ESTIMATION OF WIDELY FACTORIZABLE HYPERCOMPLEX SIGNALS WITH UNCERTAIN OBSERVATIONS

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ABSTRACT

The filtering estimation problem under uncertainty conditions is addressed for a class of improper quaternion signals, called widely factorizable, characterized because their augmented correlation function is a factorizable kernel. From the knowledge of the correlation functions involved, a recursive algorithm is designed for the computation of the widely linear (WL) filtering estimate and its associated mean squared error. The main advantage of the proposed solution is that it can be applied in situations where a state-space model is not readily at hand. The benefits of the proposed WL filtering algorithm is analyzed through a simulation example where WL filtering errors are compared with respect to the strictly linear (SL) counterparts, showing the superior behavior of the former over the latter.

***Index Terms*—** Quaternion signals, filtering algorithm, uncertainty conditions, widely factorizable signals, widely linear processing

1. INTRODUCTION

For a long time, the problem of estimating a signal in the presence of noise has been a subject undergoing intense study among signal processing researchers and it still remains of great interest in recent literature (see, e.g., [1], [2] or [3]). In general, the signal is assumed to be present in the observations. Nevertheless, in the real world the signal can be absent due to sensor failures, lack of uniformity and constancy in the data, network congestion, among many other reasons (see, e.g., [4] and [5]). In these situations, the observation equation is defined containing, besides the additive noise, a multiplicative noise defined as Bernoulli random variables which take the value one or zero depending on whether the signal is present or absent in the observation.

Optimal linear estimation with uncertain observations was first addressed by Nahi [6]. The author provides a recursive algorithm for the computation of the filtering estimate of the signal of interest by ass uming that the state-space model is completely known. As a matter of choice, in those cases where a state-space model is not readily at hand, knowledge of the correlation functions involved in the observation equation has been a key in solving estimation problems. In this framework, an alternative estimation methodology based on correlation information has been developed to solve different estimation problems for the class of factorizable signals (see, e.g., [7], [8], [9] and [10]). Note that these signals are characterized by having a factorizable kernel, involving stationary as well as nonstationary signals.

In the last decades, advances in technologies have led to the use of complex and hypercomplex signals that facilitates the modeling of multidimensional and multivariate signals. In this framework, the benefits of the so-called widely linear (WL) processing over the conventional or strictly linear (SL) processing has been extensively exploited in recent literature (see, e.g., [11], [12], [13] and [3]). Indeed, based on the correlation information, the above methodology has been extended to the complex case providing WL estimation algorithms for a class of signals with the specificity that the correlation of the augmented vector formed by the signal and its conjugate is a factorizable kernel ([1], [12]). Illustrative examples of widely factorizable signals can be found in [12]. Our interest here is to extend this approach to the quaternion field.

Quaternion signals are commonly used to represent rotations in a three-dimensional space, since they avoid the singularity problem inherent in Euler angle representations [14], and they have found applications in many different problems such as image processing [15], robotics [16], processing of polarized waves [17] or vector sensor [18, 19], among others. In this paper, we consider a class of quaternion signals which are widely factorizable, it means that the correlation function of the augmented vector formed by the signal and its three involutions is a factorizable kernel. Then, using information of the correlation functions involved, a WL recursive filtering algorithm is devised in the case of uncertainty observations. Finally, the good behavior of the proposed algorithm is nu-

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merically analyzed by means of a simulation example.

2. PRELIMINARIES

This section is devoted to introducing some basic notations and concepts that will be necessary throughout this paper.

In general, scalar quantities will be denoted by lightface lowercase letters, vectors by lowercase letters and matrices by boldface uppercase. For some specific matrices we will also use boldfaced upper case italicized letters. Additionally, row k of any matrix $\mathbf{A}(\cdot)$ will be denoted by $\mathbf{a}_{[k]}(\cdot)$. Moreover, the following notation will be adopted: $\boldsymbol{I}_m$ is the identity matrix of dimension m, $\mathbf{0}_{n\times m}$ is the $n \times m$ zero matrix, $\mathbf{0}_m$ is the m-dimensional vector whose elements are zero. Furthermore, the quaternion field will be expressed by $\mathbb{H}$, where superscripts $(\cdot)^*$, $(\cdot)^{\mathrm{T}}$ and $(\cdot)^{\mathrm{H}}$ will represent the quaternion conjugate, transpose and Hermitian, respectively. The notation $\mathbf{A} \in \mathbb{R}^{n\times m}$ (respectively $\mathbf{A} \in \mathbb{H}^{n\times m}$) means that $\mathbf{A}$ is a real (respectively quaternion) $n \times m$ matrix. Similarly, $\mathbf{a} \in \mathbb{R}^n$ (respectively $\mathbf{a} \in \mathbb{H}^n$) means that $\mathbf{a}$ is a real (respectively quaternion) n-dimensional vector.

In addition, $\mathrm{diag}(\cdot)$ will represent a diagonal matrix with the elements specified on the main diagonal, $E[\cdot]$ is the expectation operator and "$\odot$" denotes the Hadamard product[1].

Definition 1 *A quaternion random signal $x(k) \in \mathbb{H}$ is defined as a stochastic process of the form*

$$x(k) = x_r(k) + \eta x_\eta(k) + \eta' x_{\eta'}(k) + \eta'' x_{\eta''}(k)$$

where $x_\nu(k)$, $\nu = r, \eta, \eta', \eta''$, are zero-mean real random signals and $\{1, \eta, \eta', \eta''\}$ fulfills the following relations:

$$\begin{aligned}
\eta^2 = \eta'^2 = \eta''^2 &= \eta\eta'\eta'' = -1 \\
\eta\eta' &= \eta'' = -\eta'\eta \\
\eta'\eta'' &= \eta = -\eta''\eta' \\
\eta''\eta &= \eta' = -\eta\eta''
\end{aligned}$$

Definition 2 *The product $\star$ between two quaternion random signals $x(k), y(k) \in \mathbb{H}$ is defined as*

$$\begin{aligned}
x(k) \star y(l) =& x_r(k)y_r(l) + \eta x_\eta(k)y_\eta(l) \\
&+ \eta' x_{\eta'}(k)y_{\eta'}(l) + \eta'' x_{\eta''}(k)y_{\eta''}(l)
\end{aligned} \tag{1}$$

Definition 3 *The quaternion signal $x(k) \in \mathbb{H}$ is said to be factorizable if and only if there exist two n-dimensional quaternion vectors $\boldsymbol{\alpha}(k), \boldsymbol{\beta}(k) \in \mathbb{H}^n$ such that the correlation function of $x(k)$, $r_x(k,l)$, can be expressed in the form*

$$r_x(k,l) = \begin{cases} \boldsymbol{\alpha}^{\mathrm{T}}(k)\boldsymbol{\beta}^*(l), & k \geq l \\ \boldsymbol{\beta}^{\mathrm{T}}(k)\boldsymbol{\alpha}^*(l), & k \leq l \end{cases} \tag{2}$$

Note that this type of signal is very common and includes both stationary and nonstationary signals (see, e.g., [8], [12]).

In the quaternion domain, signals can be classified as proper or improper, depending on the vanishing or nonvanishing, respectively, of their complementary functions (i.e. correlation functions between the quaternion signal and its three involutions) [20]. Moreover, there exist two main types of properness: $\mathbb{Q}$-properness if the three complementary functions are zero and $\mathbb{C}^\eta$-properness if all the complementary functions cancel except that one corresponding to the involution η, and the improper case where none of the complementary functions vanish. According to the type of properness, a different kind of linear processing should be applied. In the more general case of improper signals, in order to provide a complete description of the second-order statistical properties of a quaternion random signal $x(k)$, an augmented quaternion signal vector[2] of the form $\mathbf{x}^q(k) = [x(k), x^\eta(k), x^{\eta'}(k), x^{\eta''}(k)]^{\mathrm{T}}$ is required. Then, the optimal linear processing is the WL processing which means to operate simultaneously on a four-dimensional vector whose elements are chosen among the signal, its conjugate, and the three potential involutions. WL processing has proved to outperform the conventional or SL processing, that does not need to operate on the involutions (see, e.g., [13], [2]).

Notice that the following relation can be established between the augmented signal vector $\mathbf{x}^q(k)$ and the real vector $\mathbf{x}^r(k) = [x_r(k), x_\eta(k), x_{\eta'}(k), x_{\eta''}(k)]^{\mathrm{T}}$:

$$\mathbf{x}^q(k) = \boldsymbol{\mathcal{A}}\mathbf{x}^r(k) \tag{3}$$

where

$$\boldsymbol{\mathcal{A}} = \begin{bmatrix} 1 & \eta & \eta' & \eta'' \\ 1 & \eta & -\eta' & -\eta'' \\ 1 & -\eta & \eta' & -\eta'' \\ 1 & -\eta & -\eta' & \eta'' \end{bmatrix}$$

with $\boldsymbol{\mathcal{A}}^{\mathrm{H}}\boldsymbol{\mathcal{A}} = 4\mathbf{I}_4$.

Moreover, the following property of the product $\star$ defined in (1) is verified.

Property 1 *The augmented vector of $s(k,l) = x(k) \star y(l)$ is*

$$\mathbf{s}^q(k,l) = \frac{1}{4}\boldsymbol{\mathcal{A}}\,\mathrm{diag}(\mathbf{x}^r(k))\boldsymbol{\mathcal{A}}^H\bar{\mathbf{y}}(l)$$

Next, based on the augmented quaternion signal vector, a new class of signal is introduced by imposing the condition of factorizable kernel on the correlation function $\mathbf{R}_{\mathbf{x}^q}(k,l) = E[\mathbf{x}^q(k)\mathbf{x}^{q^{\mathrm{H}}}(l)]$ of the augmented quaternion signal vector $\mathbf{x}^q(k)$. This type of quaternion signal, called *widely factorizable* , is defined as follows.

[1] The Hadamard product of two $n \times m$ matrices $\mathbf{A} = [A_{i,j}]$ and $\mathbf{B} = [B_{i,j}]$ is defined as a $n\times m$ matrix whose elements are $\mathbf{A}\odot\mathbf{B} = [A_{i,j}B_{i,j}]$.

[2] Note that, this augmented quaternion signal vector can be defined from any combination of four elements among the signal, its conjugate, and the three perpendicular quaternion involutions $x^\nu(k) = -\nu x(k)\nu'$, $\nu = \eta, \eta', \eta''$ or their conjugates.

Definition 4 *A quaternion signal $x(k) \in \mathbb{H}$ is said to be widely factorizable if and only if there exist two $4 \times n$-matrices $\mathbf{A}(k), \mathbf{B}(k) \in \mathbb{H}^{4\times n}$ such that the correlation function $\mathbf{R}_{\mathbf{x^q}}(k,l)$ of the augmented vector $\mathbf{x}^q(k)$ can be expressed as*

$$\mathbf{R}_{\mathbf{x^q}}(k,l) = \begin{cases} \mathbf{A}(k)\mathbf{B}^{H}(l), & k \geq l \\ \mathbf{B}(k)\mathbf{A}^{H}(l), & k \leq l \end{cases} \tag{4}$$

Note that all widely factorizable quaternion signals are also factorizable ((4) implies (2)), but the converse does not hold (condition (2) does not assure that the correlation function of the augmented vector satisfies (4)). Strategies for the factorization of the correlation function of an augmented quaternion vector can be found in [21].

Finally, the cross-correlation function between any two quaternion augmented signal vectors $\mathbf{x}^q(k)$ and $\mathbf{y}^q(k)$ will be denoted by $\mathbf{R}_{\mathbf{x^q y^q}}(k,l) = E\left[\mathbf{x}^q(k)\mathbf{y}^{q^{\mathrm{H}}}(l)\right]$ and $\mathbf{r}_{x\mathbf{y}^q}(k,l) = E\left[x(k)\mathbf{y}^{q^{\mathrm{H}}}(l)\right]$ will represent the cross-correlation function between $x(k)$ and the quaternion augmented vector $\mathbf{y}^q(k)$.

3. PROBLEM STATEMENT

Consider a widely factorizable quaternion signal vector $x(k) \in \mathbb{H}$ which is observed through the following linear equation:

$$y(k) = \gamma(k) \star x(k) + v(k) \tag{5}$$

where the components of $\gamma(k) \in \mathbb{H}$, $\gamma_\nu(k)$, $\nu = r, \eta, \eta', \eta''$, are independent Bernoulli random variables with parameter $p_\nu(k)$, which indicates the presence ($\gamma_\nu(k) = 1$) or absence ($\gamma_\nu(k) = 0$) of the quaternion signal component $x_\nu(k)$ in the observation. Moreover, $v(k)$ is a quaternion white noise [3].

With the purpose of a WL processing, we aim to find the linear least-mean square error estimator $\hat{x}^{WL}(k|k)$ of the quaternion signal $x(k)$ based on the augmented quaternion observations set $\{\mathbf{y}^q(1), \mathbf{y}^q(2), \ldots, \mathbf{y}^q(k)\}$. Observe that, from Property 1, the augmented quaternion observations obey the following equation:

$$\mathbf{y}^q(k) = \mathbf{\Gamma}(k)\mathbf{x}^q(k) + \mathbf{v}^q(k) \tag{6}$$

where $\mathbf{\Gamma}(k) = \frac{1}{4}\mathcal{A}\,\mathrm{diag}(\boldsymbol{\gamma}^r(k))\mathcal{A}^{\mathrm{H}}$, with $\mathcal{A}$ defined in (3) and $\boldsymbol{\gamma}^r(k) = [\gamma_r(k), \gamma_\eta(k), \gamma_{\eta'}(k), \gamma_{\eta''}(k)]^{\mathrm{T}}$.

It is well known that this estimator can be expressed as a linear function of the set of augmented quaternion observations as follows:

$$\hat{x}^{WL}(k|k) = \sum_{j=1}^{k} \mathbf{h}^{\mathrm{T}}(k,j)\mathbf{y}^q(j), \tag{7}$$

[3]The quaternion white noise $v(k)$ is defined as $v(k) = v_r(k) + \eta v_\eta(k) + \eta' v_{\eta'}(k) + \eta'' v_{\eta''}(k)$ where $v_r(k)$, $v_\eta(k)$, $v_{\eta'}(k)$ and $v_{\eta''}(k)$ are real-valued white noises.

where the four-dimensional vector $\mathbf{h}(k,j)$, is the impulse response function satisfying the equation

$$\mathbf{r}_{x\mathbf{y}^q}(k,j) = \sum_{i=1}^{k} \mathbf{h}^{\mathrm{T}}(k,i)\mathbf{R}(i,j) + \mathbf{h}^{\mathrm{T}}(k,j)\mathbf{Q}(i), \quad 1 \leq j \leq k \tag{8}$$

where $\mathbf{R}(i,j) = \mathbf{\Pi}(i)\mathbf{R}_{\mathbf{x}^q}(i,j)\mathbf{\Pi}(j)$, with $\mathbf{\Pi}(i)$ denoting the 4×4-diagonal matrix $\mathbf{\Pi}(i) = \frac{1}{4}\boldsymbol{\mathcal{A}}\,\mathrm{diag}\left(E[\boldsymbol{\gamma}^r(i)]\right)\boldsymbol{\mathcal{A}}^H$, and $E[\mathbf{v}^q(i)\mathbf{v}^{q^{\mathrm{H}}}(i)] = \mathbf{Q}(i)$.

Note that the problem is completely determined from the computation of the impulse response function by solving equation (8). Nevertheless, our objective here is to devise a recursive algorithm for the computation of such an estimate. In the following Section, the formulas for the recursive computation of the filter (7) and its associated error $p(k|k) = E[|x(k) - \hat{x}^{WL}(k|k)|^2]$ are displayed.

4. FILTERING ALGORITHM

Algorithm 1 *The WL filter $\hat{x}^{WL}(k|k)$ defined in (7) can be recursively computed as follows:*

$$\hat{x}^{WL}(k|k) = \mathbf{a}_{[1]}(k)\boldsymbol{\epsilon}(k), \quad k \geq 1$$

where the n-dimensional vector $\boldsymbol{\epsilon}(k)$ satisfies this recursive formula

$$\begin{aligned} &\boldsymbol{\epsilon}(k) = \boldsymbol{\epsilon}(k-1) + \mathbf{J}(k)\left[\mathbf{y}^q(k) - \mathbf{\Pi}(k)\mathbf{A}(k)\boldsymbol{\epsilon}(k-1)\right] \\ &\boldsymbol{\epsilon}(0) = \mathbf{0}_n \end{aligned}$$

with the $n \times 4$-matrix $\mathbf{J}(k)$ given by the expression

$$\mathbf{J}(k) = [\mathbf{B}(k) - \mathbf{A}(k)\mathbf{S}(k-1)]^{H}\,\mathbf{\Pi}(k)\mathbf{\Omega}^{-1}(k)$$

where the $n \times n$-matrix $\mathbf{S}(k)$ satisfies the recursive equation

$$\begin{aligned} &\mathbf{S}(k) = \mathbf{S}(k-1) + \mathbf{J}(k)\mathbf{\Omega}^{-1}(k)\mathbf{J}^{H}(k) \\ &\mathbf{S}(0) = \mathbf{0}_{n\times n} \end{aligned}$$

and the 4×4-matrix $\mathbf{\Omega}(k)$ is of the form

$$\mathbf{\Omega}(k) = \mathbf{\Sigma} + \mathbf{Q}(k) - \mathbf{\Pi}(k)\mathbf{A}(k)\mathbf{S}(k-1)\mathbf{A}^{H}(k)\mathbf{\Pi}(k)$$

with $\mathbf{\Sigma} = \boldsymbol{\mathcal{A}}\,\mathrm{diag}\left(E\left[\boldsymbol{\gamma}^r(k)\boldsymbol{\gamma}^{r^{T}}(k)\right] \odot E\left[\mathbf{x}^r(k)\mathbf{x}^{r^{T}}(k)\right]\right)\boldsymbol{\mathcal{A}}^{H}$.

The associated WL filtering error is obtained as follows

$$p^{WL}(k|k) = r_x(k,k) - \mathbf{a}_{[1]}(k)\mathbf{S}(k)\mathbf{a}^{H}_{[1]}(k) \tag{9}$$

5. NUMERICAL EXAMPLE

In this section, the efficiency of the proposed WL filtering algorithm is numerically analyzed. Specifically, the better behavior of the WL filter in the improper case with respect to the SL counterpart is illustrated.

With this purpose, the signal of interest is considered to be an improper quaternion Wiener process [22] whose augmented correlation function can be expressed in the form (4), where

$$A(k) = \begin{bmatrix} a & b & c & d \\ b^{\eta} & a^{\eta} & d^{\eta} & c^{\eta} \\ c^{\eta'} & d^{\eta'} & a^{\eta'} & b^{\eta} \\ d^{\eta''} & c^{\eta''} & b^{\eta''} & a^{\eta''} \end{bmatrix} \qquad B(k) = k\mathbf{I}_{4\times 4}$$

with $a = 1.4676$, $b = -0.2876 + 0.2720\mathrm{j} + 1.0640\mathrm{k}$, $c = -0.0324 - 0.7096\mathrm{i} - 0.0240\mathrm{j}$, and $d = -0.1476 + 0.1096\mathrm{i} - 0.7520\mathrm{j}$. Moreover, the signal is assumed to be observed through the equation (5) where the measurement noise v_n is a $\mathbb{Q}$-proper quaternion white Gaussian noise with variance parameter $E[v(k)v^*(k)] = 0.04$, and the Bernoulli random variables $\gamma_\nu(k)$ have constant probabilities $P[\gamma_\nu(k) = 1] = p_\nu$, $\nu = r, \eta, \eta', \eta''$.

In this framework, Algorithm 1 is applied in order to compute the WL filtering errors $p^{WL}(k|k)$, which are compared with the SL counterparts $p^{SL}(k|k)$ by considering different probabilities p_r, p_η, $p_{\eta'}$ and $p_{\eta''}$ for the Bernoulli random variables. In detail, for each combination of these probabilities, the means of the SL and WL filtering errors, defined as $ME^{SL} = \frac{1}{k}\sum_{i=1}^{k} p^{SL}(i|i)$ and $ME^{WL} = \frac{1}{k}\sum_{i=1}^{k} p^{WL}(i|i)$ have been calculated. By fixing three of the Bernoulli probabilities and varying the other from 0.75 to 1, the difference of these means of the filtering errors ($DME = ME^{SL} - ME^{WL}$) are displayed in Fig. 1 for:

(a) $p_\eta = 0.95$, $p_{\eta'} = 0.8$, $p_{\eta''} = 0.75$, and p_r varying in the interval $[0.75, 1]$

(b) $p_r = 0.9$, $p_{\eta'} = 0.8$, $p_{\eta''} = 0.75$, and p_η varying in the interval $[0.75, 1]$

(c) $p_r = 0.9$, $p_\eta = 0.95$, $p_{\eta''} = 0.75$, and $p_{\eta'}$ varying in the interval $[0.75, 1]$

(d) $p_r = 0.9$, $p_\eta = 0.95$, $p_{\eta'} = 0.8$, and $p_{\eta''}$ varying in the interval $[0.75, 1]$

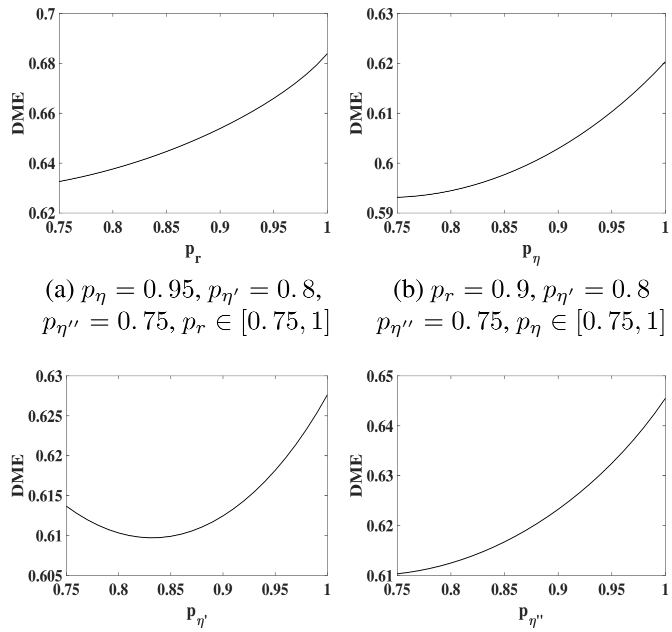
As would be expected, the superiority of WL filter over the SL one is clearly illustrated in these four figures ($DME > 0$) and also, we can observe that this superiority is higher as the probabilities of the Bernoulli random variables are close to one.

6. CONCLUSIONS

The problem of estimating a widely factorizable quaternion signal observed under uncertainty conditions has been analyzed.

Under these circumstances, the WL filtering estimate is expressed as a linear function of the augmented observations set whose impulse response is completely determined as the solution of the Wiener-Hopf equation. Then, from only the knowledge of the correlation function of the augmented vectors involved, a filtering algorithm is designed which can be applied without the necessity of postulating a state-space model.

The superiority of the proposed algorithm with respect to the conventional or SL counterpart has been illustrated on a numerical example.



(a) $p_\eta = 0.95$, $p_{\eta'} = 0.8$, $p_{\eta''} = 0.75$, $p_r \in [0.75, 1]$

(b) $p_r = 0.9$, $p_{\eta'} = 0.8$ $p_{\eta''} = 0.75$, $p_\eta \in [0.75, 1]$

(c) $p_r = 0.9$, $p_\eta = 0.95$, $p_{\eta''} = 0.75$, $p_{\eta'} \in [0.75, 1]$

(d) $p_r = 0.9$, $p_\eta = 0.95$, $p_{\eta'} = 0.8$, $p_{\eta''} \in [0.75, 1]$

Fig. 1. Difference of means of the SL and WL filtering errors.

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