ICSV13 - Vienna
The Thirteenth International Congress on Sound and Vibration
Vienna, Austria, July 2-6, 2006

FREE VIBRATION OF LAYERED ANNULAR CIRCULAR PLATE OF VARIABLE THICKNESS BY SPLINE APPROXIMATION

K.K.Viswanathan*[1], Dongwoo Sheen[2], and P.V.Navaneethakrishnan[3]

[1]Department of Mathematics, SRM Institute of Science and Technology-Deemed University, SRM Nagar, Kattankulathur, Chennai-603203., India.
[2]Department of Mathematical Sciences, Seoul National University, Seoul 151-747, Korea.
[3]Department of Mathematics, AV Institute of Technology, VMRF Deemed University, Paiyanoor-603104, India.
visu20@yahoo.com, viswa@math.snu.ac.kr

Abstract

The free vibration of layered annular circular plate of variable thickness, each layer made up of isotropic or specially orthotropic material, is studied using spline function approximation and point collocation. Three different thickness variations are considered, namely, linear, exponential and sinusoidal, along the radial directions. The equations of motion are derived as a system of coupled differential equations, in the longitudinal, circumferential and transverse displacement functions is obtained by assuming the solution in a separable form. These functions are approximated by using Bickley-type splines of suitable orders. A generalized eigenvalue problem is obtained by applying a point collocation technique and suitable boundary conditions from which the values of a frequency parameter and the corresponding mode shapes of vibration, for specific values of other parameters, are obtained.

INTRODUCTION

Extensive results are available in the literature on vibration of isotropic, orthotropic and laminated annular plates of uniform thickness. Timoshenko and Woinowsky-Krieger [1] frequently quote an exact solution for the static analysis of radially tapered disc springs developed by Conway [2]. The analysis was extended by Conway et al. [3] to the study of free vibration of tapered circular plates. These exact analyses, however, are of limited use, being applicable to materials with Poisson ratio of 1/3. A few studies have also been made on homogeneous annular plates of variable thickness. Leissa's [4] monograph on vibration of plates and surveys by Bert [5,6] contain discussions of these plates. Raju et al. [7] studied axisymmetric vibrations of linearly tapered annular plates. In the work of Vodika [8] radial non-homogeneity in circular plates is accommodated by treating the plate as a composite of homogeneous,

Eds.: J. Eberhardsteiner, H.A. Mang, H. Waubke

isotropic annuli and enforcing continuity conditions across the internal junctions. Singh and Saxena [9] and Lal and Sharma [10] studied the vibrations of annular plates of exponentially varying thickness. Zhou et al.[11] applied the Chebyshev-Ritz method for analyzing the vibration of circular and annular plates. Selmane and Lakis [12] found the frequencies of transverse vibrations of non-uniform circular and annular plates using the finite element method. Most of the above authors worked only for finding the transverse vibration of annular and circular plates.

In the current work, the general linear, exponential and sinusoidal variation in thickness of laminae along the radial direction of the plate are considered and the versatile spline function technique of solution is used. Here, a chain of lower order approximation is used which can yield greater accuracy than a global higher order approximation. This conjecture was made and tested by Bickley [13] over a two point boundary value problem with a cubic spline. Viswanathan and Navaneethakrishnan [14] have also demonstrated this, along with its attractive features of elegance in handling and convergence.

Both the axisymmetric and asymmetric vibrations are considered. Extensive parametric studies are made to provide insight into the individual and interactive influence of various geometric and material parameters. The effects on the frequency parameter, of the relative layer thickness, the radii ratio, and the other parameters characterizing the nature of variation of thickness, the number of circumferential nodes and boundary conditions are analyzed. The effect of neglecting the coupling between bending and stretching is investigated. Extensive convergence tests and comparative studies with available literature are made. The results are presented in terms of graphs and discussed.

Formulation of the problem and the Method of solution

The geometry of layered annular circular plate of linearly varying thickness are shown in Fig.1. Each individual layer is considered to behave macroscopically as a homogeneous orthotropic and linearly elastic material. The layers are assumed to be perfectly bonded. Rotatory inertia and transverse shear deformation are neglected. The plate is assumed to be thin so that the angular shifting of the axes of material symmetry of each layer due to the variation in thickness is considered to be negligible. The line or is the section of the reference surface. The thickness of the k-th layer of the plate is taken in the form

$$h_k(r) = h_{0k}\, g(r) \tag{1}$$

where h_{0k} is a constant. The thickness becomes uniform when $g(r)=1$. The elastic coefficients corresponding to layers of uniform thickness with superscript ^{c}, can be defined as

$$A_{ij} = A_{ij}^c\, g(r),\;\; B_{ij} = B_{ij}^c\, g(r),\;\; D_{ij} = D_{ij}^c\, g(r) \tag{2}$$

where A_{ij}^c, B_{ij}^c, D_{ij}^c are extensional, extensional-bending, and bending elastic coefficients respectively.

For our detailed study, the thickness variation of each layer is assumed in the form

$$h(r) = h_0\, g(r) \tag{3}$$

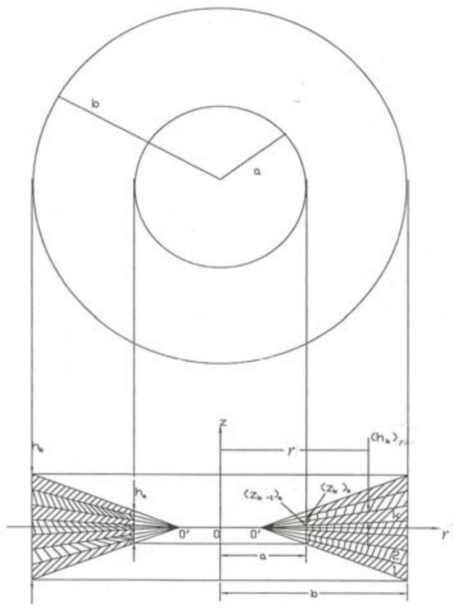

Figure 1 - Geometry of layered annular circular plate of linearly varying thickness

$$\text{where} \quad g(r) = 1 + C_\ell \left(\frac{r - r_a}{\ell} \right) + C_e \exp\left(\frac{r - r_a}{\ell} \right) + C_s \sin\left(\frac{\pi (r - r_a)}{\ell} \right) \tag{4}$$

Here ℓ is the width $(b-a)$ of the plate and r_a is the radial distance from the origin to $r=a$.

The stress resultants and moment resultants are expressed in terms of the longitudinal, circumferential and transverse displacements u, v and w of the reference surface. The displacements are assumed in the separable form given by

$$\begin{aligned} u(r,\theta,t) &= U(r)\cos n\theta \; e^{i\omega t} \\ v(r,\theta,t) &= V(r)\sin n\theta \; e^{i\omega t} \\ w(r,\theta,t) &= W(r)\cos n\theta \; e^{i\omega t} \end{aligned} \tag{5}$$

where r and θ are the polar coordinates to describe the radial and rotational directions, t is the time, ω is the angular frequency of vibration and n is the circumferential node number. When $n = 0$, the vibration becomes axisymmetric. The governing differential equations of motion of annular plate of variable thickness along the radius are obtained in the form

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = \{0\}. \tag{6}$$

The equations defining the operators $L_{ij}\,(i,j=1,2,3)$ could not be supplied here for want of space.The differential equations on the displacement functions in Eq.(6) contains the derivative of third order in U , second order in V and fourth order in W . The order of U is not suitable for the existing derivatives orders in Eq. (6). Therefore, the equations are combined within themselves and a modified system of equations on the displacement functions, of order 2 in U , order 2 in V and order 4 in W are obtained with the operators L_{3j} modified as L^*_{3j} , $(j=1,2,3)$, again not given for want of space.The length parameters, frequency parameters, radii of the plate and thickness parameters are non-dimensionalzed as

$$X=\frac{r-a}{\ell},\; a\le x\le b,\; \lambda=\ell\,\lambda',\; \beta=\frac{a}{b},\; \gamma=\frac{h_0}{r_a},\; \delta_k=\frac{h_k}{h} \tag{7}$$

Spline collocation procedure is adopted to solve this problem assuming

$$\begin{aligned}
U^*(X)&=\sum_{i=0}^{2}a_iX^i+\sum_{j=0}^{N-1}b_j(X-X_j)^3H(X-X_j)\\
V^*(X)&=\sum_{i=0}^{2}c_iX^i+\sum_{j=0}^{N-1}d_j(X-X_j)^3H(X-X_j)\\
W^*(X)&=\sum_{i=0}^{4}e_iX^i+\sum_{j=0}^{N-1}f_j(X-X_j)^5H(X-X_j)
\end{aligned} \tag{8}$$

The boundary conditions are used as follows: (i) both the edges clamped (C-C), (ii) both the edges hinged (H-H) and (iii) the inner edge clamped and the outer edge free (C-F). The resulting field and boundary conditions gives raise to the generalized eigenvalue problem of the form

$$[M]\{q\}=\lambda^2[P]\{q\} \tag{9}$$

where $[M]$ and $[P]$ are matrices of order $(3N+7)\times(3N+7)$, $\{q\}$ is a matrix of order $(3N+7)\times 1$, and $N+1$ is the number of knots of the splines on radial direction. The parameter λ is the eigenparameter and $\{q\}$ the eigenvector whose elements are the spline coefficients. Only two-layer plates are considered with $\delta=$ ratio of thickness of the first mentioned layer, to the total thickness, at the inner circular edge**.**

NUMERICAL RESULTS AND DISCUSSION

Convergence and Comparative Studies

Certain results of the convergence studies for C-C boundary conditions are carried out, whose numerical results are not presented for want of space. Based on these findings it was decided to set N , number of subintervals of the width of the annular plate, equal to 16. The fundamental frequency parameter values obtained for homogeneous plates of linearly varying thickness (treated as special cases of St-Al (Steel-Aluminium) layers with $\delta=1$) compared with those of Raju et al. [7] and the maximum difference is 1.48. The agreement is quite good which gives the confirmation of the validity of the analysis.

Results and discussion

The influence of the relative layer thickness of two layered plates is first studied. Figs. 2,3 and 4 pertain to such studies on HSG-SGE (High strength graphite – S glass epoxy) plates of the three types of variation in thickness . All three types of boundary conditions are considered for each type of thickness variation. The condition C-F, for example, implies that the inner circular boundary of the annular plate is clamped and the outer boundary is free. The plates are of medium annular width for β = 0.5. The thickness-to-inner radius ratio γ has the value 0.05. When $\delta = 0$ or 1, the plate becomes homogeneous, made up of the second or first mentioned material, accordingly.

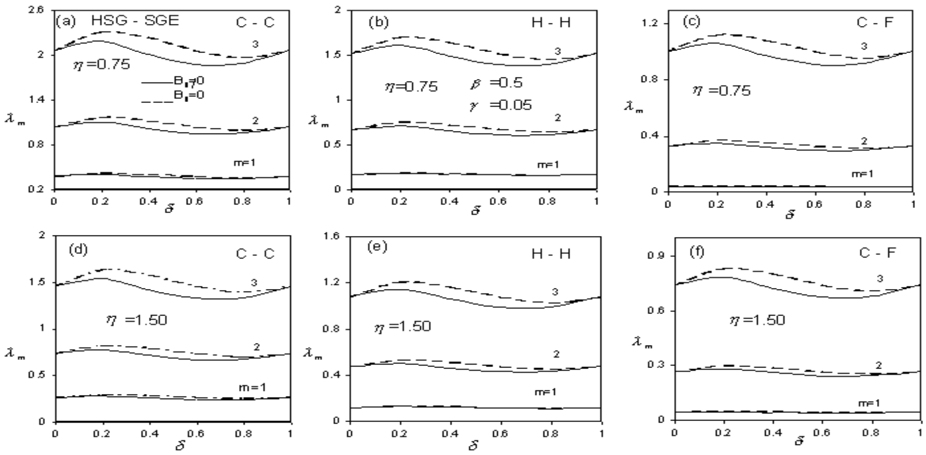

Figure 2 - Effect of relative layer thickness, coupling and boundary conditions on the frequency parameter of annular circular plates of linearly varying thickness

In Fig. 2 two types of linear variation in thickness are considered, with the taper ratio $\eta =$ 0.75 (outer thickness larger) and $\eta =$ 1.50 (outer thickness smaller). It is clearly seen that the frequency parameter values are the same for $\delta = 0$ and $\delta = 1$ (homogeneous) and vary for $0 < \delta < 1$. The range of variation of λ_m is the least for m =1 and increase with the increasing value of m . It is possible to attain frequencies higher and lower than those of homogeneous plates made up of either of the two materials. The boundary conditions do affect the frequencies. The values of λ are highest for C-C conditions, lower for C-F conditions under the same other conditions. λ is lower for higher value of η which is expected.

The continuous and dotted lines correspond respectively to the inclusion and omission of the coupling effect between the radial and transverse displacements. The effect of neglect of this coupling is seen to raise frequencies for all modes. The effect is more significant for higher modes. The percent change is however small, so that B_{ij} can be set equal to zero without introducing appreciable error, but resulting in some computational advantage. This is in agreement with Ashton's [15] findings that the reduced stiffness matrix approximation, obtained by neglecting coupling, yields reasonably accurate results. (However, throughout this study the coupling effect was not excluded). Layering with different materials of layers affect the frequency parameter differently.

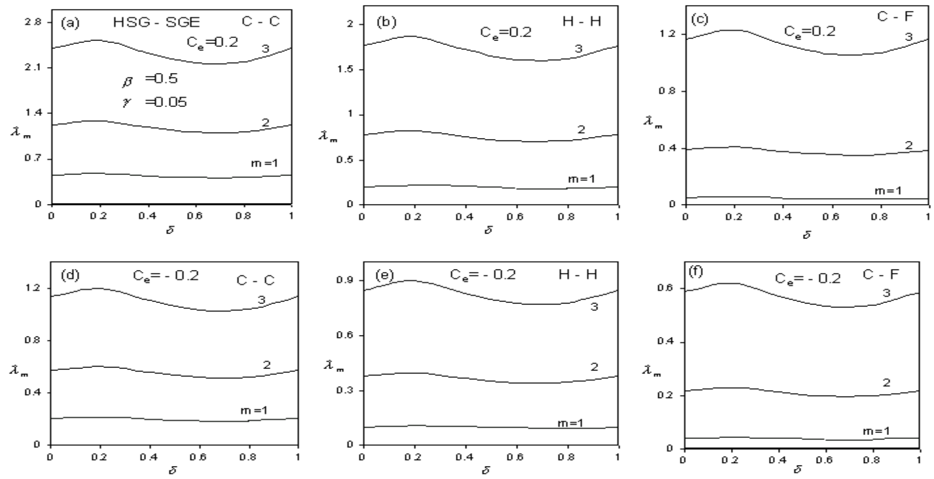

Figure 3 - Effect of relative layer thickness on frequency parameter: Exponential variations

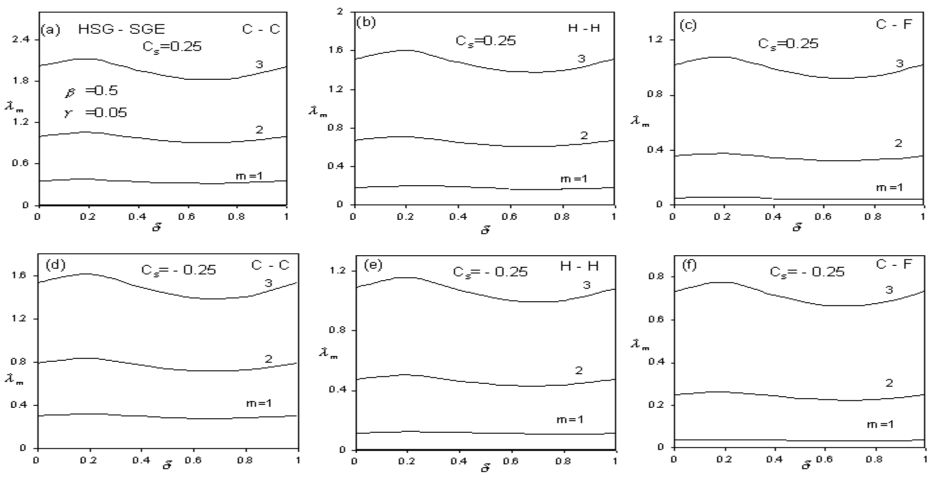

Figure 4- Effect of relative layer thickness on frequency parameter: Sinusoidal variations.

Two types of exponential variation in thickness of layers considered in Fig.3, correspond to $C_e = +0.2$ and $C_e = -0.2$. The thickness increases and decreases, respectively, as the radius increases, in these two cases. This explains why the values of λ_m for the same m are lower for $C_e = +0.2$ than those for $C_e = -0.2$. The characteristics of $\lambda \sim \delta$ relation are similar, in general, to the case of linear variation in thickness. Fig.4 pertains to sinusoidal variation in thickness of layers corresponding to $C_s = 0.25$ and $C_s = -0.25$. The thickness of the plate is same at $r=a$ and $r=b$; the surface of the plate is convex or concave for $a < r < b$, for $C_e \lessgtr 0$.

This is in keeping with the observation that the values of λ_m, for all the boundary conditions considered, are lower for $C_s = -0.25$ than for $C_s = 0.25$.

The frequencies of asymmetric vibrations of plates are affected by the circumferential node number n. Fig.5.describes the influence of n on λ_m (m=1,2,3) of HSG-SGE plates whose layers are linearly, exponentially and sinusoidally varying in thickness and which are supported at both the edges either clamped or hinged. The other parameters are as described in the figure. Three typical values for the coefficient of variation are considered. It is seen that as n increases λ_m increase for all values of m considered, in all the cases. Thus the frequencies for axisymmetric vibrations (n=0) are the least.

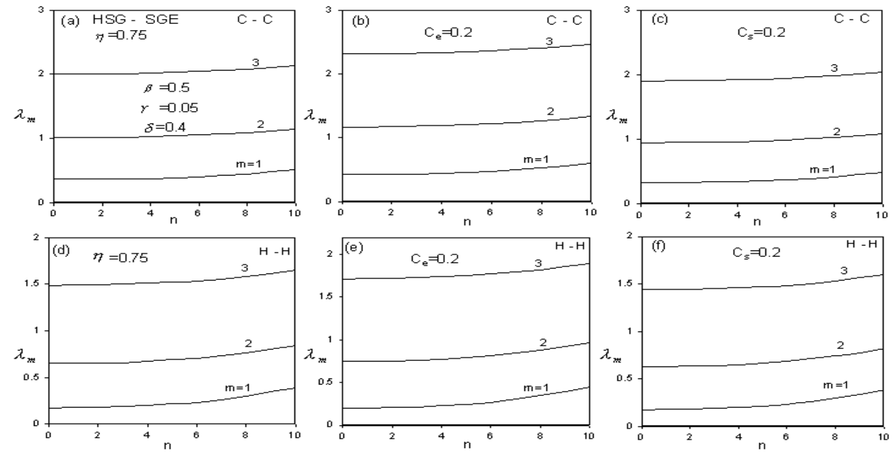

Figure 5 - Variation of frequency parameter of plates of linear, exponential and sinusoidal variation in thickness with circumferential node number under different boundary conditions.

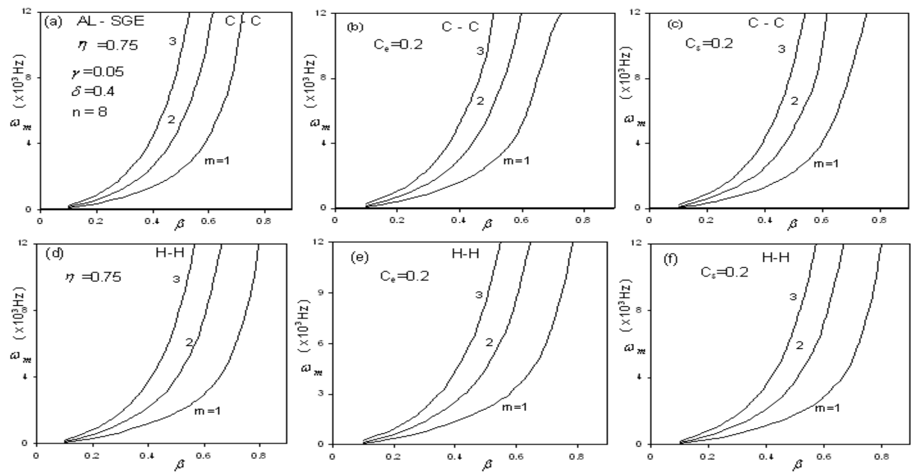

Figure 6 - Variation of frequency of asymmetric vibration of plates of linear, exponential and sinusoidal variations in thickness with length ratio under different boundary conditions.

The influence of the width of the annular plate on the frequencies of vibration for a typical asymmetric (n=8) vibration is depicted in Fig. 6. The layers are Al-SGE

(Aluminium-S glass epoxy), γ = 0.05, δ = 0.4. The boundary conditions considered are C-C and H-H. Three typical values for the coefficients of variation of the three types of variation in thickness are considered. The patterns of influence are similar to those in the axisymmetric case. However the frequencies for this asymmetric case are higher than those of the corresponding axisymmetric case of vibration, as expected. Though the case of C-F boundary conditions is not presented, the behavioral pattern is similar. The frequencies at C-C conditions are higher than those at H-H conditions which again are higher than those at C-F conditions, under identical other conditions.

CONCLUSION

Layering and variation in thickness are two factors which independently influence the vibrational behaviour of a thin plate. When the plate is both layered and of variable thickness, its behaviour is naturally more complex due to the interactive influence of these aspects. The natural frequencies of vibration of such a plate are influenced by the materials of the constituent layers, their relative thickness, the nature of variation of thickness, the value of the corresponding thickness parameter, radius ratio, circumferential node number and the boundary conditions.

REFERENCES

[1] Timoshenko, S., and Woinowsky-Krieges., *Theory of Plates and Shells*. (Mc.Graw-Hill, Second ed., London, 1959).

[2] Conway, H.D., 1948, “The bending of symmetrically loaded circular plates of variable thickness.” ASME J. Appl. Mech., **15**, pp.1-6.

[3] Conway, H.D., Becker, E.C.H., and Dubit, J.F., 1964, “Vibration frequencies of tapered bars and circular plates.” ASME J. Appl. Mech., **31**, pp.329-331

[4] Leissa, A.W., “Vibration of Plates.” NASA SP-160. Washington, D.C.(1969)

[5] Bert, C.W., “Research on dynamics of composite and sandwich plates.” Shock and Vibration Digest, **14,**7-34 (1982).

[6] Bert, C.W., “Research on dynamic behaviour of composite and sandwich plates- IV.” Shock and Vibration Digest, **17,**3-15 (1985).

[7] Raju, I.S., Prakasa Rao, I.S., and Venkateswara Rao, “Axisymmetric vibration of linearly tapered annular plates.” Journal of Sound and Vibration, **32**,507-512 (1974).

[8] Vodicka,V.,“Free vibrations of a composite circular plate.”Acta Physica Austriaca., **17,** 319-332(1964).

[9] Singh, B., and Sexena, V.,“Axisymmetric vibration of a circular plate with exponential thickness variation.” J. Sound Vib., **192,** pp.35-42(1996).

[10] Lal, R., and Sharma, S.,“Axisymmetric vibrations of non-homogeneous polar orthotropic annular plates of variable thickness.” J. Sound Vib., **272**, 245- 265(2004).

[11] Zhou, D., Au, F.T.K., Cheung, Y.K., and Lo, S.H., “Three-dimensional vibration analysis of circular and annular plates via the Chebyshev-Ritz method.” Int. J. Solids Struct., **40,** 3089-3105(2003).

[12] Selmane, A., and Lakis, A.A.,“Natural frequencies of transverse vibrations of non-uniform circular and annular plates.” J. Sound Vib., **220,** 225-249(1999).

[13] Bickley,W.G.,“Piecewise cubic interpolation and two-point boundary problems.” Comp. J., **11**, 206-208(1968).

[14] Viswanathan, K.K., Navaneethakrishnan, P.V.,“Free vibration study of layered cylindrical shells by Collocation with splines.” J. Sound Vib., **260**, 807-827.(2003).

[15] Ashton, J.E., “Approximate solutions for unsymmetrically laminated plates.” J. Comp. Mat., **3,** 189-191(1969).