ITERATIVE NUMERICAL SOLUTION OF NONLINEAR WAVE PROBLEMS

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Abstract
Nowadays, the design of phased array transducers for medical diagnostic ultrasound asks for an understanding of the nonlinear propagation of acoustic wavefields. In the last decade an imaging modality called Tissue Harmonic Imaging (THI) has become the standard for many echography investigations. THI specifically benefits from the nonlinear distortion of ultrasound propagating in tissue. Since most existing numerical models are based on a linear approximation of the underlying nonlinear physical reality, they cannot account for this kind of distortion. Several numerical models have been developed in recent years that incorporate weak nonlinear propagation. However, as yet no model has enabled the computation of large scale, full-wave nonlinear wavefields in the time domain.

In this study, we present an approach that handles weak nonlinear propagation by means of an iterative Neumann scheme. This approach enables the successive use of the solution of a linear wave problem, where the nonlinearity is treated as a contrast source. Thus, we can employ well-known linear methods for large scale wave problems to obtain the desired nonlinear wavefield.

The general formalism is outlined and applied to a one-dimensional nonlinear wave problem. The wavefield is evaluated, including harmonic frequencies up to the fifth harmonic. For each successive linear step a Green's function approach is employed. The results are validated with a solution of the lossless Burgers' equation. It is observed that already after a small number of iterations the results are in very good agreement with this exact result.

The proposed method can easily be extended to more complex problems. The Green's function approach enables us to discretize the spatiotemporal domain very efficiently, which opens the road to solving time domain nonlinear wave problems in three dimensions. Furthermore, media with attenuation and inhomogeneity can be included straightforwardly in the algorithm.

INTRODUCTION

In order to optimize the design of medical diagnostic ultrasound transducers, numerical modeling of the generated ultrasonic wavefields has become almost inevitable. For this purpose, a score of models has been developed that predict the continuous-wave or pulse-time wavefields that are generated by phased array transducers with arbitrarily steered, focused and apodized excitations [9, 12]. These models operate under a linear approximation of the basic acoustic equations, thus neglecting the nonlinear character of acoustic propagation. However, with the sound pressure levels that are common in echography, distortion of the pulse shape due to nonlinear propagation is clearly observable [10]. Moreover, in the past decade an imaging modality called Tissue Harmonic Imaging (THI) that specifically benefits from this effect has found its way into the medical ultrasound practice [8]. In a wide area of medical applications, THI has shown to give significantly improved imaging results when compared with the traditional imaging method [11, 16]. Recently, it was suggested that the image quality can be further enhanced by exploiting the nonlinear distortion more extensively [1].

As THI is explicitly based on nonlinear wave propagation, the existing linear models cannot deal with it. In the recent years, several approaches have been developed to account for the nonlinear distortion. For an overview we refer to recent publications [14, 17, 18]. Many models use a plane-wave nonlinear propagator to incorporate the nonlinear behavior, the approach of which may not be valid for acoustic wavefields propagating in a different direction than the preferred one. The challenge remains to develop a full-wave, nonlinear wave propagation model that can handle a large scale three-dimensional configuration in the time domain, with an acceptable cost in terms of memory and computation time.

In this paper, we present a novel solution strategy for the nonlinear wave propagation problem. It is based on the idea that a weak nonlinear effect can be represented by a correction to the linear wave problem. By iteratively correcting the linear wave problem by means of a Neumann scheme, we obtain the solution to the nonlinear wave problem with any desired degree of accuracy. An equivalent approach was used in [6]. For the linear wave problem, we use the Green's function method. If the Green's function is adequately regularized, we can obtain accurate wavefield results with a discretization up to the Nyquist criterion for the smallest wavelength of interest.

ITERATIVE NEUMANN SCHEME

The nonlinear acoustic wave propagation can be described by well known second order partial differential equations (PDE's) like the Westervelt equation, the KZK equation and the Burgers' equation [5]. Within the same order of approximation, an equivalent but more basic formulation is the set of first order PDE's [2]

$$\begin{aligned} \boldsymbol{\nabla} p + \rho D_t \boldsymbol{v} &= \boldsymbol{f}, & (1) \\ \boldsymbol{\nabla} \cdot \boldsymbol{v} + \kappa D_t p &= q, & (2) \end{aligned}$$

where p is the acoustic pressure, $\boldsymbol{v}$ is the fluid velocity, and $\boldsymbol{f}$ and q denote the external force and volume sources. $D_t = \partial_t + \boldsymbol{v} \cdot \boldsymbol{\nabla}$ is the total time derivative, and ρ and κ are the field-dependent mass density and compressibility, given as [7, 15]

$$\begin{aligned} \rho &= \rho_0[1 + \kappa_0 p], & (3) \\ \kappa &= \kappa_0[1 + \kappa_0(1 - 2\beta)p]. & (4) \end{aligned}$$

Here, β is the nonlinearity parameter. As a key step of our scheme, we rewrite the set of equations that result from Eqs. (1) to (4) by gathering the nonlinear terms in a contrast source term on the right hand side. The resulting equations can be cast in the general matrix form

$$\boldsymbol{D}^L_{\boldsymbol{x}} \boldsymbol{F} + \boldsymbol{M}^L \boldsymbol{D}^L_t \boldsymbol{F} = \boldsymbol{S} + \boldsymbol{S}^N(\boldsymbol{F}), \qquad (5)$$

where $\boldsymbol{D}^L_{\boldsymbol{x}}$, $\boldsymbol{D}^L_t$ denote the linearized differential operators in space and time, and $\boldsymbol{M}^L$ denotes the linear medium behavior. In the present case, the field variables, external sources and nonlinear contrast source terms are written as

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{v} \\ p \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} \boldsymbol{f} \\ q \end{bmatrix}, \quad \boldsymbol{S}^N(\boldsymbol{F}) = \begin{bmatrix} -\rho_0 \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} - \rho_0 \kappa_0 p D_t \boldsymbol{v} \\ -\kappa_0 \boldsymbol{v} \cdot \boldsymbol{\nabla} p - \kappa_0^2 (1 - 2\beta) p D_t p \end{bmatrix}. \qquad (6)$$

We assume that the contrast source term $\boldsymbol{S}^N(\boldsymbol{F})$ will be weak with respect to the other terms. If we can obtain a linear operator $\boldsymbol{L}$ that solves $\boldsymbol{F}$ in terms of $\boldsymbol{S}$ and $\boldsymbol{S}^N$, i.e.

$$\boldsymbol{F} = \boldsymbol{L}\left[\boldsymbol{S} + \boldsymbol{S}^N(\boldsymbol{F})\right], \qquad (7)$$

then we can set up the following iterative scheme to approximate the field solution of the nonlinear problem:

$$\begin{aligned} \boldsymbol{F}^0 &= \boldsymbol{L}\left[\boldsymbol{S}\right], & (8) \\ \boldsymbol{F}^j &= \boldsymbol{L}\left[\boldsymbol{S} + \boldsymbol{S}^N(\boldsymbol{F}^{j-1})\right], & (9) \end{aligned}$$

representing our iterative Neumann scheme.

THE LINEAR OPERATORS AND THEIR DISCRETIZATION

One method to solve a linear partial differential equation, i.e. to obtain the linear operator $\boldsymbol{L}$ in Eq. (7), is the Green's function or impulse response method. With this method, the PDE can be translated to an integral representation of the form

$$\boldsymbol{F} = \boldsymbol{G} *_{t,\boldsymbol{x}} \boldsymbol{S}, \tag{10}$$

where $\boldsymbol{G}$ denotes the Green's function for the specific wave problem, $\boldsymbol{S}$ accounts for the relevant source terms and $*_{t,\boldsymbol{x}}$ denotes the convolution with respect to the temporal and spatial dimensions. The convolution integral can be discretized by a midpoint rule approximation, yielding a linear convolution sum in each dimension.

For simplicity, let us restrict our analysis in this paragraph to the temporal dimension. Then we can write the convolution integral and its discrete form as

$$\boldsymbol{G} *_t \boldsymbol{S} = \int_{-\infty}^{\infty} \boldsymbol{G}(t-t')\boldsymbol{S}(t')\mathrm{d}t' \approx \Delta t \sum_{n=-\infty}^{\infty} \boldsymbol{G}(t-n\Delta t)\boldsymbol{S}(n\Delta t), \tag{11}$$

where Δt is the step size. In order to accurately reproduce the (non)linear propagation up to a certain frequency $f_{\max}$, the Nyquist limit prescribes step sizes of at most $\Delta t = 1/2f_{\max}$. To prevent aliasing, we need to remove the frequency content in the spectra of $\boldsymbol{G}$ and $\boldsymbol{S}$ for $|f| > f_{\max}$. This is done by analytically filtering both functions with an ideal filter. To efficiently perform the convolution operation by means of the Fast Fourier Transform (FFT) and the discrete convolution theorem [13], we need to translate the linear convolution into a cyclic one. This encompasses the selection of a window of interest for $\boldsymbol{F}$, say $t \in [0, T]$. The source term $\boldsymbol{S}$ is windowed to the same region, and the Green's function $\boldsymbol{G}$ needs to be windowed to $t \in [-T, T]$. When both the filtering and windowing operations have been performed before discretization, then the result of the convolution sum is an accurate approximation of $\boldsymbol{F}$ for $|f| \leq f_{\max}$ on points $t = n\Delta t$ for $t < T$.

Finally, as can be seen in Eq. (6), we will need to perform several numerical differentiations with respect to space and time to obtain the contrast source. To obtain these on the coarse grid we revert to high-order finite difference (FD) schemes [4].

APPLICATION TO A NONLINEAR PLANE WAVE PROBLEM

In order to illustrate and evaluate the presented method, we will apply it to a nonlinear one-dimensional wave propagation problem in (x, t) in a homogeneous, lossless medium. The set of equations described by Eqs. (5) and (6) reduces to two scalar equations. We define the source to be a volume impulse source at $x = 0$, excited with a harmonic signal with a gaussian envelope, i.e. $\boldsymbol{S} = [\ 0 \ \ q(x,t)\ ]^T$ with

$$q(x,t) = \delta(x)q(t) = \delta(x)Q_0 \exp\left[-\left(\frac{t-t_d}{t_w}\right)^2\right] \sin[2\pi f_0(t-t_d)]. \tag{12}$$

Here, $Q_0,\ t_d,\ t_w$ and f_0 are the source amplitude, pulse delay, pulse width and center frequency, respectively. For the linear propagation problem, the pressure level at the source is related to Q_0 via $p = \frac{1}{2}\rho_0 c_0 Q_0$.

It can be shown that the Green's function for this problem is

$$\boldsymbol{G}(x,t) = \begin{bmatrix} \kappa_0\partial_t & -\partial_x \\ -\partial_x & \rho_0\partial_t \end{bmatrix} g(x,t), \quad g(x,t) = \frac{c_0}{2} H\left(t - \frac{|x|}{c_0}\right), \tag{13}$$

where $g(x,t)$ is the one-dimensional free space Green's function [3], and $H(t)$ is the Heaviside step function.

The source is filtered in the x-dimension using an ideal filter with a cut-off frequency of $1/2\Delta x$. In the temporal dimension, the source function is not filtered, as it is already sufficiently bandlimited at grids with a step size $\Delta t < 4/f_0$. Subsequently, $q(x,t)$ is windowed to $x \in [0,X],\ t \in [0,T]$ and discretized with step sizes Δx and Δt. This results in $q(m\Delta x, n\Delta t) = q(n\Delta t)/\Delta x$ for $m = 0$ and $q = 0$ for $m \neq 0$.

Regarding the filtering of $\boldsymbol{G}$, we will focus on the function $g(x,t)$. Filtering it in both x and t with an ideal filter gives an expression for which no closed form is available. Instead, we take an alternative route by interchanging the operations of filtering and windowing $g(x,t)$ and performing them in the transform domain. For this specific Green's function, the windowing operation can be expressed and simplified as

$$\begin{aligned} g^{X,T}(x,t) &= g(x,t)[H(x+X) - H(x-X)][H(t+T) - H(t-T)] \\ &= g(x,t)[H(t) - H(t-T)]. \end{aligned} \tag{14}$$

We study the windowing and filtering operations in the spatial Fourier and temporal Laplace domain. The transform of $g(x,t)$ is obtained as

$$\hat{g}(k,s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,t)\exp(-ikx - st)\mathrm{d}x\mathrm{d}t = \frac{1}{k^2 + c_0^{-2}s^2}, \tag{15}$$

where k is the spatial Fourier parameter and $s = s_0 + i2\pi f$ is the temporal Laplace parameter. If $s_0 \downarrow 0$ the temporal Fourier transform of $g(x,t)$ is obtained, which has singularities at $k = \pm 2\pi f/c_0$.

The transform domain equivalent of the windowed form gives

$$\hat{g}^{X,T}(k,s) = \hat{g}(k,s)\left[1 - s\exp(-sT)\frac{\sin(c_0 kT)}{c_0 k} - \exp(-sT)\cos(c_0 kT)\right]. \tag{16}$$

With this, we obtain the windowed form of $\hat{\boldsymbol{G}}^{X,T}$. The subsequent filtering operation is a trivial task in the transform domain, and we discretize the result with step sizes $\Delta k = 2\pi/2X,\ \Delta f = 1/2T$. To avoid the singular points in $\hat{g}$, we keep a small positive s_0 in the transform parameter. In the FFT's, this is accounted for by multiplying the function with $\exp(-s_0 t)$ before the forward FFT and by $\exp(s_0 t)$ after the inverse FFT. These terms are kept close to 1 by setting $s_0 = 0.1/T$.

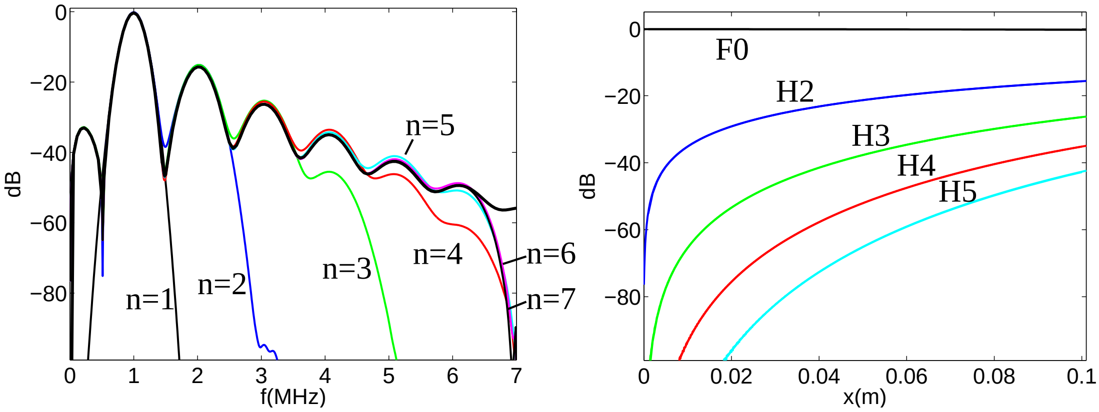

Figure 1: Left: Power spectrum (in dB*) of the signal pressure* p *at* $x = 0.1$ m *as obtained by the Burgers' equation solution (thick line) and by the algorithm after* $n = 1$ *to* $n = 7$ *iterations with* $f_{\max} = 7f_0$*. We observe that each iteration gives a better estimate of increasingly higher harmonics. Right: development of the fundamental (F0) and four of the higher harmonics (H2 to H5) during propagation.*

RESULTS

The algorithm is compared with an implicit solution of the Burgers' equation [5]. For the medium we take water, with parameters $\rho_0 = 998$ kg m^{-3}, $c_0 = 1480$ m s^{-1}, $\kappa_0 = 1/\rho_0 c_0^2$ and $\beta = 3.5$. For the source excitation we use parameters that represent typical values for diagnostic ultrasound, $f_0 = 1$ MHz, $Q_0 = 1$ s^{-1}, $t_d = 6/f_0$ and $t_w = 1.5/f_0$. This results in a source pressure of about 740 kPa.

In Fig. 1 we illustrate the convergence of the Neumann scheme. We are interested in the frequency content of the field up to the fifth harmonic. In the figure the power spectrum of the propagating pulse is plotted at $x = 0.1$ m for the successive iterations, along with a plot of the development of the higher harmonics during propagation obtained after $n = 6$ iterations. It turns out that in this case it is sufficient to use step sizes $\Delta t = \Delta x/c_0 = 1/14f_0$, i.e. we sample the seventh harmonic frequency with two points per period. Each iteration improves the nonlinear estimate at the higher harmonics, and after six iterations the spectrum of the fifth harmonic (H5) at $x = 0.1$ m is reproduced with a relative square error of less than 2% .

In Fig. 2 we investigate the stability of the algorithm close to the theoretical shock formation distance $\bar{x} = 2c_0^2(\beta\omega_0 Q_0)^{-1}$. For $Q_0 = 4$ s^{-1} we have $\bar{x} = 0.050$ m. In the figure, the waveform and power spectrum are plotted at $x = 0.98\bar{x}$. It appears that as long as $X < \bar{x}$ the iterative scheme is stable and results in a waveform that is as accurate as is permitted by the number of included harmonics. The highest harmonic components are slightly deviated due to the aliasing of the (now significant) harmonic components at frequencies larger than $f_{\max}$.

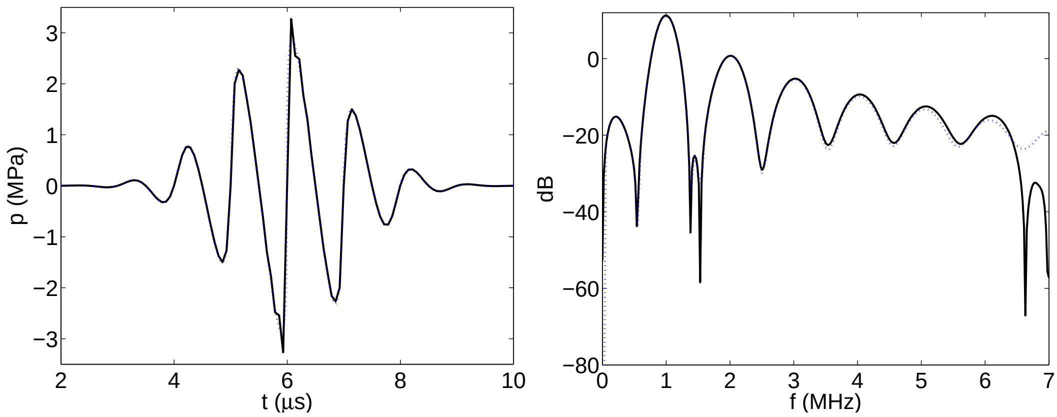

Figure 2: Waveform (in MPa*, left) and power spectrum (in* dB*, right) of the signal pressure* p *close to shock formation distance (*$x = 0.98\bar{x},\ \bar{x} = 0.05\ \mathrm{m}$*) as obtained by the Burgers' equation solution (*$\cdots$*), and by our algorithm (–), with* $f_{\max} = 7f_0$ *and* $n = 8$ *iterations. We only observe a small deviation in the waveform at the steeper slopes and deviations in the highest frequency components that are accounted for.*

DISCUSSION AND CONCLUSION

In this paper, we have introduced an iterative Neumann scheme as an algorithm to account for nonlinear wave propagation. The method involves a successive solution of a linear wave problem. We have shown that with the Green's function method the problem can be efficiently discretized. The method was applied on a one-dimensional nonlinear wave problem, of which the results already showed to be accurate after a small number of iterations. The stability of the iterative scheme was also shown to be good.

The main advantages of the proposed method are first that we can account for an arbitrary nonlinear medium, as long as the nonlinear distortion is weak compared to the linear behavior. Secondly, the nonlinear operator is independent of the direction of the field, which will be important for addressing problems with steered phased arrays. Thirdly, the method permits an efficient discretization up to the Nyquist limit for all dimensions. The method has two drawbacks. Firstly, the used operators are relatively expensive in terms of computation time, but fortunately good algorithms are available that can effectively reduce this. Secondly, the method requires the storage of the field in all spatial and temporal dimensions. Both issues are countered by the efficient discretization. In the further development of the method, these issues will become our main challenge.

In the future, we will extend the method to three dimensions and to media with dispersion. A possible development will be the inclusion of inhomogeneous media, as the method is inherently suited for that.

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