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ROTATING SHAFT ANALYTICAL RESPONSE IN ADIMENSIONAL FORM

Juan Fayos, Andrés Rovira*, Luis Baeza and Javier Carballeira

CITV - Dpto. Ingeniería Mecánica y Materiales Universidad Politécnica de Valencia
Camino de Vera, 14, E-46022 Valencia, Spain
juafasan@dimm.upv.es (e-mail address of lead author)

Abstract

With the aim of characterising the vibratory response of rotating shafts, a preliminary model is adopted from which the expressions of the free and forced responses are reformulated. By means of selecting a small number of adimensional parameters obtained from the system, the dynamic response have been characterised, obtaining the orbits described by a point of the beam in its free response (hypotrochoid) or the expressions of the frequency response functions.

INTRODUCTION

In some mechanical problems it is sometimes uncertain which hypothesis should be considered in modelling systems that contain rotating shafts. For example, in the case of railway dynamics the diversity in the wheelset model ranges from those in which shaft flexion and rotation are not taken into account, to those which consider flexion but not rotation and those which implement both effects [1]. In any case, the model limitations cannot be established from the conclusions associated with advanced applications nor from those based on Beam Theory presented in the bibliography.

Although there are many models of beams that rotate around a perpendicular axis (rotating blade problem) and of rotating discs that vibrate in a direction parallel to their own axis (disc brakes), works relating to beams that rotate around their own axis are less frequent. Brown and Shabana in [2] developed a refined model of this problem which included a general method in which, on assuming the beam modes, the formulation of the dynamic response of the system is obtained.

A more advanced version was recently presented by Sheu and Yang in [3]. In this Reference a methodology for obtaining analytically the modal properties of the Rayleigh beam was proposed. The number of parameters that defines the problem was reduced using a set of adimensional ones. Thanks to this advantage, the present study is based on Sheu and Yang's model with the aim of establishing the characteristics of the free and forced response of this type of system as a function of a more reduced set of parameters.

DYNAMIC MODEL OF THE ROTATING RAYLEIGH BEAM

The dynamic model is based on a general approach on the dynamics of Rayleigh beams, adapted for the case in which the beam turns around its own axis. The equation of motion obtained from [3] is

$$\rho\, A\, \ddot{\mathbf{U}} - \rho\, I\, \frac{\partial^2}{\partial y^2}\ddot{\mathbf{U}} + 2\rho\,\omega\, I\, \mathbf{J}\, \frac{\partial^2}{\partial y^2}\dot{\mathbf{U}} + EI\frac{\partial^4}{\partial y^4}\mathbf{U} = \mathbf{F} \tag{1}$$

where $\mathbf{U}(y,t)$ is the transversal displacement at the centre of the section and $\mathbf{F}(y,t)$ is the external forces per unit of beam length; both vectors are expressed according to two orthogonal directions to the beam axis, so that

$$\mathbf{U}(y,t) = \left\{ \begin{array}{c} U_v(y,t) \\ U_w(y,t) \end{array} \right\} \tag{2}$$

$\mathbf{J}$ is the antisymmetric C-matrix of order 2

$$\mathbf{J} = \left[\begin{array}{rr} 0 & 1 \\ -1 & 0 \end{array} \right] \tag{3}$$

ρ is the volumetric density of the material, A is the area of the section, I is the second moment of the beam section area, E is the Young's modulus of the material and ω the angular velocity of the beam.

The solution to equation (3) can be expressed as

$$\mathbf{U}(y,t) = \sum_{n=0}^{\infty} \mathbf{S}_n(y)\, \mathbf{q}_n(t) \tag{4}$$

For the n-th term, the equation (3) can be expressed (omitting sub-index n) as:

$$\mathbf{M}\,\ddot{\mathbf{q}} + \omega\,\mathbf{D}\,\dot{\mathbf{q}} + \mathbf{K}\,\mathbf{q} = \mathbf{Q} \tag{5}$$

in which:

$$\mathbf{q} = \left\{ \begin{array}{cc} q_u & q_w \end{array} \right\}^T \tag{6}$$

$$\mathbf{M} = \rho \left[A \int_L \mathbf{S}^T \mathbf{S} dy - I \int_L \mathbf{S}^T \frac{\partial^2}{\partial y^2}\mathbf{S} dy \right] \tag{7}$$

$$\mathbf{D} = 2\rho I \int_L \mathbf{S}^T \mathbf{J} \left(\frac{\partial^2}{\partial y^2}\mathbf{S} \right) dy \tag{8}$$

$$\mathbf{K} = EI \int_L \mathbf{S}^T \left(\frac{\partial^4}{\partial y^4} \mathbf{S} \right) dy \tag{9}$$

$$\mathbf{Q} = \int_L \mathbf{S}^T \mathbf{F} dy \tag{10}$$

For the case of a pinned-pinned beam, the modal form in (4) is

$$\mathbf{S}_n(y) = \mathbf{I}\ \sin{(n\pi y/L)} \tag{11}$$

being $\mathbf{I}$ the 2×2 identity matrix. On solving the integrals that appear in (5) and defining

$$m = \frac{1}{2}\rho \left[L\,A + I\,\frac{(n\,\pi)^2}{L} \right] \tag{12}$$

$$k = E\,I\,\frac{(n\,\pi)^4}{2\,L^3} \tag{13}$$

$$\omega_n = \sqrt{k/m} \tag{14}$$

$$d = \frac{(n\,\pi)^2}{2\left(\lambda^2 + (n\,\pi)^2\right)} \tag{15}$$

where λ is the beam slenderness, expression (5) is derived to

$$\ddot{\mathbf{q}} + 2\,d\,\omega\,\mathbf{J}\,\dot{\mathbf{q}} + \omega_n^2\,\mathbf{q} = \mathbf{Q} \tag{16}$$

FREE RESPONSE

The solution for the free response of the system ($\mathbf{Q} = \mathbf{0}$) is

$$\mathbf{q}(t) = 2 \begin{pmatrix} |Z_1| \cos{((\omega_n - d\omega)\,t + \arg Z_1)} + |Z_2| \cos{((\omega_n + d\omega)\,t + \arg Z_2)} \\ -|Z_1| \sin{((\omega_n - d\omega)\,t + \arg Z_1)} + |Z_2| \sin{((\omega_n + d\omega)\,t + \arg Z_2)} \end{pmatrix} \tag{17}$$

where

$$Z_1 = \frac{1}{4\omega_n} \left[(\omega_n + d\omega)\,q_{u0} - \dot{q}_{w0} - i\left((\omega_n + d\omega)\,q_{w0} + \dot{q}_{u0}\right) \right] \tag{18}$$

$$Z_2 = \frac{1}{4\omega_n} \left[(\omega_n - d\omega)\,q_{u0} + \dot{q}_{w0} + i\left((\omega_n - d\omega)\,q_{w0} - \dot{q}_{u0}\right) \right] \tag{19}$$

q_{u0}, q_{w0}, $\dot{q}_{u0}$ and $\dot{q}_{w0}$ are the initial conditions of the n-th mode and i is the imaginary unit. Let define C as follows

$$C = d\frac{\omega}{\omega_n} = \frac{(n\pi)^2}{2\left(\lambda^2 + (n\pi)^2\right)}\frac{\omega}{\omega_n} \tag{20}$$

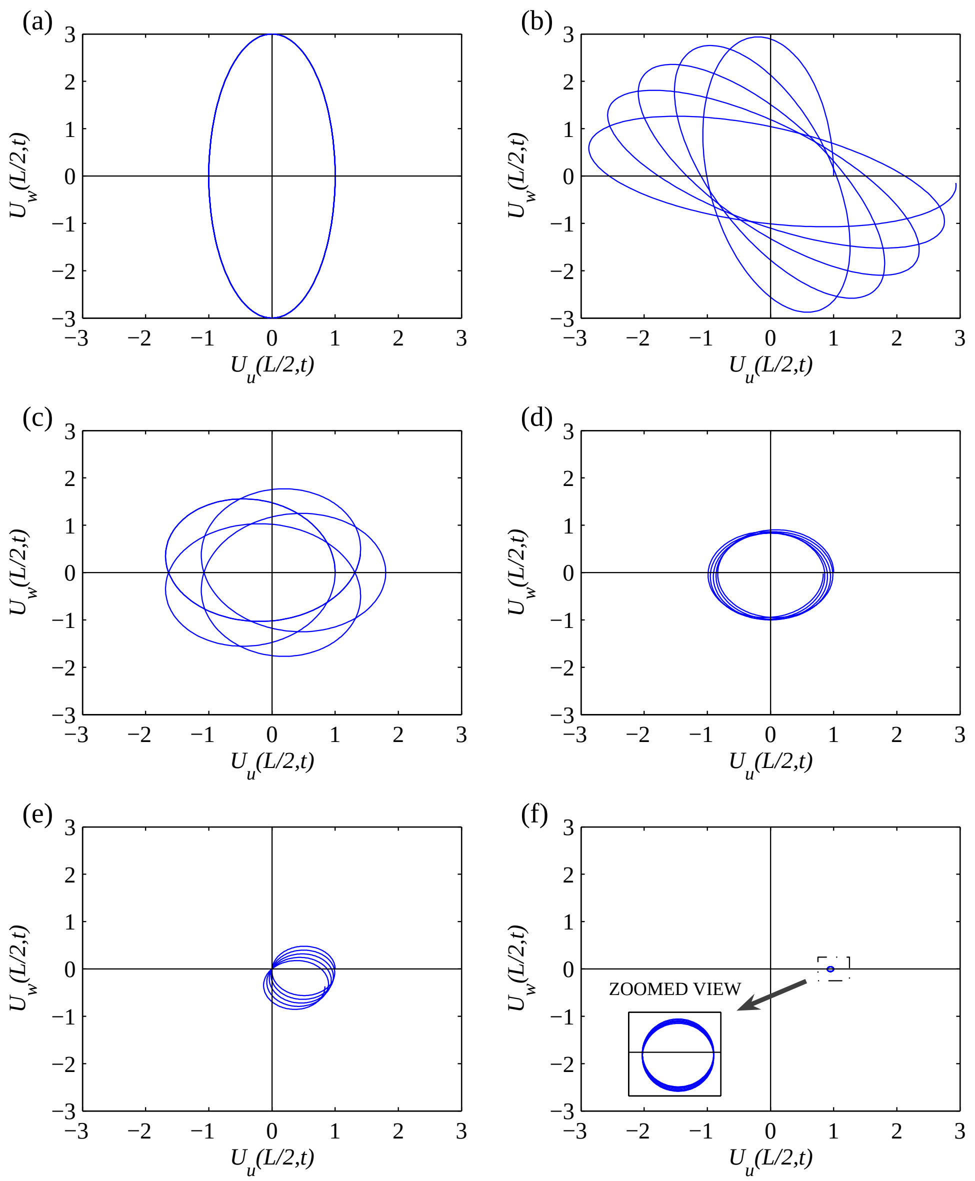

Figure 1: Orbits described by the centre point of the beam for different values of C: (a) C=0, (b) C= 0.05, (c) C= 0.75, (d) C=1.5, (e) C=3, (f) C=30.

It should be observed that this constant only depends on the slenderness and the natural frequency of the non-rotating beam to the rotating speed ratio. Transforming the initial modal velocity vector and the time so that

$$\mathbf{p}_0 = \omega_n^{-1}\ \dot{\mathbf{q}}_0 \tag{21}$$

$$s = \omega_n \left(C + \sqrt{C^2+1} \right) t \tag{22}$$

and defining the following parameters which depend only on the initial conditions and on the constant defined by (20)

$$a = \sqrt{\left(q_{u0} + \frac{p_{w0}}{\sqrt{C^2+1} - C} \right)^2 + \left(q_{w0} - \frac{p_{u0}}{\sqrt{C^2+1} - C} \right)^2} \tag{23}$$

$$b = \left(\frac{C}{2\sqrt{C^2+1}} + \frac{1}{2} \right) \sqrt{\left(q_{u0} + \frac{p_{w0}}{\sqrt{C^2+1} - C} \right)^2 + \left(q_{w0} - \frac{p_{u0}}{\sqrt{C^2+1} - C} \right)^2} \tag{24}$$

$$c = \left(\frac{C}{2\sqrt{C^2+1}} + \frac{1}{2} \right) \sqrt{\left(q_{u0} - \frac{p_{w0}}{C + \sqrt{C^2+1}} \right)^2 + \left(q_{w0} + \frac{p_{u0}}{C + \sqrt{C^2+1}} \right)^2} \tag{25}$$

$$\phi_0 = -\arg\left(\left(C + \sqrt{C^2+1} \right) q_{u0} - p_{w0} - i \left(\left(C + \sqrt{C^2+1} \right) q_{w0} + p_{u0} \right) \right) \tag{26}$$

$$s_0 = -\arg\left(\left(-C + \sqrt{C^2+1} \right) q_{u0} + p_{w0} + i \left(\left(-C + \sqrt{C^2+1} \right) q_{w0} - p_{u0} \right) \right) \tag{27}$$

equation (16) can be written as

$$\mathbf{q}(s) = \begin{pmatrix} (a-b)\cos(s-s_0) + c\cos\left(\frac{a-b}{b}s - \phi_0\right) \\ (a-b)\sin(s-s_0) - c\sin\left(\frac{a-b}{b}s - \phi_0\right) \end{pmatrix} \tag{28}$$

This solution describes the orbits of the beam section centre, which corresponds to the parametric equation of a hypotrochoid curve. This curve is the trajectory described by a point P rigidly attached at a distance c from the centre of a moving circle of radius b which rolls without slip in the interior of a fixed circumference of radius a (with $a > b$). ϕ_0 is the angle that forms the radius of the moving circle that passes through point P with the horizontal axis when the parameter $s = s_0$. Note that all the systems with the same value of the constant C have the same response to the same initial conditions, when the time and initial velocity conditions have been transformed.

Figure (1) shows the orbits of the centre of the beam section at $y = L/2$ for different values of the characteristic constant C corresponding to the n-th mode, under the following initial conditions

$$\mathbf{q}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{p}_0 = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{29}$$

As can be seen in Figure (1.b), for small C values the response is similar to that of the non-rotating beam, but with the vibration plane turning. When the value of C is increased (Figure (1.c), (1.d) and (1.e)), the orbit increases its curve radius, evolving towards a circumference not centred on the beam axis (Figure (1.f)) which changes its position very slowly.

HARMONIC FORCED RESPONSE

In the case of harmonic excitation, from the form $\mathbf{F}(y,t) = \mathbf{F}_0(y)\cos(\omega_e t)$ is obtained $\mathbf{Q}(t) = \mathbf{Q}_0 \cos \omega_e t$, with $\mathbf{Q}_0 = \int_L \mathbf{S}(y)^T \mathbf{F}_0(y)\, dy$. In the steady state response the solution to (5) is

$$\mathbf{q}(t) = \frac{1}{(\omega_n^2 - \omega_e^2)^2 - (\omega_e 2\, C\, \omega_n)^2} \;\mathrm{Re}\left(\begin{pmatrix} \omega_n^2 - \omega_e^2 & -i\, \omega_e 2\, C\, \omega_n \\ i\, \omega_e 2\, C\, \omega_n & \omega_n^2 - \omega_e^2 \end{pmatrix} \mathbf{Q}_0 \, \mathrm{e}^{i\omega_e t} \right) \qquad (30)$$

It can be seen that the response between planes appears coupled through the cross term $\omega_e 2\, C\, \omega_n$. Deriving (30) twice gives the generalised acceleration. Particularising for an excitation of unitary amplitude in one of the axes, the modal inertances in the excitation axis A_D (direct response) and in the transversal axis A_C (cross response) are provided. Let ξ be the excitation frequency to natural frequency ratio ω_e/ω_n. The amplitudes of the inertances are then

$$A_D(\xi) = \left| \frac{\xi^2\left(1-\xi^2\right)}{(1-\xi^2)^2 - (2\, C\, \xi)^2} \right| \qquad (31)$$

$$A_C(\xi) = \left| \frac{2\, C\, \xi^3}{(1-\xi^2)^2 - (2\, C\, \xi)^2} \right| \qquad (32)$$

Figure (2) shows the graphs of these amplitudes as a function of ξ in the range [0,7] for values of C of 0, 0.05, 0.75, 1.5 and 3. For $C > 0$ we can see that the direct inertance A_D presents an antiresonance for $\xi = 1$ and consequently the cross inertance is bigger. Can be easily proved that the crossed response is greater than the direct one if the next condition is satisfied

$$\sqrt{C^2+1} - 1 < \xi < \sqrt{C^2+1} + 1 \qquad (33)$$

CONCLUSIONS

This work deals with the analysis of the vibratory response of a Rayleigh beam that rotates around its own principal axis, considering the free and forced response to harmonic excitation. In both cases a characteristic constant C is identified which depends on the beam slenderness and the angular velocity to the natural frequency ratio. In the free response the trajectories of a free point of the beam correspond to a hypotrochoid curve whose basic characteristics depend exclusively on the characteristic parameter C.

The analysis of the inertances shows a direct relationship between these functions and the C parameter. For each vibration mode the inertance amplitude has two resonances whose relative position depends on C. An analysis of the inertances lets us to conclude the existence of a significant coupling between crossed plane movements in such a way that the response of the cross FRF may be even greater than the direct response.

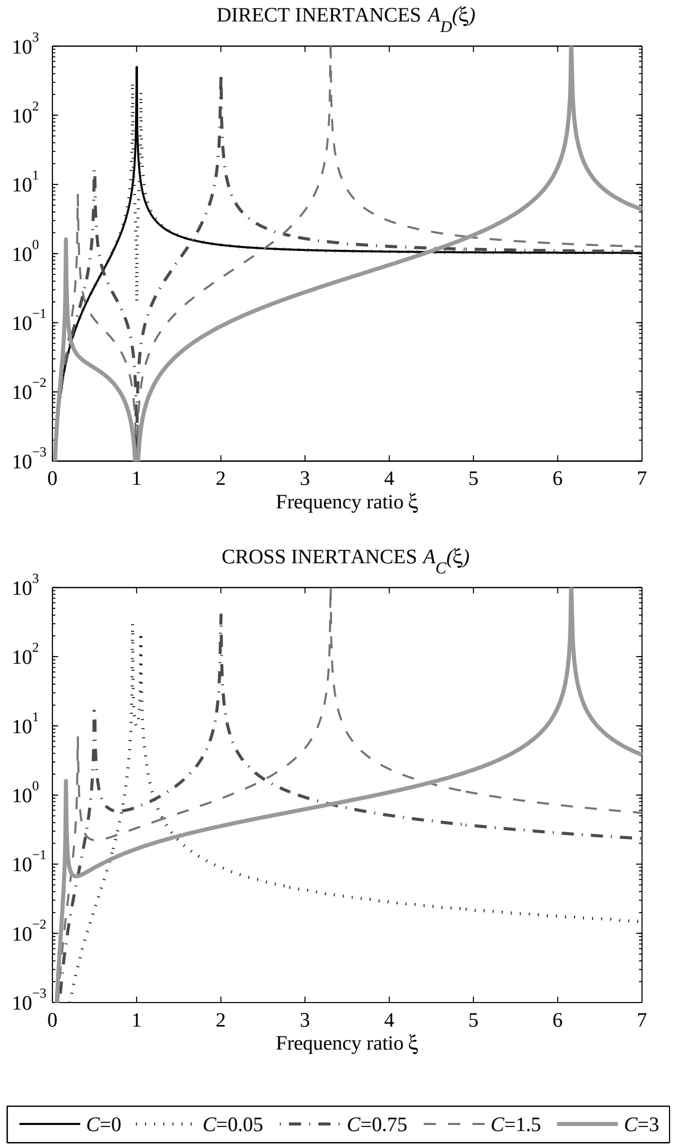

Figure 2: Direct and cross modal inertances

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