ICSV13 - Vienna
The Thirteenth International Congress on Sound and Vibration
Vienna, Austria, July 2-6, 2006

WAVE PROPAGATION, REFLECTION AND TRANSMISSION IN NON-UNIFORM BEAMS

Seung-Kyu Lee*[1], Brian Mace[1], and Michael Brennan[1]

[1]Institute of Sound and Vibration Research, University of Southampton
University Road, Highfield, Southampton, SO17 1BJ, UK
skl@isvr.soton.ac.uk

Abstract
A generalised approach based on reflection, transmission and propagation of waves is applied to the analysis of non-uniform Euler-Bernoulli beams whose properties vary rapidly but deterministically. The variation of the properties with position is such that no wave reflection occurs. Examples are given that include an Euler-Bernoulli beam with geometric variation that can be described by a polynomial. The state vector in the physical domain is transformed to the wave domain using displacement and internal force matrices. The wave amplitudes at one point are then related to those at another point by a propagation matrix, which is diagonal for the cases considered. By normalising the elements of the propagation matrix with respect to energy, their magnitudes are less than or equal to unity, so that the problem is always well-posed. The energy transport velocity, at which energy is transported by the waves, is derived using the relationship between power and energy. It is shown that this energy velocity decreases as the cross-section decreases in size. A numerical example for wave transmission through a rectangular connector with linearly tapered thickness and constant width is presented. This well-conditioned approach can be used to predict the transmission of vibration through the connector without any approximation errors and at a low computational cost, irrespective of the frequency.

INTRODUCTION

The dynamic behaviour of a structure may be described in terms of waves and their propagation, reflection and transmission. This wave approach is especially suitable in the high frequency region since it does not require powerful computing resources and is well conditioned. However, most real structures are too complicated to apply the wave approach easily. One such case is that of a non-uniform one-dimensional structure which has continuous variation in geometric and/or material properties.

Eds.: J. Eberhardsteiner, H.A. Mang, H. Waubke

In non-uniform waveguides with rapid variation in cross-section, the energy of one wave component is generally transferred to another, i.e. a positive-going wave is reflected to produce a negative-going, back-scattered, wave. However, previous work has showed that there are classes of non-uniform waveguides where such reflection does not occur. Cranch and Adler [1] considered the case of non-uniform Euler-Bernoulli beams of rectangular cross-section. When the thickness varies with distance x along the beam as x, x^2 or x^3 while the width varies as an arbitrary power of x, they showed that the motion can be exactly described in terms of Bessel functions. When the cross-sectional area and moment of inertia of a non-uniform beam vary together as x^4, the equation of motion can be transformed into the wave equation [1],[2]. It has also been found that the motion of non-uniform beams with exponentially varying properties along the distance can be expressed simply in terms of exponential functions [1],[3]. Banerjee and Williams [4] used the solutions to obtain the exact dynamic stiffness matrices of some non-uniform beams. Petersson and Nijman [5] studied dynamic characteristics of the bending wave horn, featured by a broad-banded transition from vibrations governed by the properties at the mouth to vibrations governed by those at the throat. Krylov and Tilman [6] showed, using the geometrical acoustic approach, that incident flexural waves are trapped near the edge of the wedges, the thickness of which varies in a polynomial manner and the waves are therefore never reflected back.

In this paper, a generalised wave approach based on reflection, transmission and propagation of waves is applied to an Euler-Bernoulli beam whose geometric properties can be described by a polynomial. The state vector in the physical domain is transformed to the wave domain using displacement and internal force matrices. The wave amplitudes at one point are then related to those at another point by a diagonal propagation matrix. The energy transport velocity, at which energy is transported by the waves, is derived using the relationship between power and energy. It is seen that this energy velocity decreases as the cross-section decreases in size. A numerical example for wave transmission through a rectangular connector with linearly tapered thickness and constant width is presented.

BENDING WAVES IN A NON-UNIFORM BEAM

Mace [7] developed a wave approach for bending motion of uniform beams including nearfield effects and Harland *et al*. [8] suggested a systematic formulation of the approach and applied this to wave propagation in uniform, fluid-filled beams. In this section, the same framework is applied to bending motion of a non-uniform Euler-Bernoulli beam with geometric variation that can be described by a polynomial.

Equation of motion

The flexural displacement $w(x,t)$ for the free vibration of an Euler-Bernoulli beam at position x and time t is governed by the differential equation [1]

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 w}{\partial x^2}\right]+\rho A\frac{\partial^2 w}{\partial t^2}=0 \tag{1}$$

where E is the modulus of elasticity of the beam, I the second moment of area, ρ the density and A the cross-sectional area. Consider a non-uniform beam as shown in Figure 1 where the material properties are constant but where

$$A(x)=\alpha_A x^{\mu}, \qquad I(x)=\alpha_I x^{\mu+2} \tag{2a,b}$$

where μ is real and α_A and α_I are positive. When $\mu=1$ and the cross-sectional shape of the non-uniform beam is rectangular, the beam has linearly varying thickness and constant width. Assuming an $e^{\mathrm{i}\omega t}$ time dependence with angular frequency ω, substituting equation (2) into equation (1) gives

$$x^2\frac{\mathrm{d}^4 w}{\mathrm{d}x^4}+2(\mu+2)x\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}+(\mu+1)(\mu+2)\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}-k_b^4x^2w=0 \tag{3}$$

where $k_b(x)=\sqrt[4]{\rho A(x)\omega^2/EI(x)}$ is the flexural wavenumber at position x. An undamped structure is assumed for simplicity, so that k_b is real. Equation (3) can be factorised into the product of the Bessel equation and the modified Bessel equation [1] so that the general solution can be expressed by a linear combination of Hankel functions, $H_\mu^{(2)}(2k_bx)$ and $H_\mu^{(1)}(2k_bx)$, and modified Bessel functions, $K_\mu(2k_bx)$ and $I_\mu(2k_bx)$. The terms $H_\mu^{(2,1)}$ represent positive- and negative-going propagating waves, respectively, and the terms K_μ and I_μ the positive- and negative-going nearfield waves, respectively. Thus the solution of equation (3) is given by

$$w(x)=a^+ + a_N^+ + a^- + a_N^- \tag{4}$$

where a^+, a_N^+, a^-, a_N^- are the amplitudes of the four waves at position x given by

$$\begin{aligned} a^+ &= x^{-\frac{\mu}{2}}H_\mu^{(2)}(2k_bx)C_1, & a_N^+ &= x^{-\frac{\mu}{2}}K_\mu(2k_bx)C_2, \\ a^- &= x^{-\frac{\mu}{2}}H_\mu^{(1)}(2k_bx)C_3, & a_N^- &= x^{-\frac{\mu}{2}}I_\mu(2k_bx)C_4, \end{aligned} \tag{5a,b,c,d}$$

where $C_{1,2,3,4}$ are arbitrary constants, respectively.

State of a cross-section in the wave domain

The relationship between the state vector in the physical domain and the state vector in the wave domain is given by [8]

$$\begin{Bmatrix} \mathbf{w} \\ \mathbf{f} \end{Bmatrix} = \begin{bmatrix} \mathbf{\Psi}^+ & \mathbf{\Psi}^- \\ \mathbf{\Phi}^+ & \mathbf{\Phi}^- \end{bmatrix} \begin{Bmatrix} \mathbf{a}^+ \\ \mathbf{a}^- \end{Bmatrix} \tag{6}$$

where $\mathbf{w}$ and $\mathbf{f}$ are the generalized displacements and internal force vectors, and $\mathbf{\Psi}^\pm$ and $\mathbf{\Phi}^\pm$ are the displacement and internal force matrices describing the transformation. For bending motion, $\mathbf{w} = \left[w \quad \mathrm{d}w/\mathrm{d}x \right]^T$, $\mathbf{f} = \left[Q \quad M \right]^T$, $\mathbf{a}^+ = \left[a^+ \quad a_N^+ \right]^T$ and $\mathbf{a}^- = \left[a^- \quad a_N^- \right]^T$. Thus the displacement and internal force matrices for the non-uniform beam are given by

$$\mathbf{\Psi}^+ = \begin{bmatrix} 1 & 1 \\ -k_b \dfrac{H_{\mu+1}^{(2)}(2k_b x)}{H_{\mu}^{(2)}(2k_b x)} & -k_b \dfrac{K_{\mu+1}(2k_b x)}{K_{\mu}(2k_b x)} \end{bmatrix}, \quad \mathbf{\Psi}^- = \begin{bmatrix} 1 & 1 \\ -k_b \dfrac{H_{\mu+1}^{(1)}(2k_b x)}{H_{\mu}^{(1)}(2k_b x)} & k_b \dfrac{I_{\mu+1}(2k_b x)}{I_{\mu}(2k_b x)} \end{bmatrix}, \tag{7a,b,c,d}$$

$$\mathbf{\Phi}^+ = EI \begin{bmatrix} -k_b^3 \dfrac{H_{\mu+1}^{(2)}(2k_b x)}{H_{\mu}^{(2)}(2k_b x)} & k_b^3 \dfrac{K_{\mu+1}(2k_b x)}{K_{\mu}(2k_b x)} \\ k_b^2 \dfrac{H_{\mu+2}^{(2)}(2k_b x)}{H_{\mu}^{(2)}(2k_b x)} & k_b^2 \dfrac{K_{\mu+2}(2k_b x)}{K_{\mu}(2k_b x)} \end{bmatrix}, \quad \mathbf{\Phi}^- = EI \begin{bmatrix} -k_b^3 \dfrac{H_{\mu+1}^{(1)}(2k_b x)}{H_{\mu}^{(1)}(2k_b x)} & -k_b^3 \dfrac{I_{\mu+1}(2k_b x)}{I_{\mu}(2k_b x)} \\ k_b^2 \dfrac{H_{\mu+2}^{(1)}(2k_b x)}{H_{\mu}^{(1)}(2k_b x)} & k_b^2 \dfrac{I_{\mu+2}(2k_b x)}{I_{\mu}(2k_b x)} \end{bmatrix}$$

For a uniform structure the elements of $\mathbf{\Psi}$ and $\mathbf{\Phi}$ are independent of *x*, which is not the case for a non-uniform structure.

Propagation of waves

Consider two points x and $x+L$ on the non-uniform beam. The wave amplitudes at these points are $\mathbf{a}^+(x)$, $\mathbf{a}^-(x)$, $\mathbf{a}^+(x+L)$ and $\mathbf{a}^-(x+L)$, and are related by

$$\mathbf{a}^+(x+L) = \mathbf{F}^+ \, \mathbf{a}^+(x), \qquad \mathbf{a}^-(x) = \mathbf{F}^- \, \mathbf{a}^-(x+L) \tag{8a,b}$$

where $\mathbf{F}^\pm$ are the propagation matrices. These are found from equation (5) to be

$$\mathbf{F}^+ = \left(\frac{x}{x+L} \right)^{\frac{\mu}{2}} \begin{bmatrix} \dfrac{H_{\mu}^{(2)}\left(2k_b\sqrt{x(x+L)}\right)}{H_{\mu}^{(2)}(2k_b x)} & 0 \\ 0 & \dfrac{K_{\mu}\left(2k_b\sqrt{x(x+L)}\right)}{K_{\mu}(2k_b x)} \end{bmatrix}, \quad \mathbf{F}^- = \left(\frac{x+L}{x} \right)^{\frac{\mu}{2}} \begin{bmatrix} \dfrac{H_{\mu}^{(1)}(2k_b x)}{H_{\mu}^{(1)}\left(2k_b\sqrt{x(x+L)}\right)} & 0 \\ 0 & \dfrac{I_{\mu}(2k_b x)}{I_{\mu}\left(2k_b\sqrt{x(x+L)}\right)} \end{bmatrix} \tag{9a,b}$$

Since this case is reciprocal, it follows that $\mathbf{F}^+ = \mathbf{F}^-$ for a suitable basis.

Propagation of energy

The kinetic and potential energy densities for bending motion are given by [9]

$$\mathcal{T}=\frac{1}{2}\rho A\left\{\mathrm{Re}\left(\frac{\partial w}{\partial t}\right)\right\}^2,\qquad \mathcal{V}=\frac{1}{2}EI\left\{\mathrm{Re}\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}^2 \tag{10a,b}$$

where $\mathrm{Re}(\cdot)$ denotes the real part of the quantity. Assume that there is only a propagating positive-going wave with amplitude a^+. The displacement of the beam will then be $w(x)=a^+$. Thus the time-averaged energy density $\langle\mathcal{E}\rangle$, given by $\langle\mathcal{E}\rangle=\langle\mathcal{T}\rangle+\langle\mathcal{V}\rangle$ where $\langle\cdot\rangle$ indicates a time averaged quantity, is

$$\langle\mathcal{E}\rangle=\frac{1}{4}\rho A\omega^2\left|a^+\right|^2\left(1+\left|H_{\mu+2}^{(2)}(2k_bx)\right|^2\left|H_{\mu}^{(2)}(2k_bx)\right|^{-2}\right) \tag{11}$$

The time-averaged power for bending motion of a beam is [9]

$$\langle\Pi\rangle=-\left\langle \mathrm{Re}(Q)\cdot\mathrm{Re}\left(\frac{\partial w}{\partial t}\right)+\mathrm{Re}(M)\cdot\mathrm{Re}\left(\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x}\right)\right)\right\rangle \tag{12}$$

Thus the energy flow associated with the propagating wave is

$$\langle\Pi\rangle=\rho A\omega^2\left|a^+\right|^2\frac{c_b}{\pi k_b x}\left|H_{\mu}^{(2)}(2k_bx)\right|^{-2} \tag{13}$$

where c_b is the phase velocity of the bending wave at x. Noting that a^+ is given by equation (5a), it follows that the power is constant along the beam.

The energy transport velocity c^E, at which energy is transported by the waves, is defined by [10]

$$c^E=\frac{\langle\Pi\rangle}{\langle\mathcal{E}\rangle} \tag{14}$$

Substituting equations (11) and (13) into equation (14) gives

$$c_b^E=\frac{4c_b}{\pi k_b x}\left(\left|H_{\mu}^{(2)}(2k_bx)\right|^2+\left|H_{\mu+2}^{(2)}(2k_bx)\right|^2\right)^{-1} \tag{15}$$

The energy transport velocity associated with the propagating positive-going wave is the same as that associated with the propagating negative-going wave.

Figure 2 shows the energy transport velocity, normalised with respect to c_b, for the non-uniform beam with three different values of μ. It is seen that the velocity decreases as μ increases, i.e. as the degree of non-uniformity increases. When

$2k_b x << 1$, the velocity is approximately proportional to $(k_b x)^{2\mu+4}$. When $2k_b x >> 1$, the velocity asymptotes to the group velocity of the uniform beam, i.e. $c_b^E \to 2c_b$.

Numerical example

Consider a tapered connector of length L between two semi-infinite uniform rectangular beams with the same width but different thicknesses, h_1 and h_2, as shown in Figure 3. The material properties are invariant while the thickness of the connector varies with x (measured from the fictitious vertex at $x = 0$) as

$$h(x) = h_1 x / x_1 \tag{16}$$

where $x_1 = h_1 L / (h_2 - h_1)$ is the distance from the fictitious vertex to junction 1. Consider waves $\mathbf{a}^+$ incident from the left-hand side of junction 1. The relevant waves $\mathbf{b}^\pm$, $\mathbf{c}^\pm$ and $\mathbf{d}^+$ can then be related in terms of the reflection, transmission matrices at the junctions and the propagation matrices between the junctions [11].

Figure 4 shows the power transmission coefficient τ for the connector when a propagating wave is incident. In the figure, $k_{b,m}$ is the *effective* wavenumber in the section between the two points x_1 and x_2 and is given by

$$\frac{1}{k_{b,m}} = \frac{1}{2}\left(\frac{1}{k_b(x_1)} + \frac{1}{k_b(x_2)}\right) \tag{17}$$

Thus the *effective* wavelength in the section is simply the average of the wavelengths at each end of the section. When $k_{b,m} L >> 1$, the power transmission coefficient τ tends to 1, i.e. the power incident on the connector is totally transmitted when frequency increases or the non-uniformity decreases. When $k_{b,m} L << 1$, the results asymptote to those of the case where the two uniform beams are directly connected without the connector [9].

CONCLUDING REMARKS

The wave approach has been applied to bending motion of a non-uniform Euler-Bernoulli beam for which the cross-sectional area and the second moment of area vary as $A(x) \propto x^\mu$ and $I(x) \propto x^{\mu+2}$. The displacement, internal force and propagation matrices for the non-uniform beam were obtained. These matrices provide a foundation for the systematic wave analysis. The energy transport velocity associated with the propagating bending wave in the non-uniform beams is derived exactly. It is shown that the energy velocity decreases towards the vertex.

A numerical example for the wave transmission through a rectangular connector with linearly varying thickness and constant width is presented. This well-conditioned wave approach can be used to predict the transmission of vibration through the connector without any approximation errors and at a low computational cost, irrespective of the frequency.

REFERENCES

[1] E.T. Cranch and A.A. Adler, ‘Bending vibrations of variable section beams’, Journal of Applied Mechanics 23, 103-108. (1956).
[2] S. Abrate, ‘Vibration of non-uniform rods and beams’, Journal of Sound and Vibration 185, 703-716. (1995).
[3] E.D. Suppiger and N.J. Taleb, ‘Free lateral vibration of beams of variable cross section’, ZAMP 7, 501-520. (1956).
[4] J.R. Banerjee and F.W. Williams, ‘Exact Euler-Bernoulli dynamic stiffness matrix for a range of tapered beams’, International Journal for Numerical Methods in Engineering 21, 2289-2302. (1985).
[5] B.A.T. Petersson and E.J.M. Nijman, ‘The one-dimensional bending wave counterpart to the acoustic horn and the semi-infinite wedge’, Journal of Sound and Vibration 211, 95-121. (1998)
[6] V.V. Krylov and F.J.B.S. Tilman, ‘Acoustic ‘black holes’ for flexural waves as effective vibration dampers’, Journal of Sound and Vibration 274, 605-619. (2004).
[7] B.R. Mace, ‘Wave reflection and transmission in beams’, Journal of Sound and Vibration 97, 237-246. (1984).
[8] N.R. Harland, B.R. Mace and R.W. Jones, ‘Wave propagation, reflection and transmission in tunable fluid-filled beams’, Journal of Sound and Vibration 241, 735-754. (2001).
[9] L. Cremer, M. Heckl and E.E. Ungar, Structure-borne Sound, Berlin: Springer-Verlag. (1973).
[10] M. J. Lighthill, Waves in Fluids, Cambridge: Cambridge University Press. (1978).
[11] S. K. Lee, B. R. Mace and M.J. Brennan. 'Wave propagation, reflection and transmission in non-uniform beams'. Institute of Acoustics Spring Conference 2004, Vol 26 Pt 2 pp 147-156. (2004)

FIGURES

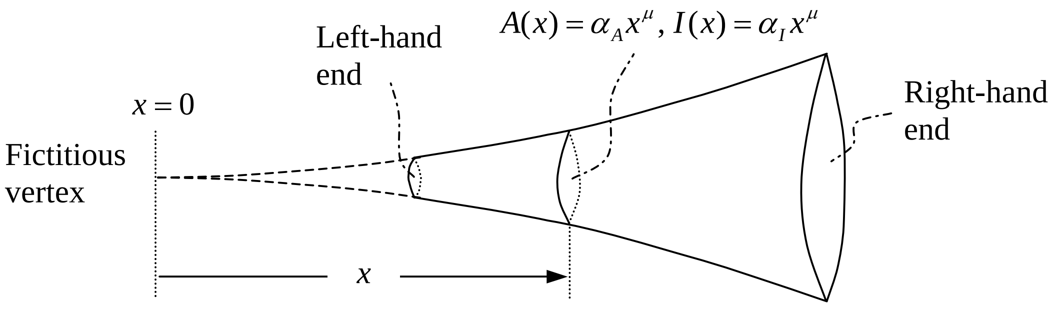

Figure 1. A non-uniform beam with a polynomial variation in geometries.

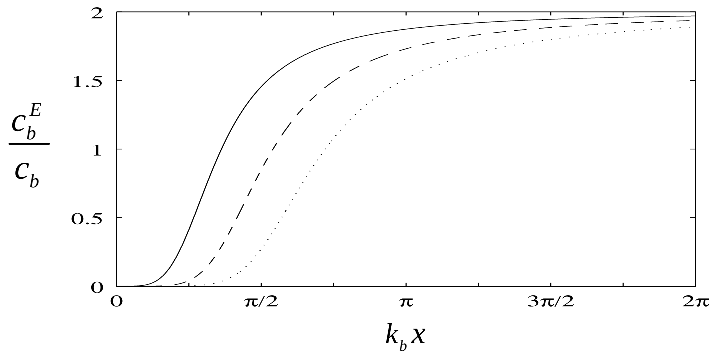

Figure 2. Energy transport velocity for the polynomially varying beams: $\mu = 1$ *(—);* $\mu = 2$ *(---);* $\mu = 3$ *(······).*

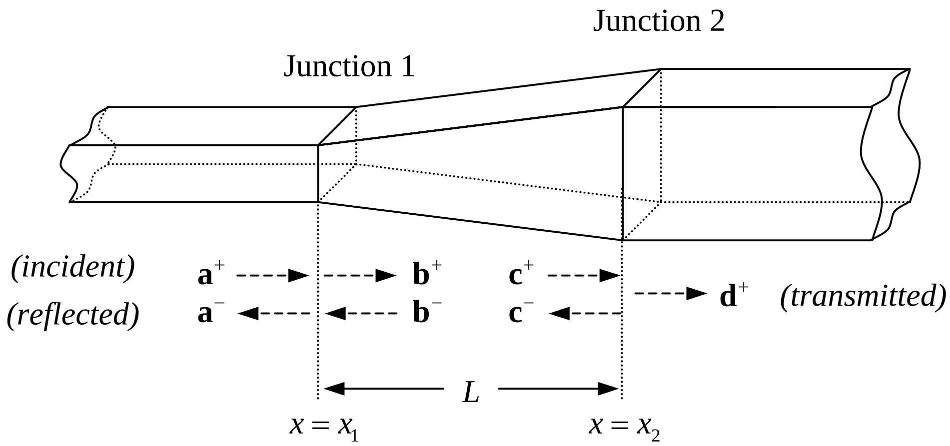

Figure 3. A rectangular connector tapered in thickness.

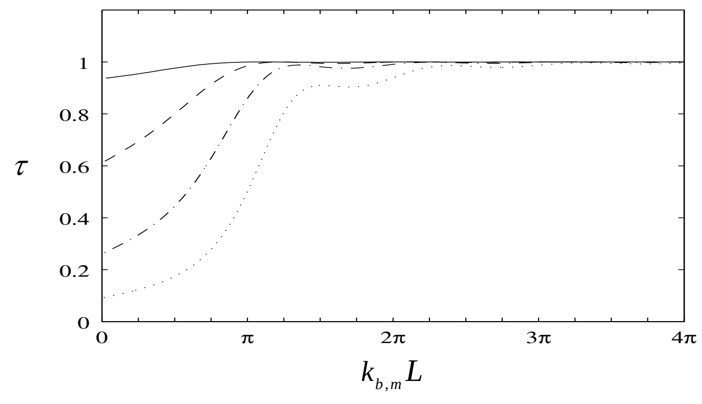

Figure 4. Power transmission coefficient τ *for the connector when the propagating bending wave is incident:* $h_2/h_1 = 2$ *(—);* $h_2/h_1 = 4$ *(---);* $h_2/h_1 = 8$ *(-·-·);* $h_2/h_1 = 16$ *(······).*