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VIBRATION MODES OF SHORT CYLINDERS: FREQUENCY CROSSINGS AND MODE SHAPES

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Abstract
A study of the axisymmetric vibration modes of a short cylinder is presented. The Ritz method is applied to calculate the non-dimensional frequency and its dependence on both the slenderness *L/D* and Poisson's ratio. The plot of the non-dimensional frequency versus *L/D* is analysed to determine the frequency crossings and the mode shapes. In the case of Poisson's ratio equal to zero, it is verified that there are multiple frequencies for the antisymmetric modes and that the mode shapes below and above the first crossing are interchanged. The natural frequencies and the mode shapes are accurately calculated for stainless-steel cylinders with a Poisson's ratio of 0.298. It is found that although actually no frequency crossing occurs, the mode shapes below and above the false crossing are interchanged. The theoretical results are experimentally verified by using a laser speckle interferometer as detector of vibration of two stainless-steel cylinders with *L/D*=1.2 and 1.5, respectively. Free vibration is induced by an impact and the out-of-plane component of the displacements is detected. An analysis of the spectrum provides the natural frequencies. Forced vibration is then excited by adhering two piezoelectric transducers to the cylinder's ends. The out-of-plane and in-plane displacement components are detected along one generatrix on the lateral surface. An excellent concordance between the theoretical and the experimental results is found.

INTRODUCTION

The theory of wave propagation and vibrations in rods based on the classical theory of elasticity is well established [1]. The free vibration solutions of the Pochhammer equations for rods are exact. These solutions are applicable to circular cylinders with stress-free surface and infinite length; but traction-free ends are not permitted. Natural frequencies and normal modes of finite cylinders can be accurately determined by means of numerical methods. From Hamilton's principle applied to a system whose displacements are harmonic in time, it is concluded that the difference between the maximum kinetic and potential energies is a minimum when the cylinder is vibrating in a normal mode. A widely used methodology is the Ritz method, based on assuming approximate solutions, suitable for the system, which satisfied the boundary conditions. Convergence towards more accurate frequencies and mode shapes is obtained as the number of terms in the approximating expressions is increased. This methodology is applied by Leissa and So [3] to finite cylinders, the displacement functions are in the form of algebraic polynomials in the cylindrical coordinates.

In the present work, the Ritz method is used to study axially symmetric vibration modes of finite length cylinders. Power series are assumed for the displacements. For axisymmetric displacements, the displacement functions only have radial *u* and axial *w* components; both functions of time *t*, the distance *r* to the revolution axis, and of the distance *z* to the plane perpendicular to the axis at the central point of the rod. Standing-wave solutions are sought of the form

$$u = U(r,z)\sin(\omega t); \qquad w = W(r,z)\sin(\omega t). \tag{1}$$

The amplitudes *U* and *W* are assumed as polynomial functions,

$$U(r,z) = \sum_{i}^{I}\sum_{j}^{J} A_{ij} r^i z^j; \qquad W(r,z) = \sum_{p}^{P}\sum_{q}^{Q} C_{pq} r^p z^q, \tag{2}$$

with i=1,2,…I; j=0,1,2,...J; p=0,1,2,…P; q=0,1,2,...Q; where i=0 is not considered in order to avoid singularities in the stresses in r=0. In symmetric modes *j* only takes even values and *q* odd values, and in antisymmetric ones *j* takes odd values and *q* even values.

Expressed the functional of the maximum potential energy and the maximum kinetic energy for a mode of the axisymmetric vibration, Hamilton's principle obliges that $\partial(V_{\max} - T_{\max}) / \partial A_{ij} = 0$, $\partial(V_{\max} - T_{\max}) / \partial C_{pq} = 0$, for all A_{ij} and C_{pq}. These conditions constitute a homogenous set of linear algebraic equations in A_{ij} and C_{pq}. The eigenvalues of this system are the square of the nondimensional frequencies and the eigenvectors are the coefficients of the polynomials. For economy of calculations the nondimensional frequency $\Omega = \pi f D\sqrt{\rho / G}$ is used, where $f=\omega/2\pi$ is the ordinary frequency measured in Hz, *D* the diameter of the cylinder, and *G* the shear modulus. Shear modulus is related to Young's modulus *E* and Poisson's ratio ν by $G=E/[2(1+\nu)]$. The frequencies Ω depend on Poisson's ratio and the quotient of the length and diameter of the cylinder, i.e. the slenderness *L/D*. Therefore, nondimensional natural frequencies can be expressed as

$$\Omega = \Omega(\nu, L / D). \tag{3}$$

In the elemental theory for longitudinal vibrations of slender rods, the plot of Ω versus *L/D*, for a given Poisson's ratio, is a set of equilateral hyperbolas. In such a plot, no crossing occurs, neither between symmetric modes nor between antisymmetric ones. On the other hand, for short cylinders the nondimensional frequency is a complicated function of both ν and *L/D*. The purpose of this work is to study, theoretically and experimentally, such a dependence and to analyse the frequency crossings and the mode shapes. The study can contribute to a better understanding of the complete frequency spectrum of a rod.

FREQUENCY CROSSINGS IN THE PLOT OF Ω VERSUS SLENDERNESS

The Ritz method is applied to determine the natural frequencies of axisymmetric vibrating cylinders whose lengths are of the same order as their diameters. The results of Ω versus *L/D* for different Poisson's ratios are plotted in order to determine the frequency crossings. Then, the mode shapes are obtained in the neighbourhood of the crossing points.

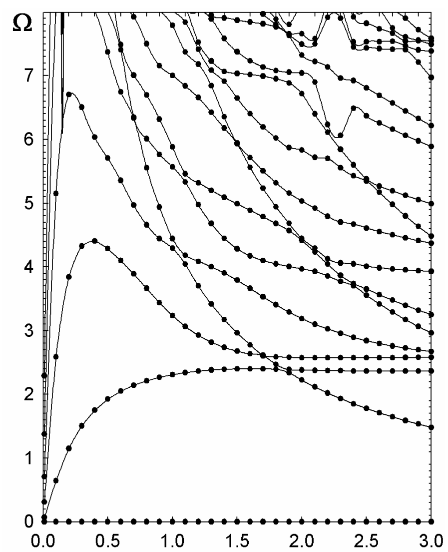

Figure 1. The lowest nondimensional frequencies in terms of slenderness for the antisymmetric vibration of short cylinders with ν=0.

Firstly, the study is concerned with ideal materials with a Poisson's ratio of zero. The lowest axisymmetric nondimensional frequencies Ω for the symmetric and antisymmetric modes are calculated for values of *L/D* ranging from 0 to 3. The values of Ω are arranged for each value of *L/D* according to the increasing value of the frequency. The plot of Ω versus *L/D* for the symmetric modes shows that there are several crossings points apart from the universal one. The latter is called universal point [5] because first-symmetric-mode curves for materials with any Poisson's ratio pass through it. Let us analyse the results for the antisymmetric modes shown in Fig.1. For small values of the slenderness, the frequency of a cylinder with a constant diameter increases with L, as expected as the cylinder can be considered as a disc of thickness L. Whereas, for values of $L/D \geq 1.9$, the curve for the lowest frequency is similar to an equilateral hyperbola of the form $\Omega L/D$=constant, as in the elemental theory for slender rods. A detailed calculation shows that the two lowest curves intersect for L/D=1.844 and Ω=2.412, which is a multiple frequency since such a cylinder can vibrate in two different mode shapes with the same frequency. An analysis of the mode shapes in the surroundings of such a point confirms that the mode shapes below and above the crossing point are interchanged. There exist more crossing points for $L/D \approx 1.0$ and $L/D \approx 1.7$ for higher frequencies. Therefore, in the case of a Poisson's ratio of zero, there are many multiple frequencies in the plot of Ω versus L/D.

Secondly, the results presented refer to the first symmetric and antisymmetric modes. In the elemental theory for slender rods, there are crossings neither between the symmetric modes themselves, nor between the antisymmetric ones, nor between the symmetric and antisymmetric modes. Since for short cylinders the dependence of Ω on L/D is complicated, multiple frequencies are expected to be found. An study is carried out for the first symmetric and first antisymmetric modes for stainless-steel cylinders with slendernesses ranging from 0.1 to 2.0 and with a Poisson's ratio of 0.298. A multiple frequency is found for a slenderness L/D=0.786; which is experimentally verified in the laboratory. For this purpose, a steel cylinder, 31.35 mm in length and 39.90 mm in diameter, its slenderness being L/D=0.7857, is put under free vibration by applying an axial percussion. A laser interferometer is used to detect the out-of-plane component of the resulting vibration at the center of one end. The maximum amplitudes of the spectrum, obtained from the fast Fourier transform of the detected signal, correspond to the natural frequencies. Fig.2 shows a detail of the spectrum obtained for the steel sample. There is an unique natural frequency of 67875 Hz, which confirms the theoretical predictions.

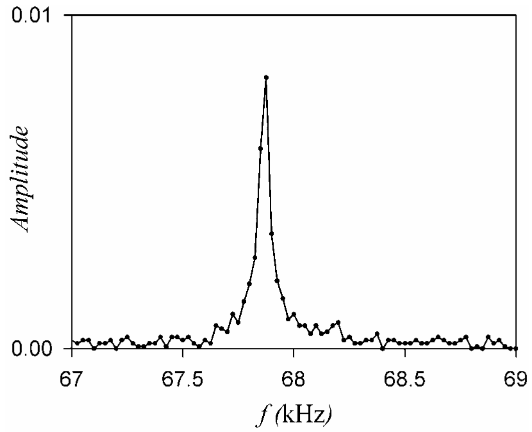

Figure 2. Detail of the spectrum for a steel cylinder with L/D=0.7857.

Finally, the frequency spectra for short cylinders with Poisson's ratio other than zero are determined. Hutchisson [2] accurately determined the curves of Ω versus *L/D* trying a series solution of the general three-dimensional equations of linear elasticity. He concluded that there are no frequency crossing in any of the plots of the frequency in terms of the slenderness, and that no frequency crossing should occur when the plotting is done for even and odd modes separately. This statement seems to contradict the above results. In order to confirm Hutchison's results, the spectra for stainless-steel cylinders are calculated. Fig.3 shows the plot of Ω versus *L/D* for the lowest antisymmetric modes of steel cylinders with a Poisson's ratio of 0.298. It appears that the two lowest modes intersects in the surroundings of *L/D*=1.3. This apparent crossing is analysed in detail by calculating numerically the nondimensional frequencies for values of slenderness close to 1.3. It is found that for all the values of the slenderness there is a difference between the frequencies Ω_{a2}-Ω_{a1}, and, therefore, such apparent crossing does not occur.

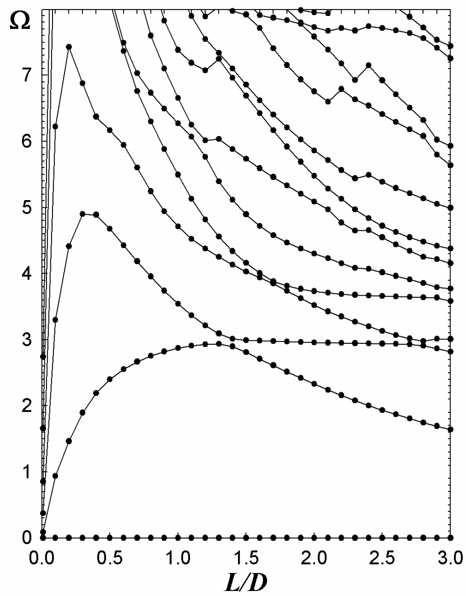

Figure 3. Ω-L/D curves for the antisymmetric modes of steel cylinders with a Poisson's ratio of 0.298.

MODE SHAPES IN THE NEIGHBOURHOOD OF THE CROSSING. NUMERICAL RESULTS VERSUS EXPERIMENTAL ONES.

The mode shapes for stainless-steel cylinders in the neighbourhood of the apparent crossing are also calculated by means of the Ritz method. The purpose is to determine the variation of the mode shapes as *L/D* increases. The mode shapes for the two lowest antisymmetric modes corresponding to cylinders with slendernesses from 1.2 to 1.6 are analysed. It is found that as *L/D* increases, the mode shapes gradually change, resulting in almost an interchange between the modes below and above the apparent crossing at *L/D*=1.3. Fig.4 shows the numerical results for the slendernesses

L/D=1.2 and 1.5, i.e. aspect ratios below and above the crossing. Note that there is some similarity between the mode shape for the first antisymmetric mode for *L/D*=1.2 (Fig.4a1) and that for the second mode corresponding to *L/D*=1.5 (Fig.4b2). In the same way, the second mode for *L/D*=1.2 (Fig.4a2) bears a remarkable resemblance to the first mode for *L/D*=1.5 (Fig.4b1).

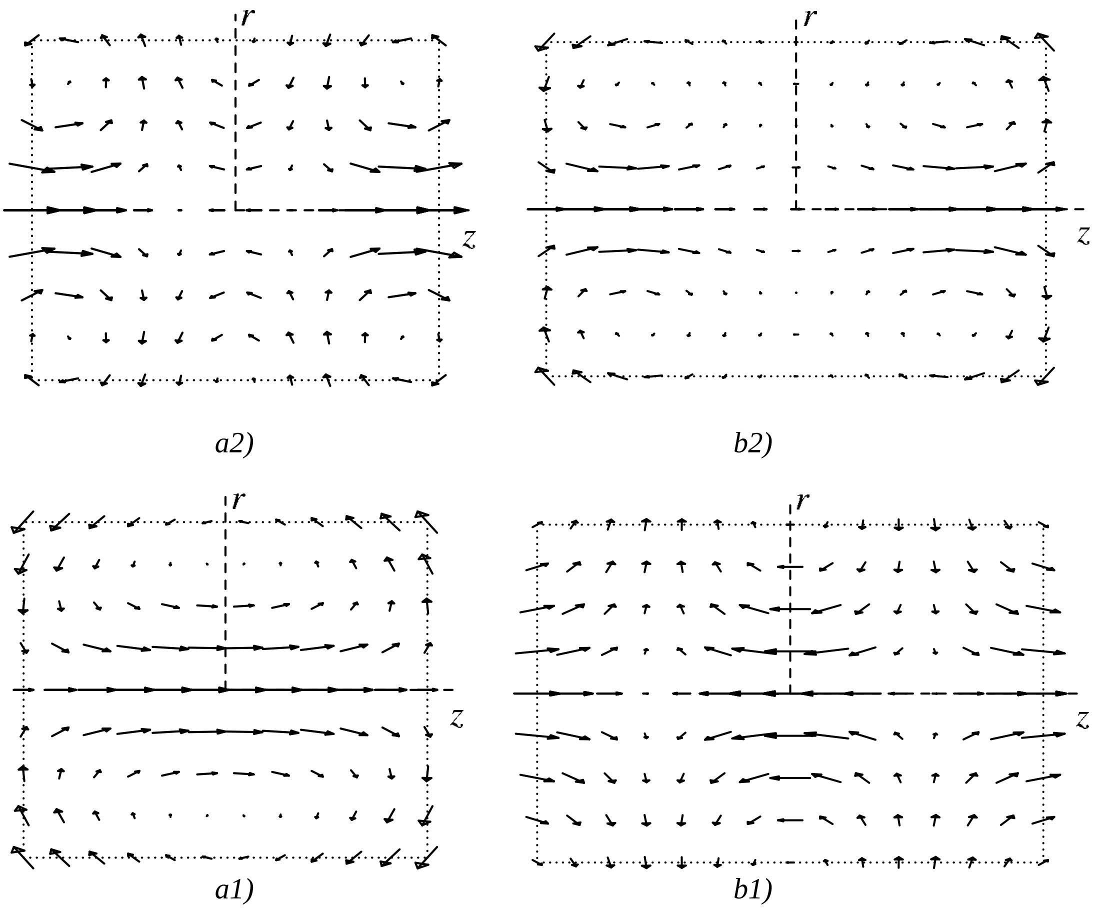

Figure 4. a) Mode shapes for a cylinder with L/D=1.2, figure a1) corresponds to the first antisymmetric mode and figure a2) to the second one. b) The same as a) for the slenderness L/D=1.5

The above numerical results are verified experimentally. The samples used are two stainless-steel cylinders with diameter *D*=49.00 mm and slendernesses *L/D*=1.2 and 1.5, respectively, whose physical properties are: density ρ=7894 kg/m3, shear modulus *G*=77.32 GPa, and Poisson's ratio ν=0.288. In a first experiment, the axisymmetric natural frequencies for both cylinders are determined. The procedure used to generate the vibration of the sample and the posterior detection has been described previously [4]. The cylinder is positioned horizontally and supported at its center on a small rubber block. A steel sphere is used to apply an axial impact to the center of one of its ends. Then, the cylinder is left to vibrate freely. An optical interferometer is used as detector of the vibration at the center of the opposite end.

The interferometer used can detect both the normal and tangential components of the displacement with a resolution of about 1 nm. The electronic demulation circuit produces an output proportional to the surface displacement component. In order to determine the natural frequencies the fast Fourier transform of the out-of-plane component is calculated. The analysis of the spectrum yields the lowest antisymmetric frequencies: f_1=59075 Hz, f_2=65175 Hz for *L/D*=1.2, and f_1=56650Hz, f_2=60200 Hz for *L/D*=1.5.

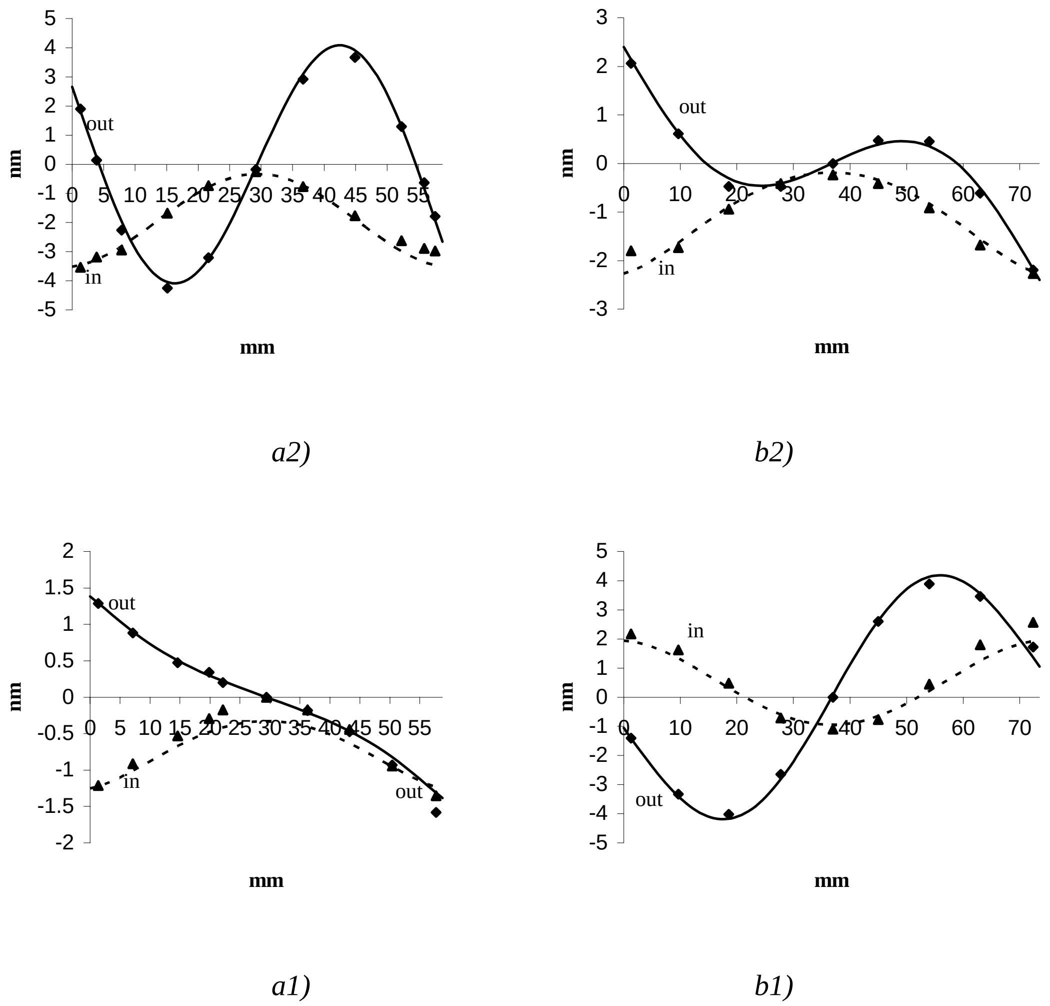

Figure 5. a) The out-of-plane and in-plane displacements along a generatrix of a cylinder with L/D=1.2. a1) For the first antisymmetric mode and a2) for the second one. b) The same as a) for a cylinder with L/D=1.5. The solid and dashed lines represent the numerical results and the squares and triangles the experimental ones for the out-of- plane and in-plane components, respectively.

In a second experiment, forced vibration is induced by adhering two small piezoelectric transducers to the cylinder's ends. The resulting vibration is detected along one generatrix of the cylinder. The out-of-plane and in-plane components are detected by the laser interferometer at points on such a generatrix, whereby both amplitude and phase are measured. A sinusoidal signal generator drives the transducers, the exciting frequency being near to the resonance for the mode to be excited. Therefore, the displacements can be easily detected since a large amplitude is expected. However, if the exciting frequencies are too close to the resonance, any parasitic variation of frequency causes sharp changes of amplitude and phase.

The out-of-plane and in-plane components of the displacement for points on one generatrix of the cylinder with L/D=1.2 are represented in Fig.5a. The first antisymmetric mode corresponds to Fig.5a1, and the second one to Fig.5a2. The solid and dashed lines represent the numerical results shown in Fig.4 for the out-of-plane and in-plane components of the displacements along the generatrix. The squares and triangles indicate the amplitudes for the out-of-plane and in-plane components, respectively, measured with the interferometer. Fig.5b shows the results for the cylinder with slenderness L/D=1.5. Note that the experimental results are in complete agreement with the numerical ones. Fig.5a1 for the first antisymmetric mode for L/D=1.2 bears some resemblance to Fig.5b2 corresponding to the second mode for L/D=1.5. Similarly, Fig.5a2 resembles Fig.5b1. Therefore, it is found that although no crossing occurs, the mode shapes gradually tend to become similar resulting in almost an interchange of the mode shapes.

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