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DYNAMIC RESPONSE OF BEAMS WITH A FLEXIBLE SUPPORT UNDER A MOVING LOAD

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Abstract

This paper deals with the linear dynamic responses of beams with a flexible support under a moving load with a constant speed. The entire system is modeled as a two-span beam and each span of the continuous beams is assumed to obey the Euler-Bernoulli beam theory. Considering the compatibility requirements on the flexible constraint, the relationships between two segments can be obtained. By using a transfer matrix method, the characteristic equation of the system can then be determined. The forced responses of the system under a moving load can then be obtained through modal expansion theory. Some numerical results are also presented.

1. INTRODUCTION

The dynamic responses of beam structures subjected to moving loads or masses have been studied extensively. There are numerous references available in the monographs of Fryba [1] and most of the cases treat a uniform simply supported beam of a single span. The earliest work on the behavior of a single span beam subjected to a constant moving load was reported by Timoshenko [2]. Subsequent studies considering the effects of an elastic foundation, moving masses, etc. Cai *et al.* [3] investigated the dynamic interactions between the vehicle and guideway of magnetically levitated vehicles by modeling the vehicle as a moving force and as a two-degree-of-freedom model.

There are not so many studies on the dynamic analysis of a multi-span continuous beam subjected to moving loads or masses. Lee [4] analyzed the transverse vibration of a beam with intermediate point constraints subjected to a moving load by the assumed mode method. Wang [5] investigated the response of multi-span Timoshenko beams. Yang *et al.* [6] presented impact formulas for vehicles over continuous beams. Chatterjee *et al.* [7] investigated the dynamic behavior of multi-span continuous bridges under a moving vehicular load which was modeled as a sprung mass. Lin *et al.* [8] analyzed the dynamics of beams with multiple intermediate supports.

This study presents an analytical method that permits an efficient computation of the eigensolutions for beams with an intermediate flexible support. The method is based on the use of the Euler-Bernoulli beam theory in each span, and by the compatibility requirements at the support, the relationships of the four integration constants of the eigenfunctions between two spans of the system can be determined [8, 10-11]. After the eigensolutions are obtained, the forced responses of the entire system under a moving load can then be obtained through modal expansion theory.

2. THEORETICAL MODEL

An Euler-Bernoulli beam of length L and with an intermediate flexible support and a moving load F_1 with constant speed V is considered as in Fig.1.

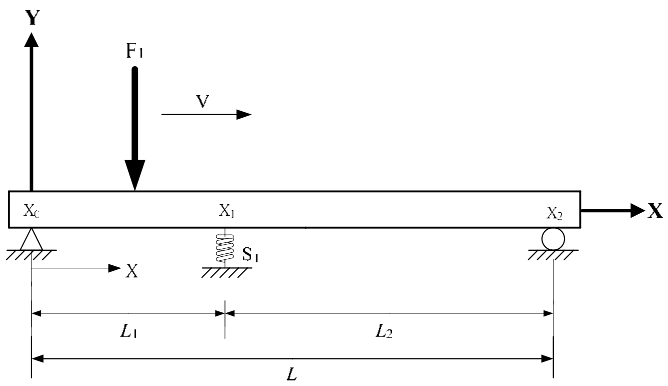

Figure 1: A beam with a flexible intermediate support with stiffness S_1 located at position X_1 and the lengths of sub-sections are L_1 and L_2 where $L_1 + L_2 = L$. The load F_1 moves with constant speed V.

It is assumed that the support is located at point X_1 and with stiffness S_1 and X_0 and X_2 represent end points. The vibration amplitude of the transverse displacement of the beam is denoted by $Y(X,T)$. By using the Euler-Bernoulli beam theory [9], the equation of motion, assumed to have a uniform cross section, is:

$$EI\frac{\partial^4 Y(X,T)}{\partial X^4} + \rho A\frac{\partial^2 Y(X,T)}{\partial T^2} = F_1\,\delta(X - VT), \tag{1}$$

where E is Young's modulus of the material, I is the moment of inertia of the beam cross-section, ρ is the density of material, A is the cross-section area of the beam, $\delta(X - VT)$ denotes the Dirac delta distribution and T is time. The boundary conditions of the beam for a simply-supported case are:

$$Y(0,T) = Y(L,T) = 0, \tag{2a}$$
$$Y''(0,T) = Y''(L,T) = 0, \tag{2b}$$

where the symbol $(\,'\,)$ denotes the derivative with respect to the space coordinate *X*.

The "compatibility conditions" enforce continuities of the displacement field, the slope, the bending moment and the shear force, respectively, across the support and can be expressed:

$$Y_{(1)}(X_1^-,T) = Y_{(2)}(X_1^+,T)\,, \tag{3a}$$
$$Y'_{(1)}(X_1^-,T) = Y'_{(2)}(X_1^+,T)\,, \tag{3b}$$
$$Y''_{(1)}(X_1^-,T) = Y''_{(2)}(X_1^+,T)\,, \tag{3c}$$
$$Y'''_{(1)}(X_1^-,T) = Y'''_{(2)}(X_1^+,T) + \frac{S_1}{EI} Y_{(1)}(X_1^-,T)\,, \tag{3d}$$

where the symbols X_1^+ and X_1^- denote the locations immediately above and below the support position X_1 and the sub-index in the parenthesis represents the segments (sub-beams) of the system.

In the above, the following quantities are introduced:

$$y = \frac{Y}{L}\,,\ x = \frac{X}{L}\,,\ x_i = \frac{X_i}{L}\,,\ l_1 = \frac{L_1}{L}\,,\ l_2 = \frac{L_2}{L}\,,\ t = \frac{T}{\sqrt{L}}\,,\ v = \frac{V}{\sqrt{L}}\,. \tag{4a-4g}$$

Thus, the governing Eq. (1) and the non-dimensional "compatibility conditions" Eqs. (3a~3d) can then be expressed as:

$$\frac{\partial^4 y(x,t)}{\partial x^4} + \frac{\rho A L^3}{EI}\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{F_1 L^3}{EI}\delta(x - vt)\,. \tag{5}$$
$$y_{(1)}(x_1^-,t) = y_{(2)}(x_1^+,t)\,, \tag{6a}$$
$$y'_{(1)}(x_1^-,t) = y'_{(2)}(x_2^+,t)\,, \tag{6b}$$
$$y''_{(1)}(x_1^-,t) = y''_{(2)}(x_1^+,t)\,, \tag{6c}$$
$$y'''_{(1)}(x_1^-,t) = y'''_{(2)}(x_1^+,t) + s_1\, y_{(1)}(x_1^-,t)\,, \tag{6d}$$

where $s_1 = \dfrac{S_1 L^3}{EI}$ is the non-dimensional support stiffness.

3. FREE RESPONSE

Following the procedures in [8] and using the separable solutions: $y_{(i)}(x,t)$ = $w_{(i)}(x)\ e^{j\omega t}$ in Eq. (5) lead to the associated eigenvalue problem:

$$w''''_{(i)}\ (x)\ -\ \lambda^4\ w_{(i)}(x)\ = 0, \qquad x_{i-1} < x < x_i\,,\ i = 1, 2 \tag{7a}$$

where $\lambda^4 = \dfrac{\rho A L^3 \omega^2}{EI}$. (7b)

The corresponding compatibility conditions across the flexible support lead to:

$$w_{(1)}(x_1^-)\ =\ w_{(2)}(x_1^+)\,, \tag{8a}$$

$$w'_{(1)}(x_1^-)\ =\ w'_{(2)}(x_1^+)\,, \tag{8b}$$

$$w''_{(1)}(x_1^-)\ =\ w''_{(2)}(x_1^+)\,, \tag{8c}$$

$$w'''_{(1)}(x_1^-)\ =\ w'''_{(2)}(x_1^+)\ +\ s_1\ w_{(1)}(x_1^-)\,. \tag{8d}$$

The general solution of Eq. (7a), for each segment, is:

$$w_{(i)}(x) = A_i\ \sin\lambda(x - x_{i-1}) + B_i\ \cos\lambda(x - x_{i-1}) + C_i\ \sinh\lambda(x - x_{i-1}) + D_i\ \cosh\lambda(x - x_{i-1})$$
$$x_{i-1} < x < x_i\,, \qquad i = 1,2 \tag{9}$$

where $A_i\,, B_i\,, C_i$ and D_i are constants associated with the i-th segment ($i = 1,2$). These constants in the second segment ($A_2\,, B_2\,, C_2$ and D_2) are related to those in the first segment ($A_1\,, B_1\,, C_1$ and D_1) through the compatibility conditions in Eqs. (8a~8d) and can be expressed as:

$$\begin{Bmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{Bmatrix} = \begin{bmatrix} t_{11}\ t_{12}\ t_{13}\ t_{14} \\ \vdots \\ \\ \ldots\ldots\ldots\ldots.t_{44} \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{Bmatrix} = \underline{T}_{4\times4} \begin{Bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{Bmatrix}, \tag{10}$$

where $\underline{T}_{4\times4}$ is the 4×4 transfer matrix which depends on the eigenvalue λ and the elements are derived as:

$$t_{11} = \cos\lambda l_1 + \frac{1}{2}\,\frac{s_1}{\lambda^3}\ \sin\lambda l_1\,,\quad t_{12} = -\ \sin\lambda l_1 + \frac{1}{2}\,\frac{s_1}{\lambda^3}\ \cos\lambda l_1\,,\quad t_{13} = \frac{1}{2}\,\frac{s_1}{\lambda^3}\ \sinh\lambda l_1\,,$$

$$t_{14} = \frac{1}{2}\,\frac{s_1}{\lambda^3}\ \cosh\lambda l_1\,,\quad t_{21} = \sin\lambda l_1\,,\quad t_{22} = \cos\lambda l_1\,,\quad t_{23} = 0,\quad t_{24} = 0,\quad t_{31} = -\ \frac{1}{2}\,\frac{s_1}{\lambda^3}\ \sin\lambda l_1$$

$$t_{32} = -\ \frac{1}{2}\,\frac{s_1}{\lambda^3}\ \cos\lambda l_1\,,\quad t_{33} = \cosh\lambda l_1 - \frac{1}{2}\,\frac{s_1}{\lambda^3}\ \sinh\lambda l_1\,,\quad t_{34} = \sinh\lambda l_1\ \frac{1}{2}\,\frac{s_1}{\lambda^3}\ \cosh\lambda l_1\,,$$

$$t_{41} = 0,\quad t_{42} = 0,\quad t_{43} = \sinh\lambda l_1\,,\quad t_{44} = \cosh\lambda l_1\,. \tag{11a~11p}$$

For the case of a simply-supported beam, after substituting the corresponding boundary conditions of Eqs. (2a) and (2b), the following result can be obtained:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} \sin\lambda l_2 & \cos\lambda l_2 & \sinh\lambda l_2 & \cosh\lambda l_2 \\ -\sin\lambda l_2 & -\cos\lambda l_2 & \sinh\lambda l_2 & \cosh\lambda l_2 \end{bmatrix} \begin{Bmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{Bmatrix} = \underline{B}_{2\times4} \begin{Bmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{Bmatrix} \tag{12a}$$

where $\underline{B}_{2\times4} = \begin{bmatrix} \sin\lambda l_2 & \cos\lambda l_2 & \sinh\lambda l_2 & \cosh\lambda l_2 \\ -\sin\lambda l_2 & -\cos\lambda l_2 & \sinh\lambda l_2 & \cosh\lambda l_2 \end{bmatrix}$. (12b)

Substitution of Eq. (10) into Eq. (12a) leads to

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \underline{B}_{2\times4} \begin{Bmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{Bmatrix} = \underline{B}_{2\times4}\, \underline{T}_{4\times4} \begin{Bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{Bmatrix} = \underline{R}_{2\times4} \begin{Bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{Bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \end{bmatrix} \begin{Bmatrix} A_1 \\ 0 \\ C_1 \\ 0 \end{Bmatrix}, \tag{13a}$$

where $\underline{R}_{2\times4} = \underline{B}_{2\times4}\, \underline{T}_{4\times4} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \end{bmatrix}$. (13b)

Thus, the characteristic equation of this system can be obtained from the existence of non-trivial solutions of Eq. (13a) and can be expressed explicitly as:

$$s_1\,[\sin\lambda(1-l_1)\sin\lambda l_1 \sinh\lambda - \sin\lambda \sinh\lambda(1-l_1)\sinh\lambda l_1)] + 2\lambda^3 \sin\lambda\ \sinh\lambda = 0. \tag{14}$$

The coefficients of the eigenfunctions, $w_n(x)$, are obtained by back substitution into Eqs. (13a), (10) and then Eq. (9).

4. FORCED RESPONSES

Using the modal expansion theory, the forced response $y(x,t)$ due to the moving load of the constant speed in Eq. (5) can be expressed as:

$$y(x,t) = \sum_{k=1}^{N} w_k(x) q_k(t)\,, \tag{15}$$

where $w_k(x)$ are normalized eigenfunctions of the system and which are obtained from the free response analysis (**section 3**), $q_k(t)$ are generalized coordinates and N is the number of terms used to approximate the solution.

Substituting Eq.(15) into Eq. (5), multiplying by $w_j(x)$, integrating from 0 to 1 and using the othogonality relationship of the eigenfinctions lead to

$$\ddot{q}_k(t)+\omega_k^2 q_k(t)=\frac{F_1}{\rho A}\int_0^1 w_k(x)\delta(x-vt)\,dx=\frac{F_1}{\rho A}\,w_k(vt)=Q_k(t)\,,\quad k=1,2,\ldots,N\,. \tag{16}$$

The generalized coordinate $q_k(t)$ are solved from Eq. (16) by the convolution theory:

$$q_k(t)=q_k(0)\cos\omega_k t+\frac{\dot{q}_k(0)}{\omega_k}\sin\omega_k t+\frac{1}{\omega_k}\int_0^t \sin\omega_k(t-\tau)Q_k(\tau)d\tau\,, \tag{17}$$

where $q_k(0)=\int_0^1 y_0(x)w_k(x)dx$, $\dot{q}_k(0)=\int_0^1 \dot{y}_0(x)w_k(x)dx$, $k=1,2,\ldots,N$ (18a,b)

and $y_0(x)=y(x,0)$, $\dot{y}_0(x)=\dot{y}(x,0)$ are initial conditions of the original system.

The *k*-th eigenfunctions $w_k(x)$ used in Eq. (16) are from Eq.(9) as:

$$w_k(x)=\begin{cases} f_{k1}(x)=A_{k1}\sin\lambda_k(x-x_0)+B_{k1}\cos\lambda_k(x-x_0)+C_{k1}\sinh\lambda_k(x-x_0)+D_{k1}\cosh\lambda_k(x-x_0), & x\le x_1 \\ f_{k2}(x)=A_{k2}\sin\lambda_k(x-x_1)+B_{k2}\cos\lambda_k(x-x_1)+C_{k2}\sinh\lambda_k(x-x_1)+D_{k2}\cosh\lambda_k(x-x_1), & x>x_1. \end{cases}$$

The generalized forcing term $Q_k(t)$ in Eq.(16) can be written as

$$Q_k(t) = \frac{F_1}{\rho A}w_k(vt)=\begin{cases} \dfrac{F_1}{\rho A}f_{k1}(vt), & 0<vt\le x_1 \quad (19a) \\ \dfrac{F_1}{\rho A}f_{k2}(vt), & vt>x_1. \quad (19b) \end{cases}$$

The term $\frac{1}{\omega_k}\int_0^t \sin\omega_k(t-\tau)Q_k(\tau)d\tau$ in Eq. (17) can thus be expressed as

$$\frac{1}{\omega_k}\int_0^t \sin\omega_k(t-\tau)Q_k(\tau)d\tau$$
$$=\begin{cases} \dfrac{F_1}{\rho A}\dfrac{1}{\omega_k}\displaystyle\int_0^t \sin\omega_k(t-\tau)f_{k1}(v\tau)d\tau, & 0<\tau\le\dfrac{x_1}{v}, \quad (20a) \\ \dfrac{F_1}{\rho A}\dfrac{1}{\omega_k}\Big[\displaystyle\int_0^{x_1/v}\sin\omega_k(t-\tau)f_{k1}(v\tau)d\tau+\int_{x_1/v}^t \sin\omega_k(t-\tau)f_{k2}(v\tau)d\tau\Big], & \tau>\dfrac{x_1}{v}. \quad (20b) \end{cases}$$

After the generalized coordinates $q_k(t)$ in Eq. (17) are obtained, the forced response $y(x,t)$ can then be reconstructed from Eq. (15).

5 NUMERICAL RESULTS AND DISCUSSION

In order to show the method used in this article, some numerical examples are presented. For a case of a simply-supported beam structure with an intermediate flexible support with stiffness $S_1 = 46{,}000$ N/m , the beam is square cross-section with width $B = 0.037$ m, height $H = 0.006$ m, total length $L = 1.0$ m, material density $\rho = 7860$ Kg/m^3, Young's modulus $E = 2.06 \times 10^{11}$ N/m^2.

Figure 2 shows the lowest four natural frequencies of the above system as the support position x_1 ($x_1 = \dfrac{X_1}{L}$) varies. The curves in Fig. 2 are symmetric because of the symmetry of the system, i.e., the case for $x_1 = 0.3$ is exactly the same as the case for $x_1 = 0.7$. Figure 3 represents the forced responses (mid-point response, x=0.5) of a simply-supported beam with different support positions. The non-dimensional support positions used here are $x_1 = 0.1$, 0.3, 0.5, 0.7 and 0.9. The moving load speed is $V = 0.9 \times V_{crit}$ where V_{crit} is the critical speed and is defined as $V_{crit} = v\sqrt{L} = \dfrac{\pi}{L}\sqrt{\dfrac{EI}{\rho A}} =$ 27.86 m/sec. From these results, it is observed that the maximum deflections will be smaller as the support positions close to the center of the beam.

All the above examples used are cases for beams of simply-supported boundary conditions. The solutions for other types of different boundary conditions can also be obtained using the method presented in this research.

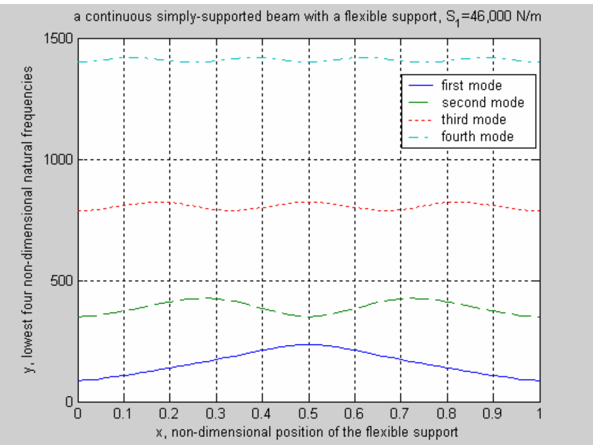

Figure 2: The lowest four non-dimensional natural frequencies of the constraint beam system with support stiffness $S_1 = 46{,}000$ N/m *as the support position* x_1 *(*$x_1 = \dfrac{X_1}{L}$*) varies.*

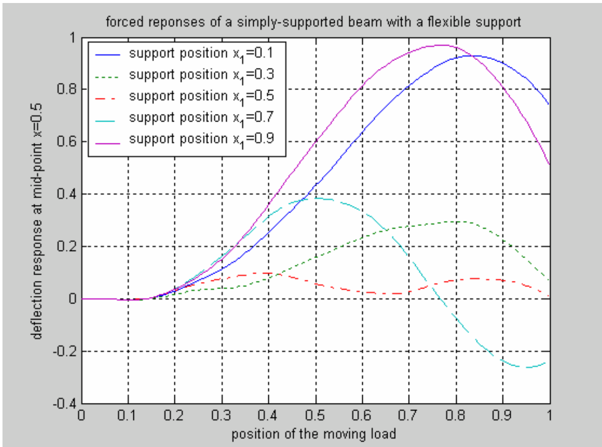

Figure 3: Forced responses (at the position $x = 0.5$*) for different support positions* $x_1 = 0.1$*, 0.3, 0.5, 0.7 and 0.9. The moving load speed is* $V = 0.9 \times V_{crit}$ *and the support stiffness is* $S_1 = 46{,}000$ *N/m.*

6. CONCLUSIONS

An analytical method is developed to present the dynamic responses of a constrained simply-supported beam subjected to a traveling load of constant speed. The constrained beam system is modeled as a two-span beam and each span of the continuous beam is assumed to obey the Euler-Bernoulli beam theory. Considering the compatibility requirements on the flexible support, the relationships between these two spans can be obtained. By using the analytical transfer matrix method, eigensolutions of this constrained system are obtained. The eigenfunctions obtained in this article are analytical solutions and forced responses can be obtained by the modal expansion of eigenfunctions. The solutions converge very rapidly. Some numerical results are also shown and are studied for different load speeds.

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