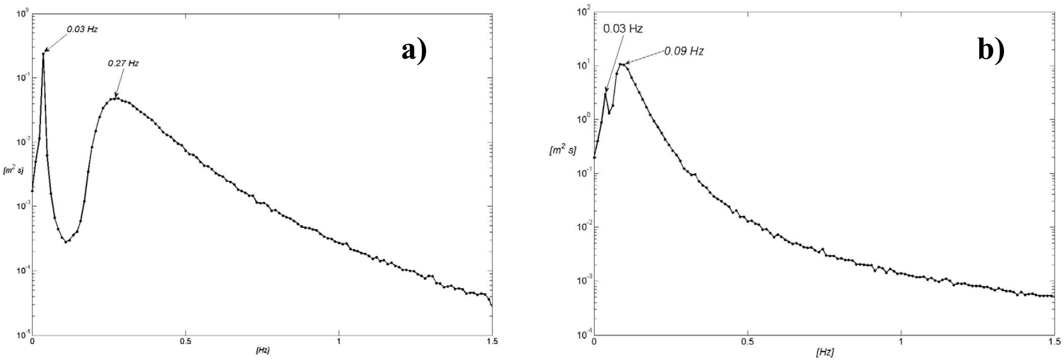

Figure 3 – Power spectral density of the displacement: a) V=5m/s, b) V=15m/s

CONCLUSIONS

Two linearization techniques are developed to solve a nonlinear stochastic differential equation. A cascade of two perturbation techniques and a statistical linearization with a perturbation technique constitute the first and the second method, respectively. Equations (20) and (27) show that the rough solution of the second method is equal to the solution of the first one. The statistical linearization technique gives to the solution some terms which should improve the result.

The numerical results synthesized in table 1 show that the theoretical solution has a good agreement with the numerical integration when the wind velocity is low, i.e. the force amplitude is small. In fact, this condition allows to respect the hypotheses of the perturbation technique. In the studied cases, the relative errors start from 2% and increase up to 115%. The terms introduced by the statistical linearization are not able to give a relevant improvement of the solution for high wind velocity.

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