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ON ALTERNATIVE THEORIES FOR ANALYSIS OF TIME HARMONIC BEHAVIOUR OF ELASTIC PLATES WITH AND WITHOUT HEAVY FLUID LOADING

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Abstract

The talk is concerned with applicability ranges of the three theories (Kirchhoff theory, Timoshenko theory and elasto-dynamics) to describe time harmonic behaviour of elastic plates with and without heavy fluid loading. Both the free wave propagation in an unbounded plate and the eigenfrequency spectra of plates of finite length are addressed. The findings for an elementary case of the absence of heavy fluid loading are compared with those in the case of a plate with heavy fluid loading. Some useful observations are made and their physical interpretation is given. The role of fluid's compressibility is also explained.

INTRODUCTION

Two issues, which have been addressed rather unevenly in the literature, are considered in this paper. The first one is the relevance of the second spectrum of eigenfrequencies predicted by Timoshenko beam theory. Although this aspect might be regarded as a classical one, there is still a room for its misinterpretation. To gain a physical insight into the origin of such a misinterpretation and to clarify the subject, the problem of determination of eigenfrequencies is put into the context of analysis of dispersion curves predicted by Timoshenko beam theory and by a solution of the problem in elasto-dynamics. To remove any ambiguities, the interpretation is also extended to the assessment of the validity of Bernoulli-Euler model. The second issue considered here is the role of heavy fluid loading, particularly, the role of fluid's compressibility, in dynamics of thin plates, described by Kirchhoff plate theory and by Timoshenko theory. To the best of the author's knowledge, this aspect has not yet been highlighted in the literature.

As regarding the first issue discussed in the previous paragraph, it is most expedient to refer to the paper [5], which contains a comprehensive list of 21 publications on the subject. The wave propagation in Kirhhoff plates under heavy fluid loading is discussed, for example, in a classical text [1]. A problem of wave propagation in a fluid-loaded sandwich plate, which has several common features with the problem for a fluid-loaded Timoshenko plate, is considered in [3-4].

ON THE SECOND SPECTRUM OF TIMOSHENKO BEAM THEORY

For definiteness, consider the example from [5], and take the beam of depth $h = 0.125m$, Young's modulus $E = 210 \cdot 10^9 \frac{N}{m^2}$, density $\rho = 7850 \frac{kg}{m^3}$, Poisson ratio $\nu = 0.3$ and unit thickness consistent with plane stress conditions. Furthermore, select the shear coefficient $\kappa = \frac{5}{6}$. In order to facilitate discussion of results from [5] and their interpretation, the dispersion curves in Figure 1 are presented in the following manner: the wave number is non-dimensional, it is $k = k_d h$, whereas the frequency is dimensional, it is measured in rad/s (the dimensional scaling factor f_S to transform abscissas in Figure 1 to the non-dimensional form $\Omega \equiv \frac{\omega h}{c}$ equals $f_S = \frac{1}{41377.6} \frac{s}{rad}$). The propagating waves are characterized by purely imaginary wave numbers.

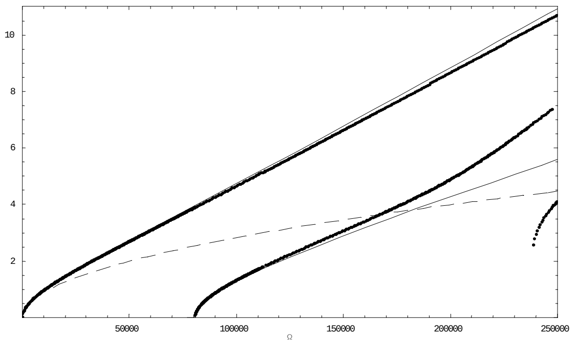


Figure 1. Dispersion curves for an infinitely long layer

The dispersion curve predicted by Bernoulli-Euler theory has only one branch, which is plotted as a dashed line. Its cut-on frequency is $\Omega_{cut-on,1} = 0$. Timoshenko theory predicts two branches, shown in Figure 1 by thin lines. The theory of elasticity predicts infinitely many branches. In Figure 1, the first three are plotted by dots. The cut-on frequencies of the second and the third ones are found to be $\Omega_{cut-on,2} = 80617 \frac{rad}{s}$ and $\Omega_{cut-on,3} = 241852 \frac{rad}{s}$. As is well known the tangent to a dispersion curve at its cut-on frequency is vertical so that the third brunch in Figure 1 does not 'hit' the coordinate axis. As is seen, the picture is well described by

Miklowitz, see [2]. The Bernoulli-Euler theory fails, when the frequency approaches the cut-on of the second branch and the Timoshenko theory becomes inaccurate, when the frequency parameter approaches the cut-on of the third branch.

Exactly as reported in [5], in the case of a very short beam, the difference between the eigenfrequencies predicted by Timoshenko theory and elasto-dynamics in the lowest branch is negligible whereas the eigenfrequencies predicted by the second spectrum of Timoshenko theory are very different from the eigenfrequencies found by use of the theory of elasticity, see Figure 2a. However, it is expedient to notice that the deviation of the second branch of eigenfrequencies from its counterpart predicted by elastodynamics occurs at the frequency range (around the 6th -7th eigenfrequency), which is not too far from the third cut-on frequency, see Figure 1. The graph in Figure 2a might suggest that the Bernoulli-Euler theory is totally irrelevant. However, a closer inspection into deviation of 'Bernoulli-Euler curve' from the other two reveals that the error of Bernoulli-Euler theory becomes large as soon as its error in prediction of the magnitude of the wave number is large. In this particular case, it is, roughly speaking, $\Omega \approx 70000 \frac{rad}{s}$ (i.e., around $\Omega_{cut-on,2}$). Simultaneously, the large error in eigenfrequencies predicted by Timoshenko theory occurs, when a frequency exceeds the value of approximately $\Omega \approx 200000 \frac{rad}{s}$ (i.e., around $\Omega_{cut-on,3}$).

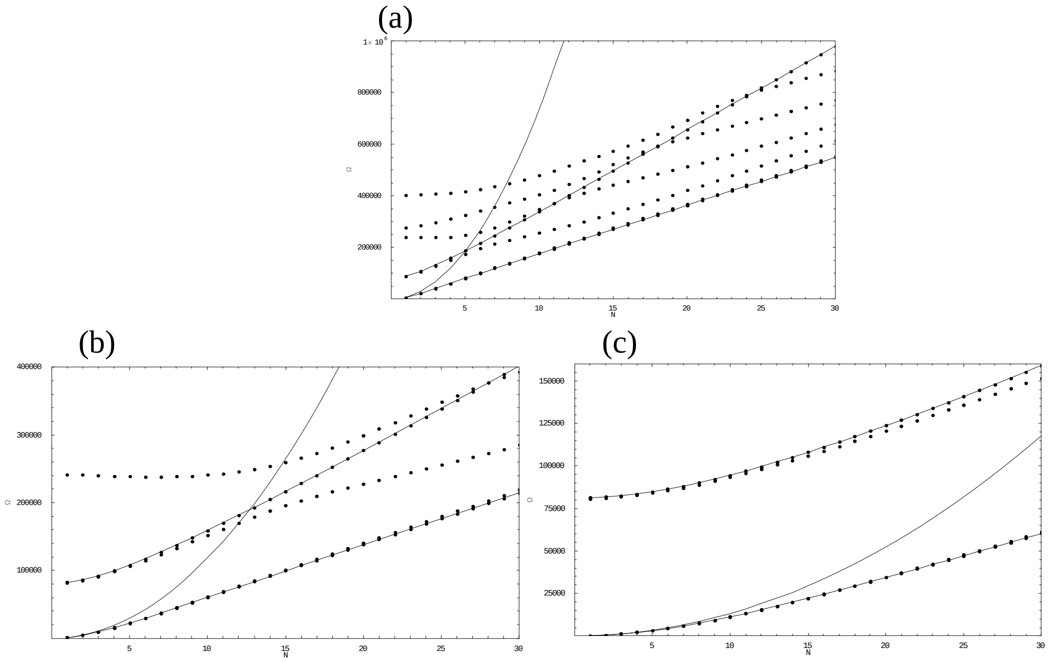


Figure 2. Eigenfrequencies of vibrations of a simply supported layer of the length (a) $L = 0.5m$ (b) $L = 1.25m$ (c) $L = 3.75m$

If a longer beam is considered, $L = 10h = 1.25m$, the eigenfrequencies set up a somewhat different pattern, see Figure 2b. Timoshenko second spectrum fits very well the second branch of curve predicted by elastodynamics as long as the eigenfrequency is sufficiently far from the third cut-on, approximately up to the 18^{th} eigenfrequency (which, naturally, lies around $\Omega \approx 200000 \frac{rad}{s}$). Simultaneously, Bernoulli-Euler theory becomes reliable up to approximately 5^{th}-7^{th} eigenfrequency of the beam, which agrees very well with the previous estimation of $\Omega \approx 70000 \frac{rad}{s}$. Moreover, as a long beam is considered, $L = 30h = 3.75m$ (see Figure 2c), the second Timoshenko spectrum becomes fairly accurate for the whole considered range of eigenfrequencies (from the first to the 30^{th}) belonging to the second spectrum predicted by elasto-dynamics and the curve which displays them asymptotically matches the curve predicted by elasto-dynamics as $\Omega \to 0$. The same tendency is clearly seen for the Bernoulli-Euler model – in this case, some 12-15 eigenfrequencies are predicted more or less adequately and there is the asymptotic matching of all three curves as $\Omega \to 0$.

The physical explanation of a failure of the Timoshenko theory to predict eigenfrequencies of the second spectrum of flexural vibrations of an elastic layer as their magnitudes approach the third cut-on frequency, is the same as the explanation of a failure of the Bernoulli-Euler theory to predict eigenfrequencies of the first spectrum of flexural vibrations of an elastic layer as their magnitudes approach the second cut-on. The formation of a standing wave in a beam is controlled by the interaction between direct and scattered waves at the boundaries. In the particular case of a simply supported structure, the evanescent waves are not involved in this process and the shape of a standing wave is purely sinusoidal in the axial direction. Eigenmodes in the first spectrum are generated dominantly due to the interaction of the travelling waves characterized by wave numbers predicted by the first branch of dispersion curve. An agreement between these wave numbers, in effect, determines the agreement between eigenfrequencies in the first spectra of Timoshenko or Bernoulli-Euler theory with the exact solution. The same holds true regarding the second Timoshenko spectrum.

WAVE PROPAGATION IN A PLATE UNDER HEAVY FLUID LOADING. THE ROLE OF COMPRESSIBILITY

The discussion of validity of Timoshenko and Bernoulli-Euler beam theories becomes particularly relevant, when a surface loading is concerned. Such a case is provided when wave propagation in a plate lying at the surface of a semi-infinite volume of an acoustic medium is considered. In other words, a plate is located in the plane $y = 0$ and an acoustic medium occupies the volume $y < 0$. For simplicity (which, however, does not undermine the generality of results), consider the case of cylindrical bending of a plate, so that its displacement is independent upon one of coordinates. Then one obtains the dispersion equation, which reads as ($\Omega_1^2 = \frac{(1-\nu^2)\rho\omega^2 h^2}{E}$, $k = k_{\dim} h$)

$$k^4 + \left[1 + \frac{2}{\kappa(1-\nu)}\right] k^2 \Omega_1^2 - 12\Omega_1^2 + \frac{2}{\kappa(1-\nu)}\Omega_1^4$$
$$= -i\frac{\rho_{fl}}{\rho}\frac{1}{\sqrt{k^2 + \left(\frac{\omega h}{c_{fl}}\right)^2}}\left[\frac{2}{\kappa(1-\nu)}k^2\Omega_1^2 - 12\Omega_1^2 + \frac{2}{\kappa(1-\nu)}\Omega_1^4\right] \tag{1}$$

In the case of Kirchhoff theory, it is reduced to

$$k^4 - 12\Omega_1^2 = 12i\frac{\rho_{fl}}{\rho}\frac{\Omega_1^2}{\sqrt{k^2 + \left(\frac{\omega h}{c_{fl}}\right)^2}} \tag{2}$$

Consider a limit case of an incompressible fluid and introduce the small parameter $\varepsilon = \frac{\rho_{fl}}{\rho}$. Then the wave number of a propagating wave in a Kirchhoff plate ($k = i\hat{k}$) is found from the equation of the fifths order

$$\hat{k}^5 - 12\Omega_1^2\hat{k} - 12\varepsilon\Omega_1^2 = 0 \tag{3}$$

The simple asymptotic formula for this wave number is

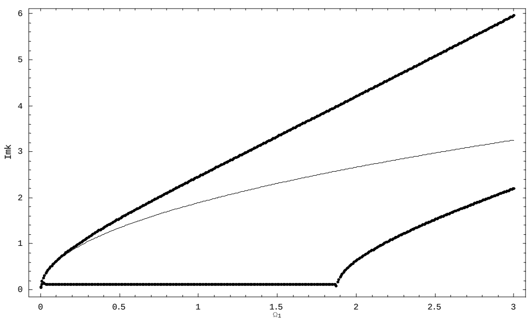
$$\hat{k} = \sqrt[4]{12}\sqrt{\Omega_1} + \frac{\varepsilon}{4} \tag{4}$$

It suggests that in the limit of an incompressible fluid the wave number is very slightly affected by the presence of a fluid. Similarly, the asymptotic formula for wave numbers of propagating waves in a Timoshenko plate reads ($k = i\hat{k}$)

$$\hat{k}_j = \hat{k}_{j0} + \varepsilon\left(\frac{2}{\kappa(1-\nu)}\hat{k}_{j0}^2\Omega_1^2 - 12\Omega_1^2 + \frac{2}{\kappa(1-\nu)}\Omega_1^4\right)\left(5\hat{k}_{j0}^4 + 3\left(1 + \frac{2}{\kappa(1-\nu)}\right)\hat{k}_{j0}^2\Omega_1^2\right)^{-1} \tag{5}$$

$$\hat{k}_{j0} = \sqrt{\frac{1}{2}\left(1 + \frac{2}{\kappa(1-\nu)}\right)\Omega_1^2 \mp \sqrt{\frac{1}{4}\left(1 - \frac{2}{\kappa(1-\nu)}\right)^2\Omega_1^4 + 12\Omega_1^2}}\ , \quad j = 1,2$$

(a) (b)



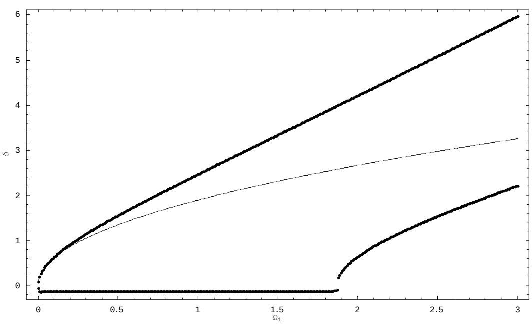


Figure 3. Dispersion curves $\mathrm{Im}\,k$ *(a) and* δ *(b) for a plate loaded by water considered as an incompressible fluid*

In Figure 3, the branches of Timoshenko dispersion curve $\operatorname{Im} k$ are shown by thick lines, the dispersion curve for a Kirhhoff plate is shown by thin line in the case of loading by water, $\varepsilon = 0.128$. The pattern of these curves differs very slightly from those shown in Figure 2. The decay rate into the fluid is characterized by the parameter $\delta \equiv \sqrt{\hat{k}^2 - \left(\frac{\omega h}{c_{fl}}\right)^2}$. Apparently, in the case of an incompressible fluid, $\delta = \hat{k}$ as is seen from comparison of Figures 3a and 3b. To sum up these results, one may conclude that the presence of an incompressible fluid does not strongly affect the wave propagation in a plate and the validity range of Kirchhoff theory is the same as in the case of vibrations in vacuum. However, the situation becomes totally different as soon as the compressibility is taken into account.

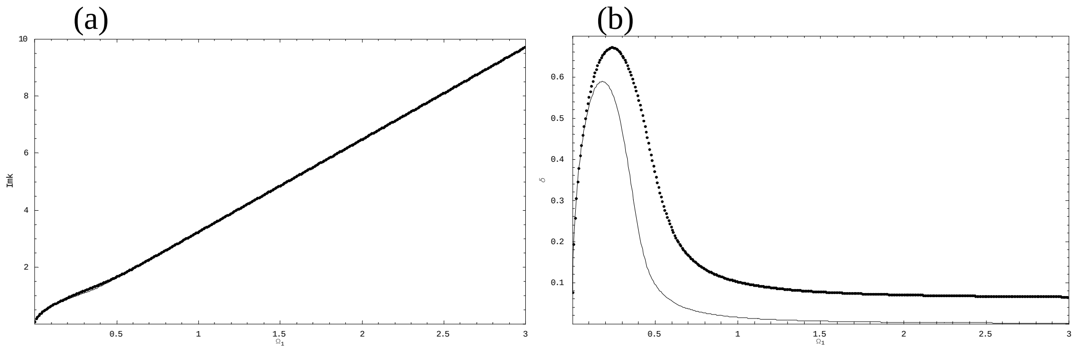


Figure 4. Dispersion curves $\operatorname{Im} k$ (a) and δ (b) for a plate loaded by water considered as an acoustic medium

In Figure 4a, the dispersion curves predicted by Timoshenko plate theory and by Kirchhoff plate theory are presented, then a sound speed in water is taken as $c_{fl} = 1440 \frac{m}{s}$, which gives $\frac{c_{fl}}{c_p} = 0.307$, $c_p = \sqrt{\frac{E}{\rho(1-\nu^2)}}$. The second branch of Timoshenko dispersion curve has disappeared and the difference between predictions of two theories can hardly be noticed (it might be visible only around $\Omega_1 = 0.3 - 0.4$). In effect, the curves are undistinguishable from each other. The parameter δ is shown in Figure 4b. There is also a relatively small difference between predictions of these two theories. These results suggest that the presence of an unbounded volume of the acoustic medium smears out almost completely the difference between Timoshenko and Kirchhoff theories, in other words, that the validity range of Kirchhoff theory is dramatically extended due to the interaction between an acoustic medium and a plate.

This phenomenon has a simple physical interpretation. The presence of an acoustic medium, especially, at high frequencies, makes the rate of decay into fluid's volume of a wave trapped at the surface of a plate very small, see Figure 4b. Therefore, a volume of fluid involved in the wave propagation along the surface of a plate is much larger when the compressibility is taken into account, than when it is ignored. Apparently, the behaviour of a fluid is the same regardless whether the Timoshenko or Kirchhoff plate model is used. Thus, the characteristics of a trapped wave propagating along the plate are determined to larger extend by the properties of a compressible fluid, than by properties of a plate. In the case of an uncompressible fluid, its motion is induced in a relatively thin layer near plate's surface (see Figure 3b) and the difference between two plate's models is more visible. This result is rather generic and may be extended to analysis of wave propagation in cylindrical shells or any other structures submerged in water.

The disappearance of the second wave has the same explanation as the disappearance of the second wave in a sandwich plate, see [3]. Indeed, the cut-on frequency of this wave is independent upon fluid loading parameter, it is
$\Omega_1^{cut-on} = \sqrt{6\kappa(1-\nu)}$.
The velocity potential field introduced by this wave must decay at infinity, $y \to -\infty$. Therefore, the following condition must be held

$$\delta = \sqrt{k_1^2 - \left(\frac{\omega h}{c_{fl}}\right)^2} > 0 \qquad (6)$$

The threshold magnitude of wave number is therefore $k_1 = \frac{c_{fl}}{c_p}\Omega_1$. This wave number and this frequency parameter must satisfy the dispersion equation (2). Then the threshold frequency becomes

$$\Omega_1^S = \sqrt{6\kappa(1-\nu)\left[1-\left(\frac{c_p}{c_{fl}}\right)^2\right]^{-1}} \qquad (7)$$

The admissible regime of shear wave propagation is formulated as $\Omega_1^S \le \Omega_1$. In the case of an incompressible fluid, $\frac{c_p}{c_{fl}} = 0$ and $\Omega_1^S = \Omega_1^{cut-on}$. However, if $c_{fl} < c_p$, then this condition cannot be fulfilled and free propagation of the 'second Timoshenko wave' is impossible. Exactly the same situation is reported in [3] regarding propagation of the shear wave in a sandwich plate.

Strictly speaking, this condition is apparently held for a plate vibrating in air and the dominantly shear wave cannot propagate in such a case. However, it should be observed that all analysis of vibrations of a plate and of an elastic layer has been conducted under an assumption of the total absence of any energy dissipation in the material. Once the internal damping is taken into account, then a propagating wave should be rather referred to as 'almost propagating' with its rate of decay controlled by the internal damping. Although such a generalisation of results reported here does not present any difficulties, it lies beyond the scope of this paper. The energy dissipation due to the acoustical emission in the case of wave propagation in water is competitive with the internal energy dissipation in the plate's material, but in the case of loading by air it is negligible. Therefore, discussion of the effect of suppression of shear wave propagation in a plate due to the energy emission into air does not seem to be practical.

CONCLUSIONS

The results reported in this paper are summarized as follows:

1. The second spectrum of eigenfrequencies predicted by Timoshenko theory is in a very reasonable agreement with the exact solution of the problem in elasto-dynamics as long as their magnitudes are 'sufficiently' lower than the third cut-on frequency predicted by solution of the problem of propagation of skew-symmetric waves in an elastic layer. The 'sufficiency' is naturally determined by the chosen tolerance level.
2. The wave propagation in a plate under heavy fluid loading is strongly influenced by fluid's compressibility. In particular, propagation of the shear wave predicted by Timoshenko theory is suppressed due to this effect.
3. The validity range of Kirchhoff plate theory is substantially extended due to heavy fluid loading effects produced by an unbounded volume of an acoustic medium in comparison with its validity range for time-harmonic behaviour of a plate in vacuum.

REFERENCES

[1] M. G. Junger, D. Feit, *Sound, structures and their interactions*. (2nd ed., MIT Press, 1993).

[2] J. Miklowitz, *The theory of elastic waves and waveguides*. (North-Hollnd, Amsterdam 1978).

[3] S. V. Sorokin, "A note on theeffect of compressibility on the propagation of free waves in sandwich plates with heavy fluid loading", J. Sound and Vibration, **270**, 433-440 (2004).

[4] S. V. Sorokin, "Analysis of wave propagation in sandwich plates with and without heavy fluid loading", J. Sound and Vibration, **271**, 1039-1062 (2004).

[5] N. G. Stephen, "The second spectrum of Timoshenko beam theory – further assessment", J. Sound and Vibration, **292**, 372-389 (2006).