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NUMERICAL INVERSE METHOD PREDICTING ACOUSTIC SPINNING MODES RADIATED BY A DUCTED FAN FROM FREE-FIELD SIMULATED TEST DATA

Serge Lewy*, Hélène Gounet

ONERA / CFD & Aeroacoustics Department, MB 72, 92322 Chatillon, France
Serge.Lewy@onera.fr

Abstract

Spinning modes generated by a ducted turbofan at a given frequency determine the free-field directivity. An inverse method starting from measured directivity patterns could provide some information on the noise sources without requiring spinning-mode analyses. This paper suggests a constrained iterative least squares fitting. Equations are based on analytical modal splitting inside a cylindrical duct, and on a Rayleigh or a Kirchhoff integral on the duct exit cross-section to get far-field directivity. Equations are equal in number to free-field measurement locations, and the unknowns are the propagating mode amplitudes. An automatic procedure using MATLAB® has been implemented. A constraint comes from the fact that squared modal amplitudes must be positive which requires an iterative process. Several numerical simulations are discussed. It is clearly shown that solution depends on the selected method, and is not unique. This means that the initial set of modes needs to be carefully chosen according to any known physical property of the acoustic sources. It is however assessed that the iterative process is very fast and efficient, and well fits the measured directivities. Issues for future improvements are raised in the conclusions.

INTRODUCTION

Deducing the modal structure generated by a ducted turbofan at a given frequency using only conventional free-field measurements is an interesting challenge because this would avoid any direct modal analysis. Such tests indeed are intrusive and expensive, requiring a rotating microphone or an array of fixed microphones. It has been shown in a previous paper how this inverse problem can be solved on the basis of analytical equations describing acoustic propagation in a hard-walled cylindrical duct, and of a Rayleigh or a Kirchhoff integral to get the directivity of sound radiation in the far field [1]. The problem is however ill-posed because there are generally much more possible propagating modes than experimental data (as soon as frequency is not very low).

Present work suggests an automatic procedure based on MATLAB® programming. Next section reminds the equations involved in a least squares fitting. The following section checks the solutions on numerical simulations. Some more detailed analyses of the results are then discussed to better conduct future tests.

LEAST SQUARES FITTING

The root mean square value of sound pressure, P_{rms} (normalized to a reference pressure, p_{re}), radiated into the far field at an angle φ ($\varphi = 0$ on the nacelle centerline) is given by:

$$P_{rms}^2(\varphi) = \sum_m \sum_\mu \frac{A_{m\mu}^2}{2} \left| \mathrm{F}_{m\mu}(\varphi) \right|^2 \tag{1}$$

(see Eq. (1) of [1]), where m is the circumferential mode and μ is the radial mode ($\mu \geq 1$). The $A_{m\mu}$ are unknown amplitudes, and the eigen-functions $\mathrm{F}_{m\mu}(\varphi)$ can be given by a Rayleigh integral (Tyler and Sofrin model [4]):

$$\mathrm{F}_{m\mu}^{(R)}(\varphi) = i^{m+1} \frac{Rk_z}{2\, d/R} e^{i(\omega t - Kd)} \, \mathrm{J}_m(k_t R) \left[\frac{(2KR \sin\varphi)\, \mathrm{J}'_m(KR \sin\varphi)}{(k_t R)^2 - (KR \sin\varphi)^2} \right]. \tag{2}$$

In this equation, R is the duct radius, d is the distance to the duct exit center, $\omega = 2\pi f$ is a given angular frequency, $K = \omega/a$ and k_t are the total and transverse wavenumbers, respectively (where a = 340 m/s is the speed of sound), and J_m is the Bessel function of first kind and of order m. Distance d will be kept constant in this paper, and is thus removed from the arguments of the various functions. The last term in square brackets is the directivity factor. Superscript (R) is added to $\mathrm{F}_{m\mu}$ in Eq. (2) to remind us that it comes from a Rayleigh integral.

A Kirchhoff integral is more general since it is also valid in a uniform flow (flight simulation). Moreover, it better represents actual configurations, mainly for lateral radiation, because there is no hypothesis of flanged inlet. In this case, neglecting any flow velocity [1]:

$$\left| \mathrm{F}_{m\mu}^{(K)}(\varphi) \right| = \left(\frac{1}{2} + \frac{K \cos\varphi}{2k_z} \right) \left| \mathrm{F}_{m\mu}^{(R)}(\varphi) \right|. \tag{3}$$

The Kirchhoff method is thus used instead of the Rayleigh integral in this paper. This allows us to take into account measurements at angles φ slightly larger than 90°.

Sound pressure levels, SPL, are measured in the far field at a given distance d of the duct exit for J radiation angles φ_j. Experimental values of the squared *rms* sound pressure (normalized to the reference pressure, p_{re}) are:

$$Z_j = 10^{\mathrm{SPL}_{test}(\varphi_j)/10} \text{ for } j = 1 \text{ to } J. \quad (4)$$

We thus need to calculate:

$$Y_j \equiv P_{rms}^2(\varphi_j) \text{ for } j = 1 \text{ to } J. \quad (5)$$

Each of these computed values is a sum on N propagating modes $n \equiv (m, \mu)$. Let us put $\Phi_{nj} \equiv |\mathrm{F}_{m\mu}(\varphi_j)|^2$ for the known eigen-functions, and $\alpha_n \equiv A_{m\mu}^2 / 2$ for the unknown amplitudes (α_n must not be negative). The Φ_{nj} are the directivities $P_{rms}^2(\varphi)$ computed for a single mode, n, with a unit amplitude, $\alpha_n = 1$. Equations (4) and (5), using Eq. (1), give a system of J equations with N unknowns:

$$Y_j = Z_j \text{ or } \sum_{n=1}^{N} \Phi_{nj}\alpha_n = Z_j \text{ for } j = 1 \text{ to } J. \quad (6)$$

A least squares fitting is the best way to find the Y_j simulating the test data Z_j because a method based on transfer matrix would be very sensitive to measurement errors [2]. It consists in searching the coefficients α_n minimizing the standard deviation between test and computed data.

Let us write the system of equations (6) in a matrix form: $\Phi\alpha = \mathrm{Z}$. Matrix Φ has J lines and N columns, vector α has N lines, and vector Z has J lines. Contrary to a conventional least squares fitting, this system generally is underdetermined. Indeed, there are usually more propagating modes, N, or unknowns, α_n, than measurement locations or equations, J.

This system is solved in a least squares sense, i.e., α is found such that the norm $\|\Phi\alpha - \mathrm{Z}\|$ is a minimum. It can be done in MATLAB® through two rather similar ways, but they do not give the same results because the solution is never unique if $N > J$:

– Computing the Moore-Penrose pseudo-inverse “pinv” of the matrix Φ, i.e., $\alpha = \mathrm{pinv}(\Phi)*\mathrm{Z}$;

– Using a QR decomposition through the backslash operator “\”, i.e., $\alpha = \Phi\backslash\mathrm{Z}$, which returns a vector α with at most J nonzero components.

Finally, data processing is performed in two main steps. (i) The eigen-functions $\mathrm{F}_{m\mu}$ in Eq. (1) are computed for the propagating modes at the measurement angles φ_j according to Eqs. (2) or (3). (ii) Operation “pinv” or “\” is repeated in an iterative process because of the constraint that α_n may not be negative. The modes $n = (m, \mu)$ giving a negative α_n are removed from the following iteration.

NUMERICAL SIMULATIONS

The method is tested in a case similar to the experiments studied in [1]. The duct diameter is $2R = 0.864$ m, free-field radiation is available at a distance $d = 18.5$ m from the duct exit every $\Delta\varphi = 5°$ from $\varphi = 0$ (on the duct center-line) to $\varphi = 100°$. There is thus a system of $J = 21$ equations to be solved. The frequency is $f = 1500$ Hz, i.e., the reduced frequency is $KR = 2\pi Rf/a = 12$. There are 24 propagating modes $(|m|, \mu)$ in these conditions: (0,1) to (10,1), (0,2) to (6,2), (0,3) to (3,3), and (0,4) and (1,4). It is assumed in the simulated test that only 5 modes out of the 24 are generated: (0,1), (1,1), (10,1), (1,2), and (1,3) whose amplitudes are $A_{m\mu} = 0.3$, 1, 8, 1, and 2, respectively (Fig. 1). It is now checked if these data can be retrieved.

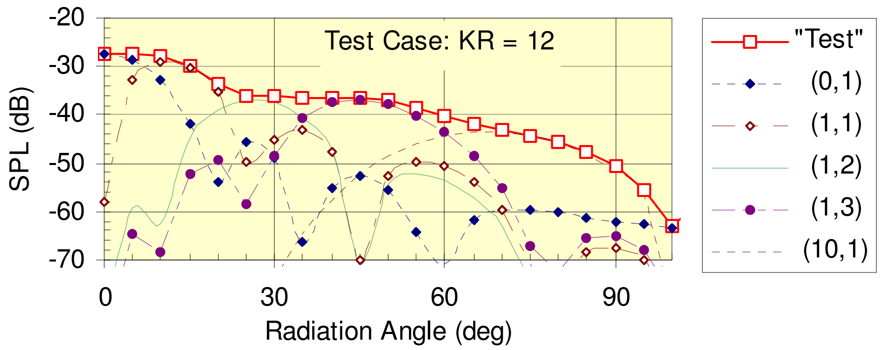

Figure 1 – Numerical simulation of test data

A first fitting uses only the 11 modes $\mu = 1$ (Fig. 2a). The overall directivity is well retrieved (curve labelled "$\mu = 1$"). The three modes with $\mu = 1$, (0,1), (1,1), and (10,1) are found as expected. Mode (1,2) is replaced by (3,1) and (1,3) by (6,1) which nearly have the same directivities. Mode (5,1) of low level is also predicted, and it improves the accuracy around 35°. It has been already pointed out that solution is not unique because several modes may generate the same directivity pattern. *The second fitting* uses the 18 modes $\mu = 1$ and 2 (not shown). The overall directivity is again very well retrieved, along with the modes (0,1), (1,1), (10,1), and (1,2). Mode (6,1) replaces (1,3) as before. Two more modes of low levels are also found, (2,1) and (8,1).

The system of equations is over-determined in these two cases, and the two methods, "pinv" or "\", give the same results. This is no longer true at the beginning of the iterations in the following cases. The pseudo-inverse function leads to the exact solution. Only inversion using the backslash operator is thus discussed below.

The third fitting uses the 22 modes $\mu = 1$, 2, and 3 (Fig. 2b). There are now more unknowns ($N = 22$) than equations ($J = 21$) for the first iteration. The result looks like the previous one with the modes (0,1), (1,1), (10,1), and (1,2). There is no mode $\mu = 3$, mode (1,3) is not predicted and is replaced by (3,2) instead of (6,1). Modes of orders (2,1) and (8,1) are changed into (4,2) which is high between 45° and 70°. *The fourth fitting* uses the 24 propagating modes (Fig. 2c). The result is nearly the same as above, again without mode (1,3). There is only another extra mode (0,2), but its amplitude is very small, more than 10 dB below the overall level.

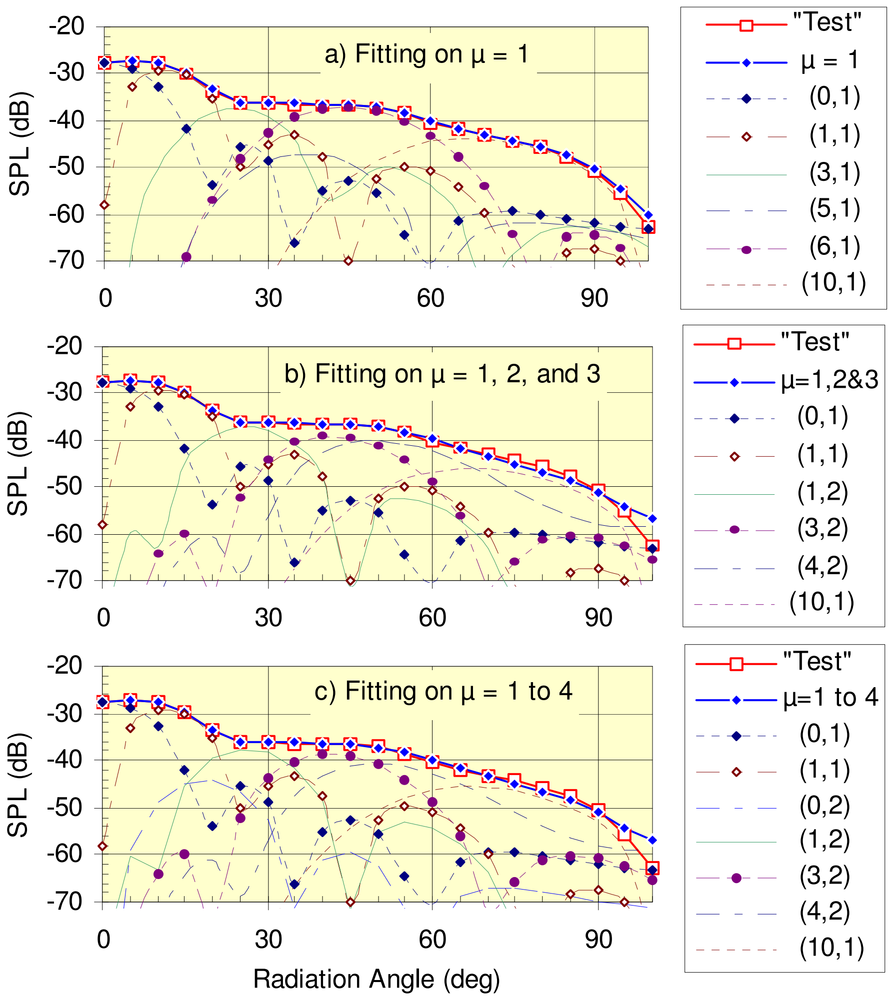

Figure 2 – Iterative backslash fitting on "test" data from Fig. 1

The last fitting based on all the propagating modes ($N = 24$) is replayed assuming now that the spatial resolution of the measurements is $\Delta\varphi = 2.5°$ (Fig. 3), i.e., the number of test data is doubled ($J = 41$). The system of equations is over-determined like in the

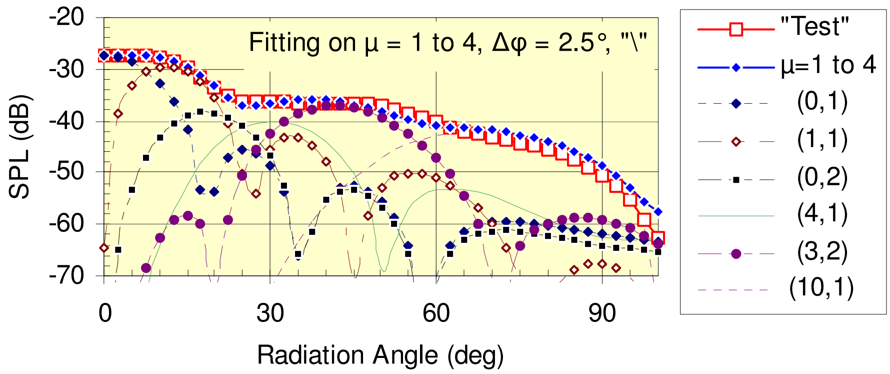

Figure 3 – Iterative backslash fitting starting from modes μ = 1 to 4, $\Delta\varphi = 2.5°$

first two previous cases, but the two methods do not give the same solution. The result using the backslash operator, plotted in Fig. 3, is not improved compared to Fig. 2c. One could think that the pseudo-inverse function would give a similar result, but this is false, and prediction (not shown) is rather poor above 70°.

COMMENTS ON THE NUMERICAL SIMULATIONS

Table 1 summarizes the modal amplitudes for the simulated test and for the four fittings with $\Delta\varphi = 5°$ described above. The second line is the cut-on ratio k_t/K. It increases from left to right, and gives an idea of the angle of maximum radiation, $\varphi_{\max}$, for each mode [3]: $\sin\varphi_{\max} \approx k_t/K$. Initial modes or those replacing them are in grey boxes. Amplitudes in brackets refer to non-useful modes, i.e., their maximum is about 10 dB lower than the overall computed level at the same angle.

Table 1 – Amplitudes $A_{m\mu}$ for the simulated test (backslash operator)

Mode	(0,1)	(1,1)	(2,1)	(0,2)	(3,1)	(1,2)	(5,1)	(6,1)	(3,2)	(1,3)	(4,2)	(8,1)	(10,1)
k_t/K	0.000	0.154	0.255	0.320	0.351	0.445	0.536	0.626	0.669	0.713	0.775	0.806	0.983
"Test"	0.30	1.00				1.00				2.00			8.00
μ=1	0.30	0.99			1.03		(0.63)	2.62					7.81
μ=1,2	0.30	0.99	(0.34)			0.88		2.64				(1.20)	6.88
1,2,3	0.30	1.00				0.97			1.47		1.81		6.03
1 to 4	0.30	0.99		(0.29)		0.91			1.54		1.72		6.36

Accuracy of the solution of the system of equations is related to the condition number of the matrix Φ. It is defined for square matrices as the product of norms $\|\Phi\|\,\|\Phi^{-1}\|$. It is given for the initial matrix Φ in Table 2 for the predictions of the previous section. Matrix Φ is badly conditioned (very large condition numbers) if there are several radial modes μ. This is explained in Fig. 4 which shows that some modes m for different values of μ have similar Φ_{nj} elements (j = 1 to J). For instance, the main directivity lobes of modes (1,3), (6,1), and (3,2) are very close together as it has been noted in the comments of the various simulations (see Table 1).

Table 2 – Condition number of the initial matrix Φ and number of iterations for the backslash operator

Modes	Number of modes	Condition number of initial matrix Φ	Number of iterations for "\" operator
μ = 1	11	5.6×10^{5}	3
μ = 1 and 2	18	2.2×10^{12}	4
μ = 1, 2, and 3	22	8.9×10^{15}	4
μ = 1 to 4	24	7.2×10^{15}	4
μ = 1 to 4, $\Delta\varphi = 2.5°$	24	3.5×10^{16}	5

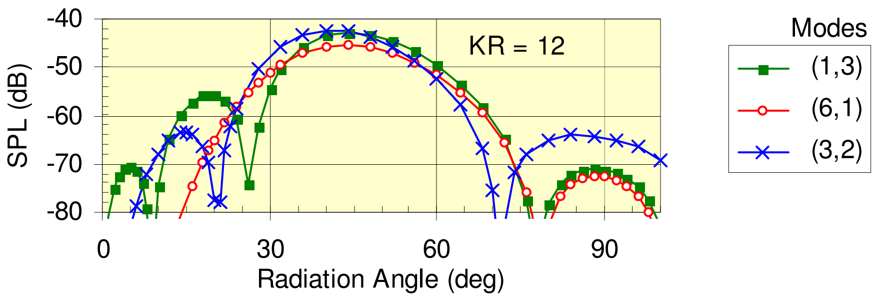

Figure 4 – Comparison of directivity patterns radiated by the modes (1,3), (6,1), and (3,2), the three amplitudes $A_{m\mu}$ being equal to 1

The condition number decreases during the iterative process because there are less and less unknowns as is shown in Table 3 in the case μ = 1 to 3 (Fig. 2b). It is only 5.3×10^6 in the second iteration using the backslash operator (on 13 modes instead of 22), and its final value is 2.4×10^4, nearly equal to the value found in the pseudo-inverse method.

Table 3 – Change of the condition number of Φ during the iterations: Case μ = 1 to 3

Pseudo-inverse method "pinv"			Backslash operator "\"		
Iteration	Number of modes	Condition number	Iteration	Number of modes	Condition number
1	22	8.9×10^{15}	1	22	8.9×10^{15}
2	15	4.8×10^{9}	2	13	5.3×10^{6}
3	8	4.2×10^{4}	3	8	2.5×10^{5}
4	5	1.8×10^{4}	4	6	2.4×10^{4}

Some comments can be drawn from these numerical simulations.

a) It was not sure that all the fittings would give a valid set of modes with positive squared amplitudes. It can also be expected that the strategy eliminating modes whose squared amplitude is negative is not always valid. A mode could recover a positive squared amplitude in a following iteration.

b) Two methods have been tested to solve the system of equations. They do not give the same result as expected, but it cannot be said that one is better than the other. A refinement could consist in implementing both methods, one for the under-determined system at the beginning of the iterations, the other when the system becomes over-determined after removal of some modes due to the constraint on positive squared amplitudes.

c) Several modes can have the same free-field directivity, and the prediction depends on the initial set of selected modes. However, most of the modes actually generated are generally found, and with a good amplitude. More test data may slightly improve the fitting, but the condition number of the system matrix can become worse.

d) Final result is not improved (and can even be worse) if a very large number of propagating modes is taken to start the iterative process. This means that a limited choice of the possible generated modes based on known physical properties helps us to get valid predictions.

CONCLUSIONS

Present study extends a previous article to estimate the modal structure and amplitude generating a measured free-field directivity, without requiring expensive experimental modal analyses. The proposed model is based on analytical equations describing ducted propagation, and on a Kirchhoff (or a Rayleigh) integral for far-field radiation. Calculations are split in two steps. Firstly, sound pressures of all the propagating modes at a given frequency are computed at the measurement locations, assuming a unit amplitude for each mode. Secondly, the system of equations with the measured data in the second member has to be solved to find the unknown actual modal amplitudes. A least squares fitting is the best way to get the expected result. Two methods have been tested, using either the pseudo-inverse function or the backslash operator of MATLAB®. An iterative process is required to satisfy the constraint that the squared modal amplitudes must be positive. The system of equations is usually underdetermined (more unknowns than measurements) at the beginning, and becomes over-determined during the iterations because the modes which do not meet the constraint are eliminated.

Numerical simulations prove that several solutions can well fit the test data because some spinning modes can generate similar free-field directivity patterns. This means that the initial set of modes used in the calculations must be carefully chosen, according to any *a priori* knowledge of the physical mechanisms generating the acoustic sources, e.g., rotor-stator interactions. An interesting result is that only a few iterations are necessary, such that computation is very fast and can be made on-line. Predictions accurately fit the test directivities, standard deviations are less than 1 dB.

Future improvements should solve two issues: (i) It has been suspected that the strategy consisting in eliminating the modes whose squared amplitude is negative is not always valid because some of them could recover a positive squared amplitude in following iterations; (ii) The two methods from MATLAB® do not give the same results (as expected), but it is not clear which is better, and they could be mixed during the iterations.

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