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IMPROVED EIGENVALUES FOR COMBINED DYNAMICAL SYSTEMS USING A MODIFIED FINITE ELEMENT DISCRETIZATION SCHEME

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Abstract

A new approach is presented to discretize an arbitrarily supported linear structure carrying various lumped attachments. Specifically, the exact natural frequencies and mode shapes of the linear structure are first used to modify its finite element mass and stiffness matrix so that the eigensolutions of the discretized system coincide with the exact modes of vibration. This is achieved by identifying a set of minimum changes in the finite element mass and stiffness matrices and enforcing certain constraint conditions. Once the updated matrices for the linear structure are found, the finite element assembling technique is then used to include the lumped attachments by adding their parameters to the appropriate elements in the modified mass and stiffness matrices. Numerical experiments show that using only a few elements, the proposed scheme returns higher natural frequencies that are nearly identical to those obtained by using a finite element model with a very fine mesh. The new method is easy to apply and efficient to use. It remains applicable for any combination of attachments, and is valid for a combined system that is either undamped or damped.

INTRODUCTION

Frequency analysis of combined dynamical systems consisting of a linear structure carrying any number of lumped attachments has been studied extensively over the years, and hence only a few selected recent references are given here. Commonly used analytical approaches include the assumed-modes method [1,2], the Lagrange multipliers formalism [3,4], dynamic Green's function approach [5,6], Laplace transform with respect to the spatial variable approach [7,8], and the analytical-and-numerical-combined method [9,10]. However, due to their complexity, these methods have been used less than the finite element method.

In this paper, a modified approach is proposed that can be effectively used to obtain the natural frequencies of a combined system consisting of a linear structure carrying various lumped attachments. To obtain the higher natural frequencies of such a system using the finite element method, one typically refines the mesh of the linear structure until the accuracy criteria are satisfied. This approach is costly and time consuming, and the slow convergence can be attributed to the fact that many elements are required to model the linear structure itself so that the higher natural frequencies of the discretized linear structure match well with the exact solutions.

To expedite convergence and to obtain sufficiently accurate results with the least cost, a new scheme is introduced to improve the finite element mass and stiffness matrices of the linear structure such that the eigendata of the updated finite element model of the linear structure coincide with the exact eigensolution. Once the system matrices of the linear structure have been updated, the finite element assembling technique is exploited and used to account for the lumped attachments.

THEORY

Berman and Nagy [11] developed a method that used test data to update the analytical mass and stiffness matrices of a structure. The method yields a set of minimum changes in the system matrices such that the eigensolutions coincide with the test measurements. In this paper, the exact eigendata of the linear structure are first used to modify its finite element mass and stiffness matrices. Once the system matrices of the linear structure have been updated, one can easily include the lumped attachments by exploiting the finite element assembling technique [12], and determine the eigenvalues of the combined system by solving a generalized eigenvalue problem. Berman and Nagy minimized the objective function

$$J_M = ||[M_0]^{-1/2}([M]-[M_0])[M_0]^{-1/2}|| + \sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_{ij} \left([U]^T[M]U] - [I]\right)_{ij}, \qquad (1)$$

where $[M_0]$ is the finite element mass matrix, $||[A]||$ denotes the sum of the squares of all elements of matrix $[A]$, λ_{ij} denotes the Lagrange multiplier that is used to enforce the orthogonality of the eigenvectors with respect to the updated mass matrix $[M]$, and $[U]$ is the exact modal matrix of the linear structure (of size $N \times N$), whose elements are obtained from the exact eigenfunctions of the linear structure. The minimization procedure results in the following expression for the updated mass matrix

$$[M] = [M_0] + [M_0][U][m]^{-1}\left([I]-[m]\right)[m]^{-1}[U]^T[M_0], \qquad (2)$$

where $[I]$ is the identity matrix, and $[m] = [U]^T[M_0][U]$, where $[U]$ is normalized such that the diagonal elements of $[m]$ are ones. Following the computation of $[M]$, an updated stiffness matrix can be determined by minimizing yet another objective function

$$J_K \;=\; ||[M]^{-1/2}([K]-[K_0])[M]^{-1/2}|| + \sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_{Kij}\left([K][U]-[M][U][\Lambda]\right)_{ij}$$

$$+ \quad \sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_{0ij} \left([U]^T[K][U] - [\Lambda]\right)_{ij} + + \sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_{Sij} \left([K] - [K]^T\right)_{ij}, \qquad (3)$$

where $[K]$ and $[K_0]$ are the updated and finite element stiffness matrices, $[\Lambda]$ denotes a diagonal matrix consisting of the exact eigenvalues of the linear structure. Here, the Lagrange multipliers are used to enforce the generalized eigenvalue problem, the orthogonality of the eigenvectors with respect to the updated stiffness matrix, and the stiffness symmetry. The updated stiffness matrix is given by

$$[K] = [K_0] + [\Delta] + [\Delta]^T, \qquad (4)$$

$$[\Delta] = \frac{1}{2}[M][U](\,[U]^T[K_0][U] + [\Lambda]\,)\,[U]^T[M] - [K_0][U][U]^T[M]. \qquad (5)$$

Equations (2) and (4) lead to updated mass and stiffness matrices whose eigensolutions coincide with the exact eigendata of the linear structure. The proposed modification scheme returns an updated model without iteration, and requires only matrix multiplications. Once these updated matrices are obtained, the lumped attachments are added to the updated system matrices to form the global mass and stiffness matrices, and the natural frequencies of the combined system are readily obtained by solving the associated generalized eigenvalue problem.

RESULTS

To validate the proposed discretization scheme, the natural frequencies of a combined system consisting of a fixed-free beam carrying various lumped attachments will be considered. In order to apply Eqs. (2) and (4), matrices $[M_0]$, $[K_0]$, $[U]$, and $[\Lambda]$ are required. The finite element mass and stiffness matrices, $[M_0]$ and $[K_0]$, of the beam can be easily obtained by superimposing the individual element matrices and enforcing the appropriate boundary conditions at the ends. Matrices $[U]$ and $[\Lambda]$ can be assembled directly from the exact modes of vibration once the boundary conditions for the beam are specified. For a fixed-free beam, its eigenfunctions and eigenvalues are given by

$$v_i(x) = \frac{1}{\sqrt{\rho L}}\left[\cos\beta_i x - \cosh\beta_i x + \frac{\sin\beta_i L - \sinh\beta_i L}{\cos\beta_i L + \cosh\beta_i L}(\sin\beta_i x - \sinh\beta_i x)\right], \qquad (6)$$

where $\beta_i L$ satisfies the transcendental equation

$$\cos\beta_i L \cosh\beta_i L = -1, \qquad (7)$$

and

$$\lambda_i = (\beta_i L)^4/(EI/\rho L^4), \qquad (8)$$

where E, I, ρ, and L denote the Young's modulus, the area moment of inertia of the cross section, the mass per unit length, and the length of the beam, respectively.

For a beam element, its generalized coordinates consist of the lateral displacement and the angular rotation (or slope) at the nodes [12]. Hence, if the fixed-free beam is discretized into n equal finite elements, there is a total of $N = 2n$ generalized coordinates. Moreover, to assemble the exact modal matrix, $[U]$, of the linear structure, the lateral deflection and the slope at each node must be specified. Fortunately, knowing the exact eigenfunctions $v_i(x)$ of beam, its slope at any point x can be easily determined by taking the derivative of $v_i(x)$ with respect to x, i.e.,

$$\theta_i(x) = \frac{d}{dx}\left[v_i(x)\right]. \tag{9}$$

Once the exact lateral displacements and angular rotations at the nodes have been computed, matrix $[U]$ can be easily assembled, where the elements of the ith column of $[U]$ are obtained by evaluating the ith eigenfunction and its derivative at the appropriate node locations. Finally, the ith element of the diagonal matrix $[\Lambda]$ is given by λ_i.

Updating the stiffness matrix requires only matrix multiplications. Updating the mass matrix, however, requires the inversion of the matrix $[m]$. When the linear structure is a simply supported Euler Bernoulli beam, $[m]$ consists of a diagonal matrix modified by a backward superdiagonal matrix, whose inverse can be obtained in closed-form. Thus, updating the mass matrix also involves the product of matrices. For a fixed-free beam, on the other hand, $[m]$ is fully populated and there is no closed-form expression for its inverse. Nevertheless, the additional computation needed to invert the $N \times N$ matrix is a relatively small price to pay for the ability to obtain the higher natural frequencies or eigenvalues that are nearly identical to the exact results, as will be illustrated.

In all the following numerical examples, the first 10 natural frequencies of the combined systems are first obtained by discretizing the linear structure into 100 equal elements. For all practical purposes, these natural frequencies can be considered exact. To illustrate the utility of the proposed discretization schemes, the first 10 natural frequencies of a combined system consisting of a fixed-free beam carrying lumped attachments are obtained by using the finite element method and the proposed discretization scheme, whereby the beam is discretized into 5 equal elements, i.e., $n = 5$.

Consider a fixed-free beam carrying a damped oscillator (with parameters $m = 0.5\rho L$, $k = 1.0 EI/L^3$ and $c = 0.2\sqrt{EI\rho/L^2}$) with a rigid body degree of freedom at $x_a = 0.8L$ (see Figure 1), whose governing equations are given by

$$[\mathcal{M}]\ddot{\mathbf{p}} + [\mathcal{C}]\dot{\mathbf{p}} + [\mathcal{K}]\mathbf{p} = \mathbf{0}. \tag{10}$$

Matrices $[\mathcal{M}]$, $[\mathcal{C}]$, and $[\mathcal{K}]$ are the $(N+1) \times (N+1)$ global mass, damping, and stiffness matrices of the combined system (the mass and stiffness matrices of the beam have already been modified using the newly developed scheme), and $\mathbf{p} = [\mathbf{q} \quad y]^T$, where $\mathbf{q}$ is the vector of generalized coordinates for the beam, and y denotes the vertical displacement of the damped oscillator. Because damping is present, the state vector approach is used to determine the eigenvalues. Introducing $\mathbf{z} = [\dot{\mathbf{p}} \quad \mathbf{p}]^T$, Eq. (10) becomes

$$[A]\dot{\mathbf{z}} - [B]\mathbf{z} = \mathbf{0}, \tag{11}$$

where matrices $[A]$ and $[B]$ are given by

$$[A] = \begin{bmatrix} [0] & [\mathcal{M}] \\ [\mathcal{M}] & [\mathcal{C}] \end{bmatrix} \qquad \text{and} \qquad [B] = \begin{bmatrix} [\mathcal{M}] & [0] \\ [0] & -[\mathcal{K}] \end{bmatrix}. \tag{12}$$

Equation (11) leads to the following $2(N+1) \times 2(N+1)$ generalized eigenvalue problem

$$[B]\bar{\mathbf{z}} = \mu[A]\bar{\mathbf{z}}. \tag{13}$$

Because the system is damped, the complex eigenvalues μ rather than the natural frequencies will be investigated. Table 1 shows the first 10 eigenvalues of the combined system. Note how well the eigenvalues obtained by using the new approach with $n = 5$ track those obtained by using the finite element method with $n = 100$, illustrating the accuracy of the proposed scheme. Thus, rather than solving a 402×402 generalized eigenvalue problem that is required of the finite element method when the beam is discretized into 100 equal elements (where $N = 200$), one only needs to invert a 10×10 matrix and then solve a generalized eigenvalue problem of size 22×22 when using the new method with $n = 5$ (where $N = 10$), which leads to dramatic computational efficiency.

Consider a uniform fixed-free Euler Bernoulli beam carrying a grounded spring, a lumped mass, an undamped oscillator with a rigid body degree of freedom, and a grounded torsional spring at $x_{a1} = 0.2L$, $x_{a2} = 0.4L$, $x_{a3} = 0.6L$, and $x_{a4} = 0.8L$, respectively (see Figure 2). The system parameters are $k_1 = 0.8EI/L^3$, $m_1 = 0.2\rho L$, $k_2 = 0.5EI/L^3$, $m_2 = 0.1\rho L$, and $k_t = 1.0EI/L$. Table 2 shows the first 10 natural frequencies of the combined assembly. The results demonstrate that the proposed discretization scheme with $n = 5$ yields higher natural frequencies that are nearly identical to those obtained using the finite element method with $n = 100$, and that it remains applicable when the beam is carrying multiple lumped attachments.

In this paper, a new discretization scheme is proposed that can be used to obtain the eigenvalues of a combined system consisting of a linear structure carrying lumped attachments. For the same number of elements, the proposed scheme returns higher eigenvalues that are substantially more accurate than those given by the finite element method. The new discretization algorithm allows one to determine the higher eigenvalues accurately without having to refine the mesh of the linear structure, as required by the finite element approach.

SUMMARY

A new finite element discretization scheme is proposed that can be used to accurately determine all of the eigenvalues, especially the higher ones, of a linear structure carrying lumped attachments. The finite element mass and stiffness matrices of the linear structure are modified or updated using the exact eigensolutions of the linear structure, such that its finite element model returns modes of vibration that coincide with the exact eigendata. Once the mass and stiffness matrices have been updated, the finite element assembling technique is exploited to include the lumped attachments. Numerical experiments show that with only a few elements,

the newly developed discretization scheme returns higher eigenvalues that are nearly identical to those obtained by using a finite element model with a very fine mesh. The new method is easy to apply and efficient to use. It remains applicable for any combination of attachments, and is valid for a combined system that is either undamped or damped.

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μ_i	FEM, $n = 100$	FEM, $n = 5$	New Scheme, $n = 5$
μ_1	-1.374e-01+j1.293e+00	-1.374e-01+j1.293e+00	-1.374e-01+j1.293e+00
μ_2	-2.729e-01+j3.815e+00	-2.729e-01+j3.815e+00	-2.729e-01+j3.815e+00
μ_3	-1.992e-03+j2.204e+01	-1.994e-03+j2.205e+01	-1.992e-03+j2.204e+01
μ_4	-6.247e-02+j6.170e+01	-6.356e-02+j6.192e+01	-6.247e-02+j6.170e+01
μ_5	-1.655e-01+j1.209e+02	-1.744e-01+j1.223e+02	-1.655e-01+j1.209e+02
μ_6	-1.443e-01+j1.999e+02	-1.487e-01+j2.030e+02	-1.443e-01+j1.999e+02
μ_7	-3.727e-02+j2.986e+02	-3.156e-02+j3.373e+02	-3.727e-02+j2.986e+02
μ_8	-5.668e-03+j4.170e+02	-1.039e-02+j4.933e+02	-5.668e-03+j4.170e+02
μ_9	-1.018e-01+j5.552e+02	-1.079e-01+j7.153e+02	-1.018e-01+j5.552e+02
μ_{10}	-1.965e-01+j7.131e+02	-1.021e-01+j1.016e+03	-1.965e-01+j7.131e+02

Table 1. The first 10 eigenvalues of a uniform fixed-free Euler Bernoulli beam carrying a damped oscillator with a rigid body degree of freedom at $x_a = 0.8L$. The oscillator parameters are $m = 0.5\rho L$, $c = 0.2\sqrt{EI\rho/L^2}$ and $k = 1.0EI/L^3$. All of the eigenvalues are normalized by dividing by $\sqrt{EI/(\rho L^4)}$.

ω_i	FEM, $n = 100$	FEM, $n = 5$	New Scheme, $n = 5$
ω_1	2.196e+00	2.196e+00	2.196e+00
ω_2	4.269e+00	4.269e+00	4.275e+00
ω_3	2.017e+01	2.018e+01	2.018e+01
ω_4	5.834e+01	5.851e+01	5.836e+01
ω_5	1.178e+02	1.191e+02	1.178e+02
ω_6	1.802e+02	1.822e+02	1.804e+02
ω_7	2.986e+02	3.370e+02	2.986e+02
ω_8	3.864e+02	4.683e+02	3.877e+02
ω_9	5.350e+02	6.862e+02	5.363e+02
ω_{10}	7.002e+02	1.015e+03	7.018e+02

Table 2. The first 10 natural frequencies of a uniform fixed-free Euler Bernoulli beam carrying a grounded translational spring, a lumped mass, an undamped oscillator with a rigid body degree of freedom, and a grounded torsional spring at $x_{a1} = 0.2L$, $x_{a2} = 0.4L$, $x_{a3} = 0.6L$ and $x_{a4} = 0.8L$, respectively. The system parameters are $k_1 = 0.8EI/L^3$, $m_1 = 0.2\rho L$, $k_2 = 0.5EI/L^3$, $m_2 = 0.1\rho L$, and $k_t = 1.0EI/L$.

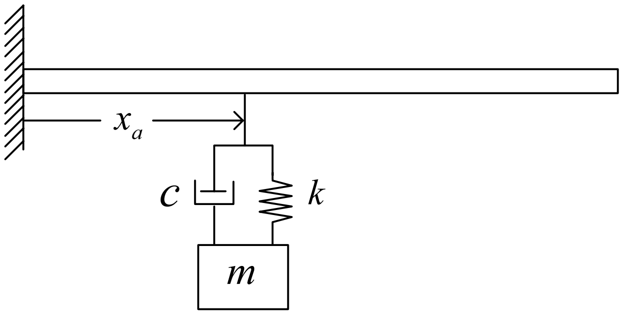

Figure 1: A fixed-free beam carrying a damped oscillator with a rigid body degree of freedom at x_a.

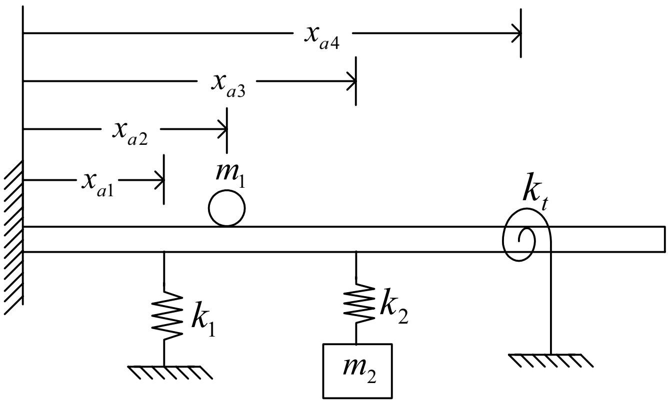

Figure 2: A fixed-free beam carrying a grounded translational spring, a lumped mass, an undamped oscillator with a rigid body degree of freedom, and a grounded torsional spring at x_{a1}, x_{a2}, x_{a3}, and x_{a4}, respectively.