Generalized and Differential Likelihood Ratio Tests with Quantum Signal Processing

Shihao Yan†, Robert Malaney‡, and Jinhong Yuan‡
†School of Engineering, Macquarie University, Sydney, NSW, Australia
‡School of Electrical Engineering and Telecommunications, The University of New South Wales, Sydney, Australia
Email: shihao.yan@mq.edu.au, {r.malaney, j.yuan}@unsw.edu.au

***Abstract*—Quantum signal processing invokes the injection of abstract quantum mechanical frameworks into classical signal processing problems. In this work we apply this idea to the notion of optimal likelihood ratio tests within the context of the location verification problem. We first draw parallels with quantum mechanical measurements and the notion of generalized likelihood ratio measurements. As we show, these quite different measurement frameworks are mathematically similar since both can be described in the language of projections into subspaces - the projections removing the nuisance parameters of the underlying system in the latter case. We then show how the imposition of an 'artificial' mathematical constraint, borrowed from a similar constraint imposed on quantum mechanics by the uncertainty principle, is likely to assist in machine-learning solutions of the location verification problem - such solutions being more useful in real-world deployments.**

I. Introduction

The term "Quantum Signal Processing" (QSP) was first coined in a series of works by Eldar and collaborators in 2001 [1]. The key idea of QSP is the introduction of conceptual quantum mechanical frameworks into the realm of classical signal processing with the idea of delivering novel algorithmic solutions that otherwise would go unnoticed. In some sense this idea borrows from a well-traveled path in engineering in the 'borrowing' from Nature of ideas and frameworks that lead to new insights. Swarming of bees and connections to routing within ad hoc mobile networks being one of many examples [2].

QSP borrows from Nature in the same sense in that it builds on Nature's most fundamental construction - Quantum Mechanics (QM). In the extended literature, QSP has been widely used to develop new (or improvements to existing) processing algorithms related to measurements, quantization, and consistency, e.g. [1, 3–6].

Within the abstract frameworks of QM lies a series of postulates (true but unprovable statements) from which physics' most successful theories are built. QM leads to a theory of measurements within which lies critical constraints that must be obeyed. These measurements can be described in the context of projections into subspaces conditioned on mathematical constraints imposed on the matrices underlying the QM system. This is perhaps most explicitly seen in the context of Gaussian CV (Continuous Variable) quantum states whose covariance matrix, V, written in terms of the quadrature operators of the electromagnetic field, must satisfy upon measurement $V + i\Omega \geq 0$, where Ω is of the symplectic form [7].

Here we apply the QSP idea to an important area in classical signal processing, the construction of a robust Location Verification System (LVS). In a LVS, a binary decision is made on an input data vector (the signal) as to whether the claimed location of a device is reliable or not (genuine or malicious). In a series of works, Yan *et al* [8–10] have constructed optimal LVSs under a series of hypothetical channel conditions and assumed signal metrics. As we see in the following, optimal decision theory within the context of a LVS in many cases collapses to a Generalized Likelihood Ratio Test (GLRT). The work of [10] further shows that such GLRTs can be constructed within the framework of a Differential Likelihood Ratio Test (D-LRT). However, optimal LVSs formed in this manner suffer from the assumptions upon which they are built, namely, idealized assumptions on the noise and channel conditions in which they operate.

In the real-world, the true channel and noise conditions are *a priori* unknown and dynamical. To combat this, a new direction for LVS construction has been recently proposed, namely, the coupling of machine learning and optimal decision theory [11]. It is this new framework for LVS that we suggest the QSP concept can be easily applied. Specifically, we suggest applying a mathematical constraint on the underlying signal matrices of the LVS similar (but less constrained) to that needed to satisfy the uncertainty principle in QM. This artificial introduction of uncertainty to the LVS turns out to be a very efficient means to overcome the issue of 'overtraining' within the context of neural-network based LVSs.

To make progress, we first draw parallels with the GLRT problem and projections into subspaces. Bear in mind in what is to follow our focus on GLRTs is largely in the context of location verification. However, the ideas we next explore can be implemented in many other problems that are based on the use of a GLRT.

II. Generalised Likelihood Ratio Tests

It is well known that the Likelihood Ratio Test (LRT) is the uniformly most powerful (UMP) test when there are no nuisance parameters in both the null and alternative hypotheses [12]. For the case in which the null hypothesis or the alternative hypothesis is composite (dependent on a nuisance parameter), the UMP test only exists for a one-sided test problem (monotonic distribution of the test statistic) [13]. For a two-sided test problem (non-monotonic distribution of the test statistic), in which the null hypothesis or the alternative hypothesis is composite, no UMP test exists [14] - although the Generalized Likelihood Ratio Test (GLRT) is known to be asymptotically optimal for such problems [13, 15]. Due to this, the GLRT has been widely adopted in many two-sided composite test problems, such as pixel (or target) detection [16, 17], spectrum sensing [18, 19], and signal detection [20, 21]. To obtain the GLRT, we first need to find the maximum likelihood estimations of all the nuisance parameters, followed by the conditional LRT. As such, from a signal processing perspective, the complexity of the GLRT is higher than that of the LRT.

We are particularly interested in the role played by nuisance parameters in the context of location verification [8]. Specifically, we are interested in the special composite hypothesis testing problem where *both* the null and alternative hypotheses are composite and the nuisance parameters are common at all elements of the *original observation vector*. In this specific problem each hypothesis is dependent on a different nuisance parameter (although our analysis will still apply when the two nuisance parameters are equal but not zero). In the context of some location verification settings, these two nuisance parameters represent the transmit powers of legitimate and malicious users. For location verification, the work of [10] also considered an LRT based on differential observations (D-LRT). Considering the property of our special problem, namely, that the nuisance parameters are common at all elements of the original observation vector, the D-LRT subtracts a reference original observation from the remaining original observations to obtain the differential observation vector - the latter being independent of the nuisance parameters. As such, the composite hypothesis testing problem can be solved by the D-LRT. As proved in [10], the performance of the D-LRT is exactly equivalent to that of the GLRT - in which the nuisance parameters first have to be estimated.

We find that using the D-LRT to solve the GLRT, problems can be constructed from within a QSP framework. Specifically, in the D-LRT we project the measurements into subspaces, which are orthogonal to the subspaces of the nuisance parameters. As such, the nuisance parameters no longer affect the system. In this work we discuss how to interpret the D-LRT from the perspective of quantum signal processing, and how we can use the QSP framework as a guide to the creation of a LVS more suitable for real-world deployment. As we progress, we will also prove some results presented in [9] with regard to the performance of D-LRT from the quantum signal processing perspective.

III. Binary Composite Hypothesis Testing Problem of Interest

We now outline the system model and state the assumptions adopted within the binary composite hypothesis testing problem. We denote the null hypothesis and the alternative hypothesis by $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively. The composite observation model is given by

$$\begin{cases} \mathcal{H}_0: \ \mathbf{y} = \theta_0 \mathbf{1}_N + \mathbf{u} + \mathbf{w} \\ \mathcal{H}_1: \ \mathbf{y} = \theta_1 \mathbf{1}_N + \mathbf{v} + \mathbf{w}, \end{cases} \tag{1}$$

where $\mathbf{y}$ is the $N \times 1$ original observation vector, θ_0 is an unknown scalar parameter under $\mathcal{H}_0$, θ_1 is an unknown scalar parameter under $\mathcal{H}_1$, $\mathbf{1}_N$ is the $N \times 1$ vector with all elements set to unity, $\mathbf{u}$ is an arbitrary signal $N \times 1$ vector (e.g., related to the timings at N base stations), $\mathbf{v}$ is another arbitrary signal $N \times 1$ vector, and $\mathbf{w}$ is the $N \times 1$ zero-mean multivariate normal random variable, of which the covariance matrix is $\mathbf{R}$ under both $\mathcal{H}_0$ and $\mathcal{H}_1$. Due to the additive noise $\mathbf{w}$, $\mathbf{y}$ under $\mathcal{H}_0$, conditional on θ_0, follows a multivariate normal distribution, which is given by

$$f\left(\mathbf{y}|\theta_0, \mathcal{H}_0\right) = \mathcal{N}(\theta_0 \mathbf{1}_N + \mathbf{u}, \mathbf{R}). \tag{2}$$

Likewise, $\mathbf{y}$ under $\mathcal{H}_1$ based on a known θ_1 also follows a multivariate normal distribution, which is given by

$$f\left(\mathbf{y}|\theta_1, \mathcal{H}_1\right) = \mathcal{N}(\theta_1 \mathbf{1}_N + \mathbf{v}, \mathbf{R}). \tag{3}$$

The problem of interest in this work is to test whether the observations are generated from $\mathcal{H}_0$ or $\mathcal{H}_1$, which is a binary hypothesis testing problem. Mathematically, the binary hypothesis testing problem is given by

$$\psi(\mathbf{y}) \underset{\mathcal{D}_0}{\overset{\mathcal{D}_1}{\gtrless}} \lambda, \tag{4}$$

where $\psi(\mathbf{y})$ is some specific function of $\mathbf{y}$, λ is a threshold corresponding to $\psi(\mathbf{y})$, and $\mathcal{D}_0$ and $\mathcal{D}_1$ are the binary decisions that infer whether $\mathbf{y}$ is from $\mathcal{H}_0$ or $\mathcal{H}_1$, respectively.

IV. Generalized Likelihood Ratio Test (GLRT) Based on Original Observations

When some parameters in the observation model are unknown, the binary detection problem presented in (4) becomes a composite hypothesis testing problem with the unknown model parameters treated as nuisances. The GLRT is asymptotically optimal in the case where no UMP test exists [13]. As such, although not necessarily optimal, the GLRT is widely used to solve the composite hypothesis testing problem. We next characterize the performance of the GLRT based on $\mathbf{y}$ by deriving its false positive rates and its detection rates.

The binary decision rule embedded in the GLRT based on the original observations obtained from (1) is given by

$$\Lambda\left(\mathbf{y}\right) \triangleq \frac{f\left(\mathbf{y}|\hat{\theta}_1, \mathcal{H}_1\right)}{f\left(\mathbf{y}|\hat{\theta}_0, \mathcal{H}_0\right)} \underset{\mathcal{D}_0}{\overset{\mathcal{D}_1}{\gtrless}} \lambda_R, \tag{5}$$

where $\Lambda\left(\mathbf{y}\right)$ is the likelihood ratio of $\mathbf{y}$, λ_R is the threshold corresponding to $\Lambda\left(\mathbf{y}\right)$, and $\hat{\theta}_0$ and $\hat{\theta}_1$ are the maximum-likelihood estimations of θ_0 and θ_1, respectively. As presented in [10], the final decision rule for the GLRT can be derived from (5), and from which the detection performance of the GLRT can be analyzed.

From the perspective of quantum signal processing, we can understand the GLRT problem in the context of location verification as a projection into subspaces. This issue is described more in the following section where we show how the GLRT can be viewed as a D-LRT with connections into the projecting out of nuisance parameters (see also [22]).

V. Projecting Measurements into Subspaces without Nuisance Parameters

A. Differential Likelihood Ratio Test

In this subsection, we present the D-LRT from the perspective of QSP.

From (1), we can see that θ_0 and θ_1 are constant at all elements of the original observation vector $\mathbf{y}$ under $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively. As such, we can obtain differential observations by subtracting a reference original observation from all the remaining observations. Such differential observations are not functions of θ_0 and θ_1 anymore. Therefore, the composite hypothesis testing problem presented in (4) can also be solved by applying the D-LRT, where we project the measurements into the null space associated with the nuisance parameters.

We achieve the $(N-1)$ basic differential observations from the N original observations by subtracting the N-th original observation from all other $(N-1)$ original observations. As such, the m-th differential observation under $\mathcal{H}_0$ is given by

$$\Delta y_m = \Delta u_m + \Delta w_m, \quad m = 1, 2, \ldots, N-1, \qquad (6)$$

where $\Delta u_m = u_m - u_N$, and $\Delta w_m = w_m - w_N$. We note that Δw_m is Gaussian with zero mean and variance $2(\sigma_R^2 - R_{mN})$. We denote the $(N-1)\times(N-1)$ covariance matrix of the $(N-1)$-dimension differential observation vector $\mathbf{\Delta y} = [\Delta y_1, \ldots, \Delta y_{N-1}]^T$ as $\mathbf{D}$. Then, we have

$$D_{mn} = R_{NN} + R_{mn} - R_{mN} - R_{nN}. \qquad (7)$$

where D_{mn} is the (m,n)-th element of $\mathbf{D}$ and R_{mn} is the (m,n)-th element of $\mathbf{R}$. As such, we have

$$f\left(\mathbf{\Delta y}|\mathcal{H}_0\right) = \mathcal{N}(\mathbf{\Delta u}, \mathbf{D}), \qquad (8)$$

where $\mathbf{\Delta u} = [\Delta u_1, \ldots, \Delta u_{N-1}]^T$ is the mean vector.

Likewise, the m-th differential observation under $\mathcal{H}_1$ is

$$\Delta y_m = \Delta v_m + \Delta w_m, \qquad (9)$$

where $\Delta v_m = v_m - v_N$. Noting $\mathbf{\Delta v} = [\Delta v_1, \ldots, \Delta v_{N-1}]^T$, we have

$$f\left(\mathbf{\Delta y}|\mathcal{H}_1\right) = \mathcal{N}(\mathbf{\Delta v}, \mathbf{D}). \qquad (10)$$

Since $f\left(\mathbf{\Delta y}|\mathcal{H}_0\right)$ and $f\left(\mathbf{\Delta y}|\mathcal{H}_1\right)$ are both normal functions, the false positive rates and the detection rates of the D-LRT are derived as [8]

$$\alpha_D = \mathcal{Q}\left(\frac{\ln\lambda_D + \frac{1}{2}\left(\mathbf{\Delta v}–\mathbf{\Delta u}\right)^T \mathbf{D}^{-1}\left(\mathbf{\Delta v}–\mathbf{\Delta u}\right)}{\sqrt{\left(\mathbf{\Delta v}–\mathbf{\Delta u}\right)^T \mathbf{D}^{-1}\left(\mathbf{\Delta v}–\mathbf{\Delta u}\right)}}\right), \quad (11)$$

$$\beta_D = \mathcal{Q}\left(\frac{\ln\lambda_D - \frac{1}{2}\left(\mathbf{\Delta v}–\mathbf{\Delta u}\right)^T \mathbf{D}^{-1}\left(\mathbf{\Delta v}–\mathbf{\Delta u}\right)}{\sqrt{\left(\mathbf{\Delta v}–\mathbf{\Delta u}\right)^T \mathbf{D}^{-1}\left(\mathbf{\Delta v}–\mathbf{\Delta u}\right)}}\right), \quad (12)$$

where λ_D is the threshold corresponding to the likelihood ratio of $\mathbf{\Delta y}$, which is $\Lambda\left(\mathbf{\Delta y}\right) \triangleq f\left(\mathbf{\Delta y}|\mathcal{H}_1\right)/f\left(\mathbf{\Delta y}|\mathcal{H}_0\right)$.

B. Properties of the Considered Measurement Projection

As proved in [10], the detection performance of the D-LRT (measurement projection) is exactly the same as that of the GLRT, which is asymptotically optimal. In this subsection, we present some properties of this projection method together with the associated proofs in the following propositions.

***Proposition** 1: For any $N > 1$ the detection performance of the D-LRT based on N difference measurements is at least as good as the performance of the GLRT for $N+1$ non-difference measurements.*

Proof: Following [9], in order to prove Proposition 1 we only have to prove

$$\underbrace{\sum_{i=1}^{N} g_i^2 - \frac{1}{N}\left(\sum_{i=1}^{N} g_i\right)^2}_{\varphi(N)} \leq \underbrace{\sum_{i=1}^{N+1} g_i^2 - \frac{1}{N+1}\left(\sum_{i=1}^{N+1} g_i\right)^2}_{\varphi(N+1)}, \qquad (13)$$

where the g_i are components of the vector $\mathbf{g} = \mathbf{v} - \mathbf{u}$ (for the RHS of above the vector becomes of length $N+1$). Consider

$$\begin{aligned}
&\varphi(N+1) - \varphi(N) \\
&= g_{N+1}^2 - \frac{1}{N+1}\left(\sum_{i=1}^{N} g_i + g_{N+1}\right)^2 + \frac{1}{N}\left(\sum_{i=1}^{N} g_i\right)^2 \\
&= g_{N+1}^2 - \frac{1}{N+1}\left[\left(\sum_{i=1}^{N} g_i\right)^2 + 2\left(\sum_{i=1}^{N} g_i\right) g_{N+1} + g_{N+1}^2\right] \\
&\quad + \frac{1}{N}\left(\sum_{i=1}^{N} g_i\right)^2 \\
&= \frac{N g_{N+1}^2}{N+1} + \frac{1}{N(N+1)}\left(\sum_{i=1}^{N} g_i\right)^2 - \frac{2 g_{N+1}}{N+1}\left(\sum_{i=1}^{N} g_i\right) \\
&= \frac{1}{N(N+1)}\left(N g_{N+1} - \sum_{i=1}^{N} g_i\right)^2. \qquad (14)
\end{aligned}$$

Following (14), we have $\varphi(N+1) - \varphi(N) \geq 0$. This completes the proof of (13) and therefore Proposition 1. ■

We note that in the D-LRT the equality $\varphi(N+1) - \varphi(N) = 0$ requires $g_{N+1} = \frac{1}{N}\sum_{i=1}^{N} g_i$. Such a circumstance can only occur in the most unusual of circumstances (such as when $\mathbf{v} = \mathbf{u}$). As such, in practice increases in N will effectively always lead to an improvement in the D-LRT performance.

***Proposition** 2: In the D-LRT, any of the original measurements can be selected as the reference measurement, and this selection does not effect the performance of the D-LRT.*

Proof: Suppose we select the τ-th measurement, instead of the N-th one as the reference measurement. Following [9] we have

$$(\mathbf{\Delta v}–\mathbf{\Delta u})^T \mathbf{D}^{-1} (\mathbf{\Delta v}–\mathbf{\Delta u})$$
$$= \sum_{i=1, i\neq\tau}^{N} (g_i - g_\tau)^2 - \frac{1}{N}\left[\sum_{i=1, i\neq\tau}^{N} (g_i - g_\tau)\right]^2 . \quad (15)$$

Noting $g_i - g_\tau = 0$ for $i = \tau$, following (6) we have

$$(\mathbf{\Delta v}–\mathbf{\Delta u})^T \mathbf{D}^{-1} (\mathbf{\Delta v}–\mathbf{\Delta u})$$
$$= \sum_{i=1}^{N} (g_i - g_\tau)^2 - \frac{1}{N}\left[\sum_{i=1}^{N} (g_i - g_\tau)\right]^2$$
$$= \left[\sum_{i=1}^{N} g_i^2 - \frac{1}{N}\left(\sum_{i=1}^{N} g_i\right)^2\right] . \quad (16)$$

Based on (16) we can see that $(\mathbf{\Delta v}–\mathbf{\Delta u})^T \mathbf{D}^{-1} (\mathbf{\Delta v}–\mathbf{\Delta u})$ does not depend on τ. Following (11) and (12), we know the detection performance of the D-LRT is not dependent on the selection of the reference measurement, which completes the proof of this proposition. ■

We note that although the theoretical performance of the D-LRT does not depend on the selection of the reference measurement, in practice this may not be always true. For example, if the variance of the receiver noise is not identical for all the measurements, selecting the measurement with the minimum receiver noise as the reference measurement should lead to the highest performance of the D-LRT. In addition, considering quantization errors in practice, the selection of the reference measurement may also effect the performance of the D-LRT.

Having discussed that D-LRT can be simply posed as a projection into a subspace (that projects out nuisance parameters), let us now discuss connections to our problem from the viewpoint of QSP. Introducing the bosonic annihilation and creation operators $\hat{a}_k$ and $\hat{a}_k^\dagger$, respectively, we can formally introduce the quadrature operators $\hat{q}_k = \hat{a}_k + \hat{a}_k^\dagger$ and $\hat{p}_k = i\left(\hat{a}^\dagger - \hat{a}_k\right)$ (here we have set $\hbar = 2$). Introducing the vector $\hat{x} = (\hat{q}_1, \hat{p}_1 \ldots \hat{q}_{N_n}, \hat{p}_{N_n})$ we have that in QM the commutation relation

$$[\hat{x}_i, \hat{x}_j] = 2i\Omega_{ij}$$

must be satisfied. The uncertainty principal arises fundamentally from the need to satisfy such commutation relations. It is straightforward to show how the above relations lead naturally to the so-called ‘coherent states’ that describe laser light. These states are the eigenstates satisfying $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$ and are given by

$$|\alpha\rangle = \exp\left(-\frac{1}{2}\,|\alpha|^2\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle \ ,$$

where $|n\rangle$ is the n-photon number (Fock) state. A quantum measurement can in general be described by a set of projection operators E_i, which satisfy $\sum_i E_i^\dagger E = I$. Given this, it is then possible to show that measurement of the quadratures via heterodyne measurement is nothing other than a projection into coherent states with $E(\alpha) = \pi^{-1/2}\,|\alpha\rangle\,\langle\alpha|$.

Clearly, we are already in a QSP scenario if we replace these quantum projection operators with those arising from the D-LRT. Further, our constraint $V + i\Omega \geq 0$, mentioned in the introduction arises fundamentally for the need to satisfy the commutation relations mentioned above. For our needs it is useful to note that another representation of the above constraint $V + i\Omega \geq 0$. This can be re-stated in terms of the symplectic eigenspectrum $\{\nu_k\}_{k=1}^{N_n}$ (N_n being the number of nodes under consideration). This spectrum can, in turn, be determined simply as the standard eigenspectrum of the matrix $|i\Omega V|$. In this form the uncertainty principle can be re-stated simply as $\nu_k \geq 1$, for all k.

Given the obvious similarities between measurements within the classical and quantum frameworks we take the next step and hypothesize a new form of uncertainty that enters the classical arena, by mimicking the mathematical constraint that arises from the uncertainty principle in QM. This is explicitly entered into our formalism through an additional constraint on the matrix described by (7). We suggest this constraint follows the similar one imposed on the covariance matrix of CV Gaussian states, namely, $\nu_k \geq K_{NN}$.

Note, that here we have relaxed this type of ‘uncertainty principle’ by adding a new constant K_{NN}. The value of this constant is to be ‘learned’ by a neural network based LVS.

Detailed calculations and the determination of the neural-network architectures required to support such LVSs is beyond the scope of this introductory article. However calculations along similar lines can be found in [11] where neural-network architectures have been deployed with training curtailed by theoretical constraints similar to those discussed here.

From a comparison of the discussions above with those given in [11] (especially in relation to the overtraining problem), one can at least see how the introduction of the constraint $\nu_k \geq K_{NN}$ greatly simplifies the machine-learning process. Training, relative to the training discussed in [11] becomes greatly simplified, which in turn translates into simple neural-network architectures that are easier to design and more efficient to process.

VI. Conclusion

In this work we have introduced the notion of QSP into the realm of classical location verification. Drawing parallels between quantum measurement theory and likelihood ratio tests, we have outlined how the concept of QSP may lead to improved real-world LVSs that utilize machine learning as a means to combat uncertain channel (and noise) conditions. The simplicity of the constraint imposed through the QSP framework leads to much simpler and therefore more efficient LVS designs.

Acknowledgments

This work was partially supported by the Macquarie University Research Fellowship Scheme, and Australia Research Council Projects DP160104566 and LP160100708.

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