

# TIME-BASED SAMPLING AND RECONSTRUCTION OF NON-BANDLIMITED SIGNALS

Roxana Alexandru and Pier Luigi Dragotti

Department of Electrical and Electronic Engineering,  
Imperial College London, South Kensington, London SW7 2AZ, UK  
email: roxana.alexandru12@imperial.ac.uk, p.dragotti@imperial.ac.uk

## ABSTRACT

The last two decades have seen a renewed interest in sampling theory, which is concerned with the conversion of continuous-domain signals into discrete sequences. This conversion is traditionally achieved by recording the intensity of the signal at specified time instants. Alternatively, sampling can be based on timing rather than amplitude information. In this paper, we investigate the problem of timing-based sampling of non-bandlimited signals, within the Finite Rate of Innovation (FRI) setting. We show how these signals can be non-uniformly sampled using a compact-support kernel that satisfies the generalised Strang-Fix conditions, and a comparator. We then prove that perfect input estimation is possible using a novel local reconstruction algorithm.

**Index Terms**— Analog-to-digital conversion, non-uniform sampling, finite rate of innovation, time encoding machine.

## 1. INTRODUCTION

Sampling theory, which bridges the continuous and discrete-time domains, has recently experienced renewed interest [1]. This is mostly due to the emergence of two inter-related theories, compressed sensing [2, 3] and finite rate of innovation [4–8], that have shown that it is possible to sample and perfectly reconstruct classes of sparse non-bandlimited signals. These approaches, however, are both essentially based on recording the intensity of the signal at specific time instants, as done in classical sampling.

An alternative method to classical sampling is the time encoding mechanism, which maps the amplitude information of a signal into a non-uniform time sequence. Sampling based on timing is inspired by nature as it captures the way neurones encode information, and leads to energy-efficient analog to digital conversions. In this context, typical acquisition models are based on an integrator, whilst other devices capture the timing information using a comparator. Methods for reconstructing signals from timing information have been presented in [9–12], and are based on connecting this problem to that of non-uniform sampling in shift-invariant spaces [13, 14]. Therefore, these methods mostly focus on the retrieval of bandlimited signals, and signals in shift-invariant spaces.

In this paper, we focus on particular classes of continuous-

time non-bandlimited signals such as streams of Diracs or piecewise constant signals, and show that it is possible to perfectly reconstruct them, from samples obtained using a time encoding mechanism based on a filter and a comparator. In Section 2 we show that sampling kernels which reproduce exponentials or polynomials preserve this property locally, when sampling is based on timing information. Furthermore, in Section 3 we propose a sequential algorithm for the retrieval of an input stream of Diracs, and extend this method to piecewise constant polynomial signals. Then, in Section 4 we highlight the sufficient conditions for perfect retrieval of these signals from non-uniform samples. Section 5 shows that reconstruction of these signals from their timing information is exact to numerical precision. Finally, we conclude in Section 6.

## 2. NON-UNIFORM SAMPLING OF FRI SIGNALS

In this section, we describe the acquisition model and highlight key features of the kernel used in our sampling scheme. In particular, we focus on the family of exponential and polynomial reproducing kernels, as they have compact support and have been extensively used in FRI sampling.

### A. Acquisition Model

We consider the time encoding strategy depicted in Fig. 1, which relies on a filter  $\varphi(-t)$ , and a comparator with reference  $g(t)$ . The output of the acquisition device is the sequence  $\{t_n\}$ , corresponding to the time instants when the filtered input signal crosses the reference, i.e. when  $y(t_n) - g(t_n) = 0$ . Moreover, since the value of the test function  $g(t)$  is known, we can retrieve the amplitudes of the output samples, given by  $y_n = y(t_n) = g(t_n)$ . Hence, decoding the input signal is equivalent to a non-uniform sampling problem, where we aim to reconstruct  $x(t)$  from the non-uniform samples defined as:

$$y_n = \int x(\tau)\varphi(\tau - t_n)d\tau = \langle x(t), \varphi(t - t_n) \rangle. \quad (1)$$

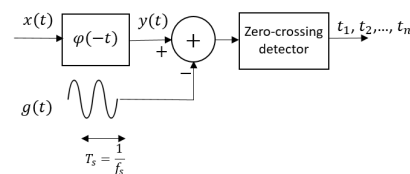


Fig. 1: Crossing Time Encoding Machine.

## B. Sampling Kernels

### 1. Polynomial reproducing kernels

A kernel is able to reproduce polynomials of maximum degree  $P$ , if together with its shifted versions, it satisfies:

$$\sum_{n \in \mathbb{Z}} c_{m,n} \varphi(t - n) = t^m,$$

where  $m \in \{0, 1, \dots, P\}$ , and for a proper choice of  $c_{m,n}$ .

Any function that satisfies the Strang-Fix conditions can be used as a polynomial reproducing kernel [15]. One example of such functions are the B-splines [16]. The zero-order B-spline can reproduce constant polynomials, and is given by:

$$\beta_0(t) = \begin{cases} 1, & -1 \leq t < 0, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, the first-order B-spline is defined as:

$$\beta_1(t) = (\beta_0 * \beta_0)(t) = \begin{cases} -t, & -1 \leq t \leq 0, \\ 2 + t, & -2 \leq t \leq -1, \\ 0, & \text{otherwise.} \end{cases}$$

We show that the polynomial reproducing property of the first-order B-spline is *locally* preserved in the context of non-uniform sampling. In other words, we prove the following condition is satisfied within a time interval  $I$ , for  $N \geq 2$ :

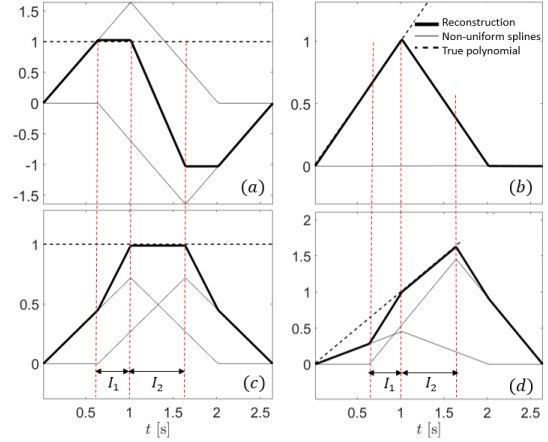
$$\sum_{n=1}^N c_{m,n}^I \beta_1(t - t_n) = t^m, \quad (2)$$

where  $m \in \{0, 1\}$ ,  $N$  is the number of overlapping splines with no discontinuities in the interval  $I$ , and  $t \in I$ .

It is sufficient to show that the condition in Eq. (2) is satisfied for  $N = 2$ , to prove that it holds for  $N > 2$ . Hence, we aim to show that it is possible to find the coefficients  $c_{m,1}^I$  and  $c_{m,2}^I$  such that  $c_{m,1}^I \beta_1(t - t_1) + c_{m,2}^I \beta_1(t - t_2) = t^m$ .

Any two non-zero regions  $v_1(t) = \beta_1(t - t_1)$  and  $v_2(t) = \beta_1(t - t_2)$  form a basis for the vector spaces of constant and linear polynomials. This is because they represent linearly independent 2D vectors, and for any constant or linear polynomial  $x$ , there is a unique sequence  $c \in \mathbb{C}^2$  such that  $x = c_1 v_1 + c_2 v_2$ . For example,  $v_1(t) = -t + t_1$  and  $v_2(t) = -t + t_2$  are clearly independent for  $t_1 \neq t_2$ . Moreover, any constant polynomial  $C$  can be written as a linear combination of  $v_1$  and  $v_2$ :  $c_{0,1}(-t + t_1) + c_{0,2}(-t + t_2) = C$ , with unique  $c_{0,1} = \frac{C}{t_1 - t_2}$  and  $c_{0,2} = \frac{C}{t_2 - t_1}$ . In addition, any linear polynomial can be represented by  $v_1$  and  $v_2$  as:  $c_{1,1}(-t + t_1) + c_{1,2}(-t + t_2) = at + b$ , and with unique  $c_{1,1} = \frac{at_2 + b}{t_1 - t_2}$  and  $c_{1,2} = \frac{at_1 + b}{t_2 - t_1}$ .

Similarly, we can show that reproduction of constant and linear polynomials is possible on any time interval spanned by continuous non-zero segments of the two kernels. As a result, Eq. (2) holds for  $N = 2$  and therefore, for  $N > 2$ . Fig. 2 illustrates the local reproduction of constant and linear polynomials, within two different intervals  $I_1$  and  $I_2$ . It



**Fig. 2:** Perfect reproduction of constant and linear polynomials in two different time intervals,  $I_1 = [0.625, 1]s$  in (a) and (b), and  $I_2 = [1, 1.625]s$  in (c) and (d), overlapped by continuous regions of two non-uniformly shifted first-order B-splines.

is evident that  $c_{m,n}^{I_1} \neq c_{m,n}^{I_2}$ , and that the polynomial reconstruction breaks outside the continuous regions.

Furthermore, the convolution  $(\beta_0 * q_\theta)(t)$  locally reproduces constant and linear polynomials, in the interval  $I$ :

$$\sum_{n=1}^N c_{m,n}^I (\beta_0 * q_\theta)(t - t_n) = t^m,$$

where  $\beta_0(t)$  is the zero-order B-spline,  $m \in \{0, 1\}$ ,  $N$  is the number of splines with no discontinuities in  $I$ , and the stretched box function is defined as:

$$q_\theta(t) = \begin{cases} 1, & 0 \leq t \leq \theta, \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

### 2. Exponential reproducing kernels

An exponential reproducing kernel is a function  $\varphi(t)$  that, together with its uniformly shifted versions, reproduces exponentials of the form  $e^{\alpha_m t}$ , with  $\alpha_m = \alpha_0 + m\lambda$ :

$$\sum_{n \in \mathbb{Z}} c_{m,n} \varphi(t - n) = e^{\alpha_m t}.$$

A family of functions suited for exponential reproduction are the E-splines [17], that satisfy the generalised Strang-Fix conditions [18, 19]. As for the polynomial case, we can prove that a linear combination of non-uniform shifts of the first-order E-spline locally reproduces exponentials.

## 3. RECONSTRUCTION OF FRI SIGNALS FROM NON-UNIFORM SAMPLES

In this section, we present a method for perfect estimation of a single input Dirac, and extend this to a sequential algorithm for reconstruction of a stream of Diracs and piecewise constant polynomials, from their non-uniform samples. Then, we present an extension of these methods to FRI reconstruction using a multi-channel sampling approach.

### A. Perfect Reconstruction of a Single Dirac

Let us consider a single input Dirac of amplitude  $|a_1| \leq 1$ :

$$x(t) = a_1 \delta(t - \tau_1). \quad (4)$$

We obtain the timing information of this signal using the model in Fig. 1, where the filter  $\varphi(t)$  is a first-order E-spline of support  $L$ . Moreover, assume that the first two output samples are located at  $t_1, t_2 \in [\tau_1, \tau_1 + \frac{L}{2}]$ , which holds provided the period of the comparator's signal satisfies  $T_s \leq \frac{2L}{5}$ , as shown in Section 4. Then, in the interval  $I = [t_2 - \frac{L}{2}, t_1]$ , where there are no knots of either  $\varphi(t - t_1)$  or  $\varphi(t - t_2)$ , we can reproduce two exponentials as described in Section 2:

$$\sum_{n=1}^2 c_{m,n}^I \varphi(t - t_n) = e^{\alpha_m t}, \text{ for } m \in \{0, 1\}. \quad (5)$$

Then, we define the signal moments  $s_m$ , as a linear combination of the measurements  $y(t_1)$  and  $y(t_2)$ :

$$\begin{aligned} s_m &= \sum_{n=1}^2 c_{m,n}^I y(t_n) \stackrel{(a)}{=} \sum_{n=1}^2 c_{m,n}^I \langle x(t), \varphi(t - t_n) \rangle \\ &\stackrel{(b)}{=} \int x(t) \sum_{n=1}^2 c_{m,n}^I \varphi(t - t_n) dt \\ &\stackrel{(c)}{=} \int a_1 \delta(t - \tau_1) \sum_{n=1}^2 c_{m,n}^I \varphi(t - t_n) dt \\ &\stackrel{(d)}{=} \int_I a_1 \delta(t - \tau_1) e^{j\omega_m t} dt = a_1 e^{j\omega_m \tau_1} = b_1 u_1^m, \end{aligned} \quad (6)$$

where  $b_1 := a_1 e^{j\omega_0 \tau_1}$ ,  $u_1 := e^{i\lambda \tau_1}$ , and the frequencies  $\omega_m = \omega_0 + \lambda m$ , for  $m \in \{0, 1\}$ .

The unknowns  $\{b_1, u_1\}$  can be uniquely retrieved from the signal moments, using the annihilating filter method [20], also known as Prony's method [21]. Then, we get the Dirac's amplitude and location, using  $b_1 = a_1 e^{j\omega_0 \tau_1}$  and  $u_1 = e^{i\lambda \tau_1}$ .

In the derivations above, (a) follows from Eq. (1), (b) from the linearity of the inner product, and (c) from Eq. (4). Moreover, (d) follows from the assumption that  $t_1, t_2 \in [\tau_1, \tau_1 + \frac{L}{2}]$ , which means  $\tau_1 \in I$ , and from the local exponential reproduction property of  $\varphi(t)$  in Eq. (5).

Finally, the derivations in Eq. (6) hold in any interval  $I = [t_N - \frac{L}{2}, t_1]$  where there are no knots of any kernel  $\varphi(t - t_n)$ , for  $t_n \in [\tau_1, \tau_1 + \frac{L}{2}]$ ,  $n = 0, 1, \dots, N$  and  $N \geq 2$ .

### B. Perfect Reconstruction of a Stream of Diracs

We now consider the case of  $K$  Diracs:

$$x(t) = \sum_{k=1}^K a_k \delta(t - t_k), \quad (7)$$

where  $|a_k| \leq 1$ .

When we filter the input signal with kernel  $\varphi(t)$ , we obtain the non-uniform output samples based on the acquisition model of Fig. 1,  $y(t_n) = \langle x(t), \varphi(t - t_n) \rangle$ , for  $n \geq 1$ . Then, using  $y(t_1)$  and  $y(t_2)$ , we can uniquely estimate the first Dirac in the stream using the method in Section A.

Furthermore, let us assume that the separation between input Diracs is larger than  $L$ , and denote with  $y(t_n)$  and  $y(t_{n+1})$

the samples located after  $\tau_1 + L$ . This means that the location of the second Dirac satisfies  $\tau_1 + L < \tau_2 < t_n$ . Moreover, we assume that  $y(t_n), y(t_{n+1}) \in [\tau_2, \tau_2 + \frac{L}{2}]$ , which holds provided the period of the comparator's signal satisfies  $T_s \leq \frac{2L}{5}$ , as shown in Section 4. Then, in the interval  $I = [t_{n+1} - \frac{L}{2}, t_n]$ , where there are no knots of any of the shifted kernels  $\varphi(t - t_n)$  or  $\varphi(t - t_{n+1})$ , perfect exponential reproduction can be achieved:

$$c_{m,n}^I \varphi(t - t_n) + c_{m,n+1}^I \varphi(t - t_{n+1}) = e^{\alpha_m t}, \text{ for } m \in \{0, 1\}.$$

Then, we can compute the signal moments as in Eq. (6):

$$s_m = c_{m,n}^I y(t_n) + c_{m,n+1}^I y(t_{n+1}) = a_2 e^{\alpha_m \tau_2}. \quad (8)$$

Finally, we can estimate the free parameters  $a_2$  and  $\tau_2$  from the signal moments, using Prony's method.

Once the second Dirac has been estimated, we use subsequent non-uniform output samples after  $\tau_2 + L$  in order to sequentially retrieve the next Diracs.

### C. Perfect Reconstruction of Piecewise Constant Signals

We show that the sampling of piecewise constant signals is equivalent to sampling a stream of Diracs [6].

Using the acquisition device depicted in Fig. 1, we filter the piecewise constant input  $x(t)$  with the kernel  $\varphi(t)$ , which is able to reproduce constant polynomials. Then, using the output samples  $y_n = \langle x(t), \varphi(t - t_n) \rangle$ , we define:

$$z_n = y_{n+1} - y_n = \langle x(t), \varphi(t - t_{n+1}) - \varphi(t - t_n) \rangle. \quad (9)$$

Denoting  $t_{n+1} - t_n = \theta_n$ , using the definition of  $q_{\theta_n}(t)$  from Eq. (3), and leveraging the proof in [6], Eq. (9) becomes:

$$z_n = \left\langle \frac{d}{dt}(x(t)), (\varphi * q_{\theta_n})(t - t_n) \right\rangle.$$

The modified outputs  $z_n = y_{n+1} - y_n$  are equivalent to the samples that would be obtained by filtering the input signal  $\frac{d}{dt}(x(t))$ , with the kernels  $(\varphi * q_{\theta_n})(t)$ , and sampling the filtered signal at times  $t_n$ . Hence, sampling the piecewise constant signal  $x(t)$  with the spline  $\varphi(t)$  is equivalent to sampling the corresponding stream of Diracs  $\frac{d}{dt}(x(t))$  with the new kernel  $(\varphi * q_{\theta_n})(t)$ , which can reproduce linear polynomials, as discussed in Section 2. We can therefore perfectly reconstruct  $x(t)$  by leveraging results in Section 3.A and 3.B.

### D. Input Estimation using a Multi-Channel Approach

The reconstruction methods above can be extended to the estimation of signals of  $K$  bursts of 2 Diracs, of the form:

$$x(t) = \sum_{k=1}^K \sum_{i=1}^2 a_{k,i} \delta(t - \tau_{k,i}),$$

where  $\tau_{k,1}$  and  $\tau_{k,2}$  are sufficiently close, and  $\tau_{k,2}$  and  $\tau_{k+1,1}$  are sufficiently separated.

This is achieved using the sampling approach in Fig. 1, with 2 filter channels,  $\varphi_1$  and  $\varphi_2$ . In this case, the signal moments give us 4 equations, from which we can retrieve the 4 free parameters of the input signal. This method can be further extended to the retrieval of input bursts of  $M$  Diracs,

using an M-channel sampling scheme. Lastly but importantly, the same results can be achieved using a multi-channel acquisition scheme, where each channel consists of an integrate-and-fire model [22].

#### 4. SUFFICIENT CONDITIONS FOR PERFECT RECONSTRUCTION OF FRI SIGNALS

In this section we present sufficient conditions, for perfect reconstruction of an input stream of Diracs, and a piecewise constant signal. First, we impose constraints on the frequency of the comparator's reference function, to guarantee the output non-uniform samples are sufficiently dense. Second, we derive the minimum separation between consecutive spikes in the input stream of Diracs and equivalently, between the discontinuities of a piecewise constant signal. This ensures sequential retrieval of the input parameters is possible.

**Proposition 1.** *The timing information  $t_1, t_2, \dots, t_n, \dots$  provided by the device shown in Fig. 1 is a sufficient representation of a stream of  $K$  Diracs as in Eq. (7), when the period of  $g(t) = \cos(w_s t)$  satisfies  $T_s \leq \frac{2L}{5}$ , with  $L$  being the support of the sampling kernel, and when the minimum spacing between Diracs is larger than  $L$ .*

*Proof.* Suppose we aim to retrieve the Dirac  $\delta_1$  of a stream of spikes, given by  $\delta_1 = a_1 \delta(t - \tau_1)$ . In order for Eq. (5) to hold, we need to ensure that the first two output samples after the Dirac  $\delta_1$  are located at  $t_1, t_2 \in [\tau_1, \tau_1 + \frac{L}{2}]$ , or equivalently that  $t_2 - \tau_1 < \frac{L}{2}$ . Moreover, the assumption  $|a_1| \leq 1$  ensures  $|y(t)| \leq 1 = \max(g(t))$ . Therefore, given the hypothesis  $T_s \leq \frac{2L}{5}$ , then the maximum separation between the input Dirac and the second output spike is  $t_2 - \tau_1 = \frac{5T_s}{4} \leq \frac{L}{2}$ , as depicted in Fig. 3. Furthermore, the sequential algorithm in Section B requires a minimum separation between the Diracs equal to  $L$ , such that the samples  $y(t_n)$  and  $y(t_{n+1})$  in Eq. (8) have contribution from one Dirac only.  $\square$

Finally, using the results of Section 3, equivalent constraints can be derived for the case of a piecewise constant polynomial:  $T_s < \frac{4L}{5}$ , and a minimum separation between discontinuities of  $2L$ , where  $L$  is the support of the zero-order B-spline used as sampling kernel.

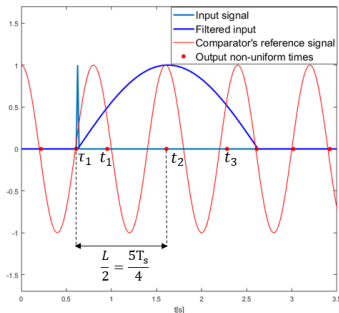


Fig. 3: Timing information of an input Dirac located at  $\tau_1$ .

#### 5. SIMULATIONS

The sampling and reconstruction of a stream of  $K = 4$  Diracs are depicted in Fig. 4. Here, the sampling kernel is a first-order E-spline, of support  $L = 2$ , shown in Fig. 4(b). The

comparator's reference signal has a frequency  $f_s = 1.25 = \frac{5}{2L}$ , and the inter-Dirac separation is larger than the kernel support  $L$ , as seen in Fig. 4(a). The amplitudes and locations of the estimated Diracs are exact to numerical precision. The reconstruction of a piecewise constant signal from non-uniform samples is depicted in Fig. 5. The sampling kernel is a zero-order B-spline, of support  $L = 1$ , and the frequency of the comparator's reference signal is  $f_s = 1.25 = \frac{5}{4L}$ . Moreover, the separation between consecutive input discontinuities is larger than  $2L = 2$ , as illustrated in Fig. 5(b). Finally, the reconstructed input signal is exact to numerical precision.

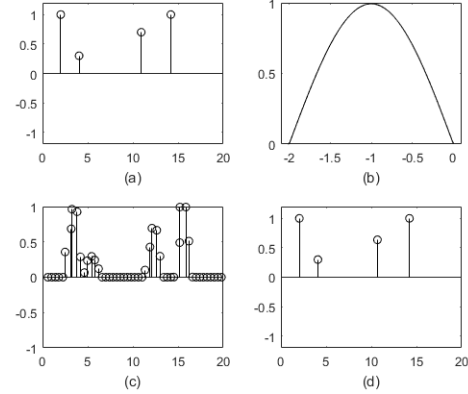


Fig. 4: Sampling of a stream of Diracs. The input signal is shown in (a), the sampling kernel in (b), the output non-uniform samples in (c), and the reconstructed signal in (d).

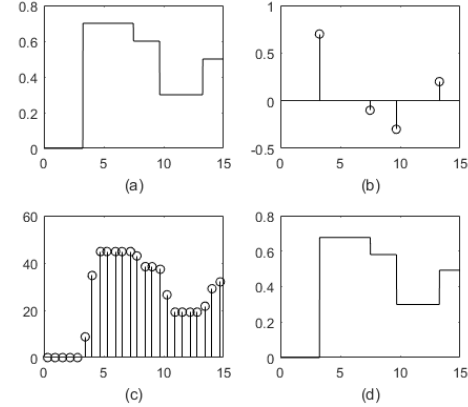


Fig. 5: Sampling of a piecewise constant signal. The input piecewise constant signal is shown in (a), the input discontinuities in (b), the output non-uniform samples in (c), and the reconstruction in (d).

#### 6. CONCLUSIONS

This paper addressed the problem of reconstructing classes of non-bandlimited signals from time-based samples, obtained by filtering the input with an exponential (polynomial) reproducing kernel, and retrieving the timing information using a comparator. We first showed that these kernels preserve locally the ability to reproduce exponentials or polynomials, in the case of non-uniform sampling. Furthermore, we designed an iterative reconstruction scheme, which can perfectly retrieve signals with finite rate of innovation from their timing information. Simulations validated the claims of the paper.

## 7. REFERENCES

- [1] M. Unser, "Sampling-50 years after Shannon," *Proceedings of the IEEE*, vol. 88, no. 4, pp. 569–587, April 2000.
- [2] E. J. Candes, J. Romberg, and T. Tao, "Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information," *IEEE Transactions on Information Theory*, vol. 52, no. 2, pp. 489–509, Feb 2006.
- [3] D. L. Donoho, "Compressed sensing," *IEEE Transactions on Information Theory*, vol. 52, no. 4, pp. 1289–1306, April 2006.
- [4] M. Vetterli, P. Marziliano, and T. Blu, "Sampling signals with finite rate of innovation," *IEEE Transactions on Signal Processing*, vol. 50, no. 6, pp. 1417–1428, June 2002.
- [5] P. L. Dragotti, M. Vetterli, and T. Blu, "Exact sampling results for signals with finite rate of innovation using Strang-Fix conditions and local kernels," in *Proceedings. (ICASSP '05). IEEE International Conference on Acoustics, Speech, and Signal Processing, 2005.*, March 2005, vol. 4, pp. iv/233–iv/236 Vol. 4.
- [6] P. L. Dragotti, M. Vetterli, and T. Blu, "Sampling Moments and Reconstructing Signals of Finite Rate of Innovation: Shannon Meets Strang-Fix," *IEEE Transactions on Signal Processing*, vol. 55, no. 5, pp. 1741–1757, May 2007.
- [7] R. Tur, Y. C. Eldar, and Z. Friedman, "Innovation Rate Sampling of Pulse Streams With Application to Ultrasound Imaging," *IEEE Transactions on Signal Processing*, vol. 59, no. 4, pp. 1827–1842, April 2011.
- [8] Y. M. Lu and M. N. Do, "A Theory for Sampling Signals From a Union of Subspaces," *IEEE Transactions on Signal Processing*, vol. 56, no. 6, pp. 2334–2345, June 2008.
- [9] A. A. Lazar, "Time encoding with an integrate-and-fire neuron with a refractory period," *NEUROCOMPUTING*, vol. 65, pp. 65–66, 2005.
- [10] A. A. Lazar and L. T. Toth, "Time encoding and perfect recovery of bandlimited signals," in *2003 IEEE International Conference on Acoustics, Speech, and Signal Processing, 2003. Proceedings. (ICASSP '03).*, April 2003, vol. 6, pp. VI–709.
- [11] A. A. Lazar and E. A. Pnevmatikakis, "Video Time Encoding Machines," *IEEE Transactions on Neural Networks*, vol. 22, no. 3, pp. 461–473, March 2011.
- [12] D. Gontier and M. Vetterli, "Sampling based on timing: Time encoding machines on shift-invariant subspaces," *Applied and Computational Harmonic Analysis*, vol. 36, no. 1, pp. 63 – 78, 2014.
- [13] M. Unser and A. Aldroubi, "A general sampling theory for nonideal acquisition devices," *IEEE Transactions on Signal Processing*, vol. 42, no. 11, pp. 2915–2925, Nov 1994.
- [14] A. Aldroubi and K. Gröchenig, "Nonuniform Sampling and Reconstruction in Shift-Invariant Spaces," *SIAM Review*, vol. 43, no. 4, pp. 585–620, 2001.
- [15] G. Strang and G. Fix, *A Fourier Analysis of the Finite Element Variational Method*, pp. 793–840, Springer Berlin Heidelberg, Berlin, Heidelberg, 2011.
- [16] M. Unser, "Splines: a perfect fit for signal and image processing," *IEEE Signal Processing Magazine*, vol. 16, no. 6, pp. 22–38, Nov 1999.
- [17] M. Unser and T. Blu, "Cardinal exponential splines: part I - theory and filtering algorithms," *IEEE Transactions on Signal Processing*, vol. 53, no. 4, pp. 1425–1438, April 2005.
- [18] C. Vonesch, T. Blu, and M. Unser, "Generalized Daubechies Wavelet Families," *IEEE Transactions on Signal Processing*, vol. 55, no. 9, pp. 4415–4429, Sept 2007.
- [19] J. A. Urigüen, T. Blu, and P. L. Dragotti, "FRI Sampling With Arbitrary Kernels," *IEEE Transactions on Signal Processing*, vol. 61, no. 21, pp. 5310–5323, Nov 2013.
- [20] P. Stoica and R. Moses, *Spectral Analysis of Signals*, 2005.
- [21] R. Prony, "Essai expérimental et analytique sur les lois de la dilatabilité de fluides lastiques et sur celles de la force expansive de la vapeur de leau et de la vapeur de lalkool, à différentes températures," *Journal de l'École Polytechnique*, vol. 1, no. 22, pp. 24–76.
- [22] R. Alexandru and P. L. Dragotti, "Time encoding and perfect recovery of non-bandlimited signals with an integrate-and-fire system," *Submitted to the 13th International Conference on Sampling Theory and Applications (SampTA '19)*, July 2019.