DESIGN OF OPTIMAL LINEAR DIFFERENTIAL MICROPHONE ARRAYS BASED ON ARRAY GEOMETRY OPTIMIZATION

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ABSTRACT

This paper presents a method to design optimal linear differential microphone arrays (DMAs) by optimizing the array geometry. By constraining the DMA beamformer to achieve a given target value of the directivity factor (DF) with a specified target frequency-invariant beampattern while achieving also the highest possible white noise gain (WNG), an optimization algorithm is developed, which consists of the following two steps. 1) The full frequency band of interest is divided into a few subbands. At every subband, the entire linear array is divided into subarrays and the number of subarrays depends on the total number of the sensors and the order of the DMA. A cost function is then defined, which is minimized to determine what subarray produces the optimal performance. 2) The subband optimal subarrays are then combined across the entire frequency band to form a fullband cost function, from which the geometry of the entire array is optimized. These two steps are repeated with the particle swarm optimization (PSO) algorithm until the desired array performance is reached. Simulation results demonstrate that the proposed method can obtain the target DF with a frequency-invariant beampattern over a wide band of frequencies while maintaining a reasonable level of WNG.

Index Terms— Differential microphone array, array geometry optimization, directivity factor, white noise gain.

1. INTRODUCTION

Generally, the design of beamformers focuses on finding the optimal beamforming filter under some criterion with a specified array geometry [1, 2], such as linear [1, 3], circular [4–6], concentric circular [7–9], and spherical [10, 11] arrays, etc. Another way to improve beamforming performance, e.g., increasing the array directivity, controlling sidelobe levels and grating lobes, and improving the robustness, is by optimizing the array geometry, which has also attracted much attention [12–18]. For example, in [19], a superdirective beamformer was developed based on sparse aperiodic planar arrays by simultaneously optimizing the sensors' positions and beamforming filters, where the obtained beamformer can achieve better performance in terms of robustness and sidelobe levels. In [20], a robust superdirective beamformer was presented based on the optimization of the array geometry and beamforming filter by using the particle swarm optimization (PSO) algorithm, which can achieve a better tradeoff between the white noise gain (WNG) and the directivity factor (DF) than the traditional superdirective beamforming methods.

Differential microphone arrays (DMAs) [1, 21] are very promising in dealing with broadband signals for their frequency-invariant beampatterns and high directivity gains. However, DMAs often suffer from white noise amplification at low frequencies and beampattern distortion at high frequencies. Clearly, the array geometry plays an important role on the DMA performance. So, this paper presents an approach to the design of DMAs of high performance by optimizing the array geometry under the constraints of minimum tolerable interelement spacing and maximum tolerable array aperture. The approach taken here is to divide the entire array into different subarrays and the optimal subarray geometries at different frequencies are identified. The array geometry is then optimized iteratively by the PSO algorithm.

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2. SIGNAL MODEL, PROBLEM FORMULATION, AND PERFORMANCE MEASURES

We consider a nonuniform linear array with M omnidirectional microphones as illustrated in Fig. 1, where the distance from the mth sensor to the reference (the first microphone) is equal to $\rho_m,\ m = 1, 2, \ldots, M$, with $\rho_1 = 0$. If we denote the azimuth angle by θ, the steering vector corresponding to θ is given by [22]

$$\mathbf{d}(\omega, \theta) = \begin{bmatrix} 1 & e^{-\jmath\rho_2\omega\cos\theta/c} & \cdots & e^{-\jmath\rho_M\omega\cos\theta/c} \end{bmatrix}^T, \quad (1)$$

where $\jmath$ is the imaginary unit with $\jmath^2 = -1$, $\omega = 2\pi f$ is the angular frequency, $f > 0$ is the temporal frequency, c is the speed of sound in air, which is generally assumed to be 340 m/s, and the superscript ^{T} is the transpose operator.

Linear DMAs have very limited steering flexibility. Therefore, in the design of differential beamformers, it is generally assumed that the signal of interest comes from the endfire direction, i.e., $\theta = 0$. In this case, the microphone array observation signal vector is written as

$$\begin{aligned} \mathbf{y}(\omega) &= \begin{bmatrix} Y_1(\omega) & Y_2(\omega) & \cdots & Y_M(\omega) \end{bmatrix}^T \\ &= \mathbf{d}(\omega) X(\omega) + \mathbf{v}(\omega), \end{aligned} \quad (2)$$

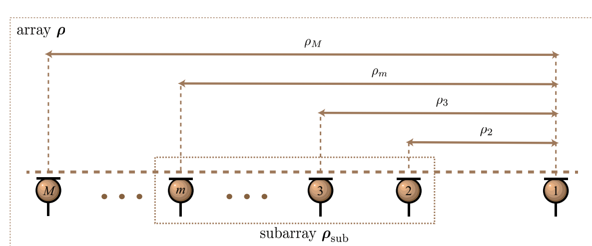

Fig. 1. Illustration of a nonuniform linear microphone array and a subarray. The distance from the mth sensor to the reference microphone is $\rho_m,\ m = 1, 2, \ldots, M$, with $\rho_1 = 0$.

where $Y_m(\omega)$ is the received signal at the mth microphone, $\mathbf{d}(\omega) = \mathbf{d}(\omega, 0)$ is the steering vector for $\theta = 0$, $X(\omega)$ is the zero-mean source signal of interest, and $\mathbf{v}(\omega)$ is the zero-mean noise signal vector defined in a similar way to $\mathbf{y}(\omega)$.

The beamforming process consists of applying a complex weight vector:

$$\mathbf{h}(\omega) = \begin{bmatrix} H_1(\omega) & H_2(\omega) & \cdots & H_M(\omega) \end{bmatrix}^T, \qquad (3)$$

to the noisy observation vector to obtain an output, i.e.,

$$\begin{aligned} Z(\omega) &= \mathbf{h}^H(\omega)\mathbf{y}(\omega) \\ &= \mathbf{h}^H(\omega)\mathbf{d}(\omega)X(\omega) + \mathbf{h}^H(\omega)\mathbf{v}(\omega), \end{aligned} \qquad (4)$$

where $Z(\omega)$ is the estimate of the signal of interest, $X(\omega)$, and the superscript ^{H} is the conjugate-transpose operator.

In our context, the distortionless constraint in the desired look direction is needed, i.e.,

$$\mathbf{h}^H(\omega)\mathbf{d}(\omega) = 1. \qquad (5)$$

With the above signal model and formulation, the problem of beamforming becomes one of designing a “good” beamforming filter, $\mathbf{h}(\omega)$, under the constraint in (5). To evaluate how good is the designed beamforming filter, we adopt three widely used performance measures: beampattern, WNG, and DF.

The beampattern, which quantifies the sensitivity of the beamformer to a plane wave impinging on the array from the direction θ, is defined as

$$\mathcal{B}[\mathbf{h}(\omega), \theta] = \mathbf{h}^H(\omega)\mathbf{d}(\omega, \theta). \qquad (6)$$

The WNG measures the robustness of a beamformer; it is defined as [23]

$$\mathcal{W}[\mathbf{h}(\omega)] = \frac{\left|\mathbf{h}^H(\omega)\mathbf{d}(\omega)\right|^2}{\mathbf{h}^H(\omega)\mathbf{h}(\omega)}. \qquad (7)$$

The DF quantifies how directive is the beamformer. It can be written as [23]

$$\mathcal{D}[\mathbf{h}(\omega)] = \frac{\left|\mathbf{h}^H(\omega)\mathbf{d}(\omega)\right|^2}{\mathbf{h}^H(\omega)\boldsymbol{\Gamma}_{\mathrm{d}}(\omega)\mathbf{h}(\omega)}, \qquad (8)$$

where the elements of the $M \times M$ matrix $\boldsymbol{\Gamma}_{\mathrm{d}}(\omega)$ are given by

$$[\boldsymbol{\Gamma}_{\mathrm{d}}(\omega)]_{ij} = \mathrm{sinc}\left[\frac{\omega(\rho_i - \rho_j)}{c}\right], \qquad (9)$$

with $i, j = 1, 2, \ldots, M$, and $\mathrm{sinc}(x) = \sin x / x$.

3. CONVENTIONAL DMA

Ideally, the frequency-independent beampattern of an Nth-order DMA is of the following form [24]:

$$\mathcal{B}_N(\theta) = \sum_{n=0}^{N} a_{N,n}\cos^n\theta, \qquad (10)$$

where $a_{N,n},\ n = 0, 1, \ldots, N$ are real coefficients determining the shape of the beampattern.

Let us adopt the DMA design method presented in [1]. If we assume that the target Nth-order DMA beampattern has N distinct nulls, which satisfy $0^\circ < \theta_{N,1} < \cdots < \theta_{N,N} \leq 180^\circ$, the problem of DMA beamforming can be converted to one of solving the following linear equations:

$$\mathbf{D}(\omega)\mathbf{h}(\omega) = \mathbf{i}_1, \qquad (11)$$

where

$$\mathbf{D}(\omega) = \begin{bmatrix} \mathbf{d}^H(\omega, 0^\circ) \\ \mathbf{d}^H(\omega, \theta_{N,1}) \\ \vdots \\ \mathbf{d}^H(\omega, \theta_{N,N}) \end{bmatrix} \qquad (12)$$

is a matrix of size $(N+1) \times M$, and $\mathbf{i}_1 = [1\ 0\ \cdots\ 0]^T$ is a vector of length $(N+1)$.

To design an Nth-order DMA, at least $N+1$ microphones are needed, i.e., $M \geq N+1$. If $M = N+1$, the solution of (11) is

$$\mathbf{h}(\omega) = \mathbf{D}^{-1}(\omega)\mathbf{i}_1. \qquad (13)$$

However, the solution given in (13) may suffer from serious white noise amplification at low frequencies. This issue can be mitigated by increasing the number of microphones so that $M > N+1$. In this case, we can obtain a minimum-norm solution of (11), i.e.,

$$\mathbf{h}_{\mathrm{MN}}(\omega) = \mathbf{D}^H(\omega)\left[\mathbf{D}(\omega)\mathbf{D}^H(\omega)\right]^{-1}\mathbf{i}_1, \qquad (14)$$

which is also referred to as the maximum WNG (MWNG) differential beamformer as it naturally maximizes the WNG [21].

4. DMA DESIGN BY OPTIMIZING THE ARRAY GEOMETRY

While it mitigates the white noise amplification problem, the MWNG beamformer may introduce beampattern distortion, such as extra nulls in the beampatterns at high frequencies [21]. In [25], a so-called zero-off unit circle (ZOU) DMA beamformer was proposed to deal with the extra-null problem with the MWNG beamformer. But the resulting beampatterns may still vary with frequency. In this paper, we attempt to design optimal nonuniform linear DMA beamformers by optimizing the array geometry.

Let us define the following vector to denote the geometry of a nonuniform linear array:

$$\boldsymbol{\rho} = \begin{bmatrix} \rho_1 & \rho_2 & \cdots & \rho_M \end{bmatrix}^T, \qquad (15)$$

where ρ_m is the spacing between the mth sensor and the reference point as shown in Fig. 1. We optimize the array geometry, i.e., the values of ρ_m, with a given maximum number of microphones, M, a pre-specified minimum tolerable interelement spacing, $\delta_{\min}$, and the maximum tolerable array aperture $L_{\max}$, to achieve the target value of the DF and the highest possible value of the WNG.

To design an Nth-order differential beamformer, we can either use all the sensors or a subset of the sensors for a given frequency band. With a given array of M microphones to design an Nth-order DMA, there are K different combinations of subarrays, i.e.,

$$K = \sum_{\mathcal{M}=N+1}^{M} \binom{M}{\mathcal{M}}, \qquad (16)$$

where $\binom{M}{\mathcal{M}}$ represents the number of all the combinations of $\mathcal{M}$ elements taken from the M different sensors. Without loss of generality, we denote the geometry of the kth subarray as $\boldsymbol{\rho}_{\mathrm{sub},k}$,

$k = 1, 2, \ldots, K$. Then, the objective is to find the optimal combination of the subarrays under certain conditions to achieve the best beamforming performance.

Once the geometry vector $\boldsymbol{\rho}$ is specified, for each subarray, $\boldsymbol{\rho}_{\text{sub},k}$, the steering vector is defined analogously to (1), and the beamforming filter, $\mathbf{h}\left(\omega, \boldsymbol{\rho}_{\text{sub},k}\right)$, is computed using the minimum-norm method given in Section 3. Then, we can define the following cost function for the kth subarray at the frequency ω:

$$J\left[\mathbf{h}\left(\omega, \boldsymbol{\rho}_{\text{sub},k}\right)\right] = \mu_1 \left\{\mathcal{D}\left[\mathbf{h}\left(\omega, \boldsymbol{\rho}_{\text{sub},k}\right)\right] - \mathcal{D}_0\right\}^2 + \mu_2 \mathcal{W}\left[\mathbf{h}\left(\omega, \boldsymbol{\rho}_{\text{sub},k}\right)\right], \quad (17)$$

where $\mathcal{D}_0$ is the desired, target value of the DF, $\mathcal{D}\left[\mathbf{h}\left(\omega, \boldsymbol{\rho}_{\text{sub},k}\right)\right]$ and $\mathcal{W}\left[\mathbf{h}\left(\omega, \boldsymbol{\rho}_{\text{sub},k}\right)\right]$ are, respectively, the DF and WNG of the kth subarray with the beamforming filter $\mathbf{h}\left(\omega, \boldsymbol{\rho}_{\text{sub},k}\right)$, and μ_1 and μ_2 are two (real) weighting coefficients.

The optimal subarray geometry at frequency ω is then determined as

$$\boldsymbol{\rho}_{\text{sub,o},\omega} = \underset{\boldsymbol{\rho}_{\text{sub},k}}{\arg\min}\ J\left[\mathbf{h}\left(\omega, \boldsymbol{\rho}_{\text{sub},k}\right)\right]. \quad (18)$$

Combining the optimal subarray geometries, $\boldsymbol{\rho}_{\text{sub,o},\omega}$, at different frequencies across the entire frequency band of interest, we obtain the subarray set:

$$\mathbf{C}_{\boldsymbol{\rho}_{\text{sub}}} = \left\{\boldsymbol{\rho}_{\text{sub,o},\omega}\right\}. \quad (19)$$

A fullband cost function based on $\mathbf{C}_{\boldsymbol{\rho}_{\text{sub}}}$ is then formed as

$$J\left(\mathbf{C}_{\boldsymbol{\rho}_{\text{sub}}}\right) = \sum_{\omega} J\left[\mathbf{h}\left(\omega, \boldsymbol{\rho}_{\text{sub,o},\omega}\right)\right]. \quad (20)$$

Finally, the optimal subarray set is determined by

$$\mathbf{C}_{\boldsymbol{\rho}_{\text{sub,o}}} = \underset{\mathbf{C}_{\boldsymbol{\rho}_{\text{sub}}}}{\arg\min}\ J\left(\mathbf{C}_{\boldsymbol{\rho}_{\text{sub}}}\right) \quad \text{s. t.} \quad \delta_{\boldsymbol{\rho}} \geq \delta_{\text{min}},\ L_{\boldsymbol{\rho}} \leq L_{\text{max}}, \quad (21)$$

where $\delta_{\boldsymbol{\rho}}$ is the minimum interelement spacing under array geometry $\boldsymbol{\rho}$, δ_{min} is the minimum tolerable interelement spacing, $L_{\boldsymbol{\rho}}$ is the array aperture, and L_{max} is the maximum tolerable array aperture. In the implementation, the optimization process is realized with the PSO algorithm, which is summarized in Table 1.

Table 1. DMA optimization algorithm based on PSO.

Parameters:
- acceleration factor, γ
- inertia weight, ϵ
- random number, $\kappa \sim U(0, \gamma)$

Initialization:
- velocity, $\boldsymbol{\xi} \leftarrow \boldsymbol{\xi}_0$
- geometry, $\boldsymbol{\rho} \leftarrow \boldsymbol{\rho}_0$
- compute $\mathbf{C}_{\boldsymbol{\rho}_{\text{sub}}}$ based on $\boldsymbol{\rho}$
- $\boldsymbol{\rho}_{\text{temp}} = \boldsymbol{\rho}$
- $\boldsymbol{\rho}_{\text{o}} = \boldsymbol{\rho}$

Repeat:
- Update the velocity $\boldsymbol{\xi}$ and the geometry $\boldsymbol{\rho}$
  - $\boldsymbol{\xi} \leftarrow \epsilon \cdot \boldsymbol{\xi} + \kappa\gamma \cdot (\boldsymbol{\rho}_{\text{temp}} - \boldsymbol{\rho}) + \kappa\gamma \cdot (\boldsymbol{\rho}_{\text{o}} - \boldsymbol{\rho})$
- If $\delta_{\boldsymbol{\rho}+\boldsymbol{\xi}} \geq \delta_{\text{min}}$ and $L_{\boldsymbol{\rho}+\boldsymbol{\xi}} \leq L_{\text{max}}$
  - $\boldsymbol{\rho} \leftarrow \boldsymbol{\rho} + \boldsymbol{\xi}$
- For each frequency ω
  - For each subarray $\boldsymbol{\rho}_{\text{sub},k}$
    - Compute the cost function $J\left[\mathbf{h}\left(\omega, \boldsymbol{\rho}_{\text{sub},k}\right)\right]$
  - End
- End
- Find $\boldsymbol{\rho}_{\text{sub,o},\omega}$
- Form the subarray set $\mathbf{C}_{\boldsymbol{\rho}_{\text{sub}}}$
- Compute the fullband cost $J\left(\mathbf{C}_{\boldsymbol{\rho}_{\text{sub}}}\right)$, $J\left(\mathbf{C}_{\boldsymbol{\rho}_{\text{temp,sub}}}\right)$
- If $J\left(\mathbf{C}_{\boldsymbol{\rho}_{\text{sub}}}\right) < J\left(\mathbf{C}_{\boldsymbol{\rho}_{\text{temp,sub}}}\right)$
  - $\boldsymbol{\rho}_{\text{temp}} = \boldsymbol{\rho}$
  - $\mathbf{C}_{\boldsymbol{\rho}_{\text{temp,sub}}} = \mathbf{C}_{\boldsymbol{\rho}_{\text{sub}}}$
- Compute the fullband cost function $J\left(\mathbf{C}_{\boldsymbol{\rho}_{\text{o,sub}}}\right)$
- If $J\left(\mathbf{C}_{\boldsymbol{\rho}_{\text{temp,sub}}}\right) < J\left(\mathbf{C}_{\boldsymbol{\rho}_{\text{o,sub}}}\right)$
  - $\boldsymbol{\rho}_{\text{o}} = \boldsymbol{\rho}_{\text{temp}}$
  - $\mathbf{C}_{\boldsymbol{\rho}_{\text{best,sub}}} = \mathbf{C}_{\boldsymbol{\rho}_{\text{temp,sub}}}$
  - $\mathbf{C}_{\boldsymbol{\rho}_{\text{sub,o}}} = \mathbf{C}_{\boldsymbol{\rho}_{\text{o,sub}}}$

End

5. SIMULATIONS

We consider a nonuniform linear array consisting of 16 microphones, the minimum tolerable interelement spacing is set to $\delta_{\text{min}} = 0.4$ cm (the value of δ_{min} should be chosen according to the size of the sensors that are used in practical applications), the maximum tolerable array aperture is set to $L_{\text{max}} = 15$ cm. The desired directivity pattern is chosen as the second order supercardioid, which has two nulls at 106° and 153°, respectively, and the corresponding DF is $\mathcal{D}_0 = 8.0$ dB.

To optimize the array geometry, we first divide the 8-kHz full frequency band into 80 uniform subbands. In every subband, the entire array is divided into subarrays based on the given 16 microphones and DMA order of 2. The optimal subarray geometry is then identified from all the possibilities using the enumeration method [26] according to the cost function defined in (17), and the fullband array geometry is optimized by minimizing the fullband cost function given in (20) using the PSO algorithm as summarized in Table 1. In the PSO algorithm, the acceleration factor and inertia weight are set to $\gamma = 1.4961$ and $\epsilon = 0.7298$, respectively [28]. In our implementation, the variables in (17) are calculated in the dB scale, and the weight coefficients in (17) are set to $\mu_1 = 1000$ and $\mu_2 = -1$, respectively. The aperture of the subarrays is limited to less than $\varsigma\lambda$, where λ is the acoustic wavelength. An empirical value of $\varsigma = 0.75$ is used in our experiment.

For comparison, the performances of the conventional DMA designed with the null-constraint method [1], the MWNG DMA [21], and ZOU DMA [25] are also presented. The conventional DMA is designed with a uniform linear array of $M = 3$ and $\delta = 1$ cm, and the MWNG and ZOU beamformers are designed with a uniform linear array of $M = 16$ and $\delta = 1$ cm, so that the array aperture is equal to L_{max} used in the simulations.

Figure 2 plots the DFs and the WNGs as a function of the frequency of the conventional, MWNG, ZOU, and proposed optimal DMA beamformers. It is seen that the conventional DMA has achieved the desired value of the DF (slightly varying with frequency), but it suffers from serious white noise amplification at low frequencies. The MWNG and ZOU DMA beamformers greatly improve the WNG; but the resulting DFs varies with frequency, indicating that the beampattern of the designed beamformer may be different from the target directivity pattern. In contrast, the proposed optimal DMA has almost frequency-invariant DF and maintains the WNG at a reasonable level in the studied frequency range. Note that practical systems can tolerate some amount of white noise amplification depending on the quality of the microphones. So, in our simulations, the WNG is controlled to be slightly smaller than 0 dB.

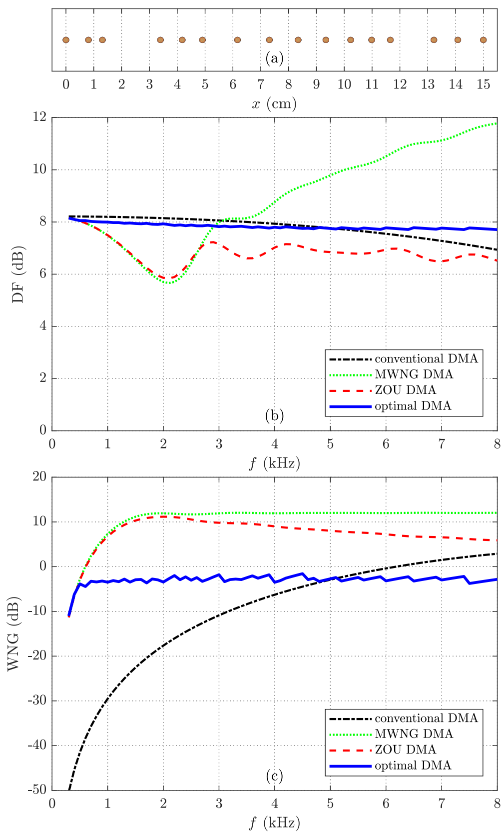

Fig. 2. The optimized array geometry and the beamforming performance: (a) the optimized array geometry, (b) DF as a function of the frequency, and (c) WNG as a function of the frequency.

This level can be adjusted by setting a proper value of μ_2.

Figure 3 plots the 3-dimensional beampatterns of the four studied methods. It is clearly seen from Fig. 3(a) that the beampattern of the DMA with the conventional method is almost the same as the target directivity pattern and is almost frequency invariant. The beampattern of the MWNG beamformer varies with frequency and it is different from the target beampattern at high frequencies as seen from Fig. 3(b). The ZOU beamformer has successfully mitigated the extra-null problem with the MWNG beamformer, but its beampattern still varies slightly with frequency. In comparison, the proposed optimal DMA has achieved frequency-invariant beampattern in the entire frequency band of interest.

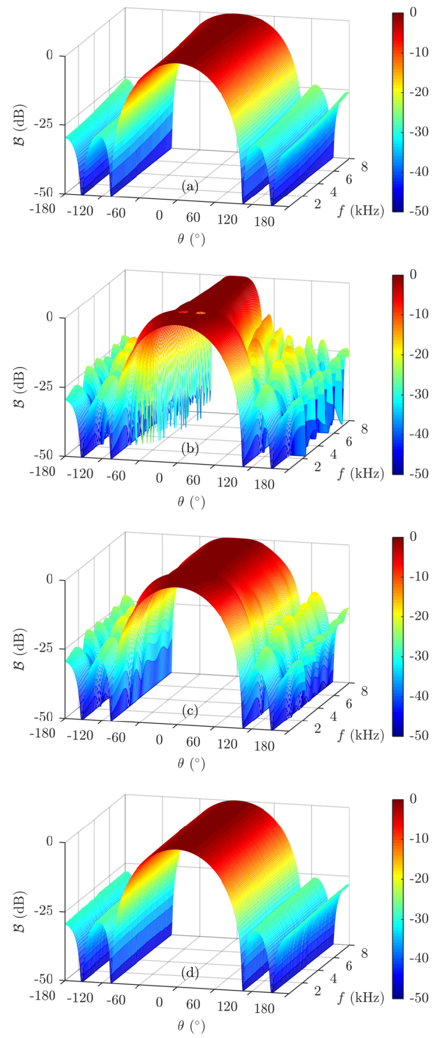

Fig. 3. Beampatterns of the conventional, MWNG, ZOU, and the developed optimal DMAs: (a) conventional, (b) MWNG, (c) ZOU, and (d) proposed optimal.

6. CONCLUSIONS

In this paper, we presented a method to design optimal nonuniform linear DMAs by optimizing subarray geometries and optimal subarray combination. With a specified target directivity pattern, this method optimizes the array geometry via two optimization processes: the first one identifies the optimal subarray geometries based on which the subarray set is formed, and the other optimizes the array geometry. In comparison with the popularly used existing approaches, the proposed method can achieve better frequency-invariant DF within the wide frequency band of interest while maintaining the WNG to a reasonable level.

7. RELATION TO PRIOR WORK

Beamforming is a critical approach to speech enhancement in complex acoustic environments. Many beamforming algorithms have been developed in the literature [29], such as the delay-and-sum beamformer [30, 31], the superdirective beamformers [32–34], and the differential beamformers [35, 36]. One important factor that may significantly affect the beamforming performance is the array geometry, whose optimization is proven to improve performance [37–42]. The DMAs, which are generally small in size and have almost frequency-invariant beampatterns, have been widely used for processing broadband signals such as speech [1, 3]. Traditionally, DMAs are implemented in a multistage way, which lacks flexibility in controlling white noise amplification [3]. Recently, a frequency-domain approach was developed to design DMAs with null constraints from the target beampattern, which offers the flexibility to design DMAs of different orders and deal with the white noise amplification problem with the MWNG method. [1, 21]. However, the MWNG differential beamformer may suffer from beampattern distortion at high frequencies [21], which makes the designed beampattern no longer resemble the target beampattern. This paper developed a method to design optimal nonuniform linear DMAs by optimizing the array geometry, which can achieve the target DF and frequency-invariant beampattern while maintaining the WNG to a reasonable level.

8. REFERENCES

[1] J. Benesty and J. Chen, *Study and Design of Differential Microphone Arrays*. Berlin, Germany: Springer-Verlag, 2012.

[2] G. W. Elko and J. Meyer, "Microphone arrays," in *Springer Handbook of Speech Processing* (J. Benesty, M. M. Sondhi, and Y. Huang, eds.), ch. 48, pp. 1021–1041, Berlin, Germany: Springer-Verlag, 2008.

[3] G. W. Elko, "Differential microphone arrays," in *Audio Signal Processing for Next-Generation Multimedia Communication Systems*, pp. 11–65, Springer, 2004.

[4] J. Meyer, "Beamforming for a circular microphone array mounted on spherically shaped objects," *J. Acoust. Soc. Am.*, vol. 109, pp. 185–193, Jan. 2001.

[5] S. Yan and Y. Ma, "Robust supergain beamforming for circular array via second-order cone programming," *App. Acous.*, vol. 66, no. 9, pp. 1018–1032, 2005.

[6] G. Huang, J. Benesty, and J. Chen, "On the design of frequency-invariant beampatterns with uniform circular microphone arrays," *IEEE/ACM Trans. Audio, Speech, Lang. Process.*, vol. 25, no. 5, pp. 1140–1153, 2017.

[7] S. Chan and H. Chen, "Uniform concentric circular arrays with frequency-invariant characteristics: theory, design, adaptive beamforming and doa estimation," *IEEE Trans. Signal Process.*, vol. 55, no. 1, pp. 165–177, 2007.

[8] G. Huang, J. Chen, and J. Benesty, "Insights into frequency-invariant beamforming with concentric circular microphone arrays," *IEEE/ACM Trans. Audio, Speech, Lang. Process.*, vol. 26, no. 12, pp. 2305–2318, 2018.

[9] G. Huang, J. Benesty, and J. Chen, "Design of robust concentric circular differential microphone arrays," *J. Acoust. Soc. Am.*, vol. 141, no. 5, pp. 3236–3249, 2017.

[10] B. Rafaely, *Fundamentals of Spherical Array Processing*. Berlin, Germany: Springer-Verlag, 2015.

[11] B. Rafaely and D. Khaykin, "Optimal model-based beamforming and independent steering for spherical loudspeaker arrays," *IEEE Trans. Audio, Speech, Lang. Process.*, vol. 19, no. 7, pp. 2234–2238, 2011.

[12] H. Schjær-Jacobsen and K. Madsen, "Synthesis of nonuniformly spaced arrays using a general nonlinear minimax optimisation method," *IEEE Trans. Antennas and Propagation*, vol. 24, pp. 501–506, 1976.

[13] S. Holm, B. Elgetun, and G. Dahl, "Properties of the beampattern of weight-and layout-optimized sparse arrays," *IEEE Trans. Ultrason., Ferroelect., Freq. Control*, vol. 44, pp. 983–991, 1997.

[14] M. Crocco and A. Trucco, "A computationally efficient procedure for the design of robust broadband beamformers," *IEEE Trans. Signal Process.*, vol. 58, pp. 5420–5424, 2010.

[15] M. Crocco and A. Trucco, "Stochastic and analytic optimization of sparse aperiodic arrays and broadband beamformers with robust superdirective patterns," *IEEE Trans. Audio, Speech, Lang. Process.*, vol. 20, pp. 2433–2447, 2012.

[16] J. Yu and K. D. Donohue, "Performance for randomly described arrays," in *Proc. IEEE WASPAA*, pp. 269–272, 2011.

[17] D. G. Kurup, M. Himdi, and A. Rydberg, "Synthesis of uniform amplitude unequally spaced antenna arrays using the differential evolution algorithm," *IEEE Trans. Antennas Propag.*, vol. 51, pp. 2210–2217, 2003.

[18] M. M. Khodier and C. G. Christodoulou, "Linear array geometry synthesis with minimum sidelobe level and null control using particle swarm optimization," *IEEE Trans. Antennas Propag.*, vol. 53, pp. 2674–2679, 2005.

[19] M. Crocco and A. Trucco, "Design of superdirective planar arrays with sparse aperiodic layouts for processing broadband signals via 3-d beamforming," *IEEE/ACM Trans. Audio, Speech, Lang. Process.*, vol. 22, pp. 800–815, 2014.

[20] S. J. Patel, S. L. Grant, M. Zawodniok, and J. Benesty, "On the design of optimal linear microphone array geometries," in *Proc. IEEE IWAENC*, pp. 501–505, 2018.

[21] J. Chen, J. Benesty, and C. Pan, "On the design and implementation of linear differential microphone arrays," *J. Acoust. Soc. Am.*, vol. 136, pp. 3097–3113, Dec. 2014.

[22] H. L. Van Trees, *Detection, Estimation, and Modulation Theory, Optimum Array Processing*. John Wiley & Sons, 2004.

[23] M. Brandstein and D. Ward, *Microphone Arrays: Signal Processing Techniques and Applications*. Springer, 2001.

[24] G. W. Elko, "Superdirectional microphone arrays," in *Acoustic Signal Processing for Telecommunication*, pp. 181–237, Springer, 2000.

[25] C. Pan, J. Chen, and J. Benesty, "Theoretical analysis of differential microphone array beamforming and an improved solution," *IEEE/ACM Trans. Audio, Speech, Lang. Process.*, vol. 23, no. 11, pp. 2093–2105, 2015.

[26] K. Bernhard and V. Jens, *Combinatorial Optimization: Theory and Algorithms, Algorithms and Combinatorics*. Berlin, Germany: Springer-Verlag, 2018.

[27] J. Kennedy, "Particle swarm optimization," in *Encyclopedia of Machine Learning*, pp. 760–766, Berlin, Germany: Springer-Verlag, 2011.

[28] M. Clerc and J. Kennedy, "The particle swarm - explosion, stability, and convergence in a multidimensional complex space," in *IEEE Trans. Evol. Comput.*, vol. 6, pp. 58–73, 2002.

[29] S. Markovich, S. Gannot, and I. Cohen, "Multichannel eigenspace beamforming in a reverberant noisy environment with multiple interfering speech signals," *IEEE Trans. Audio, Speech, Lang. Process.*, vol. 17, no. 6, pp. 1071–1086, 2009.

[30] B. Rafaely, "Phase-mode versus delay-and-sum spherical microphone array processing," *IEEE Signal Process. Lett.*, vol. 12, no. 10, pp. 713–716, 2005.

[31] Y. Zeng and R. C. Hendriks, "Distributed delay and sum beamformer for speech enhancement via randomized gossip," *IEEE/ACM Trans. Audio, Speech, Lang. Process.*, vol. 22, no. 1, pp. 260–273, 2014.

[32] E. Mabande, A. Schad, and W. Kellermann, "Design of robust superdirective beamformers as a convex optimization problem," in *Proc. IEEE ICASSP*, pp. 77–80, 2009.

[33] S. Doclo and M. Moonen, "Superdirective beamforming robust against microphone mismatch," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 15, no. 2, pp. 617–631, 2007.

[34] G. Huang, J. Benesty, and J. Chen, "Superdirective beamforming based on the Krylov matrix," *IEEE/ACM Trans. Audio, Speech, Lang. Process.*, vol. 24, pp. 2531–2543, 2016.

[35] E. D. Sena, H. Hacihabiboglu, and Z. Cvetkovic, "On the design and implementation of higher-order differential microphones," *IEEE Trans. Audio, Speech, Lang. Process.*, vol. 20, pp. 162–174, Jan. 2012.

[36] G. Huang, J. Chen, and J. Benesty, "On the design of differential beamformers with arbitrary planar microphone array," *J. Acoust. Soc. Am.*, vol. 144, no. 1, pp. 3024–3035, 2018.

[37] P. J. Bevelacqua and C. A. Balanis, "Geometry and weight optimization for minimizing sidelobes in wideband planar arrays," *IEEE Trans. Antennas Propag.*, vol. 57, pp. 1285–1289, 2009.

[38] Z. G. Feng, K. F. C. Yiu, and S. E. Nordholm, "Placement design of microphone arrays in near-field broadband beamformers," *IEEE Trans. Signal Process.*, vol. 60, pp. 1195–1204, 2012.

[39] J. Yu and K. D. Donohue, "Optimal irregular microphone distributions with enhanced beamforming performance in immersive environments," *J. Acoust. Soc. Am.*, vol. 134, pp. 2066–2077, 2013.

[40] O. Quevedo-Teruel and E. Rajo-Iglesias, "Ant colony optimization in thinned array synthesis with minimum sidelobe level," *IEEE Antennas Wireless Propag. Lett.*, vol. 5, pp. 349–352, 2006.

[41] J. Yu and K. D. Donohue, "Geometry descriptors of irregular microphone arrays related to beamforming performance," *EURASIP J. Adva. Signal Process.*, vol. 2012, p. 249, 2012.

[42] M. Bjelić, M. Stanojević, D. Šumarac Pavlović, and M. Mijić, "Microphone array geometry optimization for traffic noise analysis," *J. Acoust. Soc. Am.*, vol. 141, pp. 3101–3104, 2017.