OPTIMAL TRILATERATION IS AN EIGENVALUE PROBLEM

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ABSTRACT

The problem of estimating receiver or sender node positions from measured receiver-sender distances is a key issue in different applications such as microphone array calibration, radio antenna array calibration, mapping and positioning using UWB or using round-trip-time measurements between mobile phones and WiFi-units. In this paper we address the problem of optimally estimating a receiver position given a number of distance measurements to known sender positions, so called *trilateration*. We show that this problem can be rephrased as an eigenvalue problem. We also address different error models and the multilateration setting where an additional offset is also unknown, and show that these problems can be modeled using the same framework.

Index Terms— Trilateration, Calibration, Optimal estimation, Multilateration.

1. INTRODUCTION

Sound localization has been a topic of interest in a wide range of applications for centuries, and is well known to be a difficult problem, especially in a reverberating room environment (see e.g. [1–8], and the references therein). Measuring the time it takes for the signal to reach each sensor, the position of the source can be estimated. In the literature, this is referred to as either time of arrival (TOA) estimation, if the time of signal emission is known, or otherwise time difference of arrival (TDOA) estimation, where only the relative time delays are used. Common techniques for delay estimation include different variations on cross-correlation or canonical correlation analysis (CCA), which then allows the sources to be located in a second step using tri- and multi-lateration (see e.g. [9, 10]). Other examples of sensor from which we can get distance measurements include Ultra-Wideband, WiFi signal strength and Narrowband Radio Signals [11–14].

1.1. Trilateration and Related Work

To formalize our problem, we want to recover the position of an unknown receiver $\mathbf{x} \in \mathbb{R}^n$, given the positions of N anchors $\mathbf{s}_j$ and distance measurements d_j to these anchors. Typically if the Euclidean distances are measured we aim at having $\|\mathbf{x} - \mathbf{s}_j\| \approx d_j$ for $j = 1, 2, \ldots, N$. For Gaussian noise the Maximum Likelihood (ML) estimate is given by the following optimization problem,

Problem 1 $\mathbf{x}^\star = \operatorname{argmin}_{\mathbf{x}\in\mathbb{R}^n} \sum_{j=1}^{N} \left(\|\mathbf{x} - \mathbf{s}_j\| - d_j\right)^2 .$

This is a non-linear and non-convex optimization problem which can have several local minima.

Previously several methods have been proposed for solving Problem 1. In [15] the authors used an SDP relaxation approach. The authors solve a convex relaxation of the problem, and there is no guarantee that the solution will be optimal in the original cost. Recently [16] presented a fixed point iteration for solving Problem 1.

A standard way to derive a linear solver is to consider the equations $\|\mathbf{x}-\mathbf{s}_j\|^2 = d_j^2$. Forming differences between pairs of these equations the quadratic terms in $\mathbf{x}$ cancel, leaving only linear equations in $\mathbf{x}$,

$$2\left(\mathbf{s}_i - \mathbf{s}_j\right)^T \mathbf{x} = d_j^2 - d_i^2 + \mathbf{s}_j^T\mathbf{s}_j - \mathbf{s}_i^T\mathbf{s}_i \qquad (1)$$

which can be solved in a least squares sense. However, this does not minimize any meaningful cost (see Figure 1). Variants of this method can be constructed by e.g. by taking the difference of all points to one reference point or by taking the difference with the mean of all points, see [17].

In [15] the authors also present a globally optimal method for minimizing the following surrogate function,

$$h(\boldsymbol{x}) = \sum_j \left(\|\boldsymbol{x} - \mathbf{s}_j\|^2 - d_j^2\right)^2 . \qquad (2)$$

They consider the equivalent problem of minimizing

$$h(\boldsymbol{x}, \alpha) = \sum_j \left(\alpha - 2\boldsymbol{x}^T\mathbf{s}_j + \mathbf{s}_j^T\mathbf{s}_j - d_j^2\right)^2 , \qquad (3)$$

under the quadratic constraint $\alpha = \boldsymbol{x}^T\boldsymbol{x}$. They then set up the Lagrangian and after some manipulation end up with a single equation that only depends on the multiplier. In [15] this equation is then solved using bisection.

In [18] Zhou presented another method for the minimizing the squared distances loss. To solve the problem [18] introduce the artificial constraint $\sum_j \|\mathbf{x} - \mathbf{s}_j\|^2 = \sum_j d_j^2$, which is not satisfied in general, resulting in sub-optimal solutions.

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In [19] the authors propose to minimize the maximum likelihood cost in Problem 1 by solving a sequence of weighted versions of (2), in an IRLS-like fashion [20].

1.2. Non-Gaussian Error Models

When estimating the distances using the signal strength (see e.g. [21]) the noise often becomes Gaussian when considering the measured power

$$P_j = b + k\log\|\mathbf{x}-\mathbf{s}_j\| + \epsilon, \quad \epsilon \in \mathcal{N}(0,\sigma) \tag{4}$$

or equivalently $\log\|\mathbf{x}-\mathbf{s}_j\|^2 = 2\frac{P_j-b}{k} + \epsilon'$ where $\epsilon' \in \mathcal{N}(0, 2\sigma/k)$. Here b and k are model parameters that we assume to be known. This leads to the following ML estimator (where $m_j = 2\frac{P_j-b}{k}$),

Problem 2 $\mathbf{x}^\star = \operatorname{argmin}_{\mathbf{x}\in\mathbb{R}^n} \sum_j \left(\log\|\mathbf{x}-\mathbf{s}_j\|^2 - m_j\right)^2.$

1.3. Multilateration

In the multilateration setting we have measurements to a number of known sender positions. These measurements contain a common unknown offset which also has to be estimated. In this case the ML estimate (under the assumption of Gaussian noise) is given as the solution to

Problem 3 $\mathbf{x}^\star = \operatorname{argmin}_{\mathbf{x}\in\mathbb{R}^n} \sum_{j=1}^N \left(\|\mathbf{x}-\mathbf{s}_j\| - (d_j+o)\right)^2.$

1.4. Paper Contributions

The cost functions in Problem 1 and 2 are both on the form

$$h(\mathbf{x}) = \sum_j \left(\Psi(\|\mathbf{x}-\mathbf{s}_j\|^2) - m_j\right)^2 \tag{5}$$

with $\Psi(x) = \sqrt{x}$ and $\Psi(x) = \log x$ respectively. We can get an approximation by replacing Ψ with its first order Taylor expansion. The linearization point is chosen as the point x_j which satisfies $\Psi(x_j) = m_j$. Note that this does not depend on the receiver position. By differentiating $\Psi(x)$ and inserting the linearization points, the approximations of the cost functions in Problem 1 and 2 become

$$h_1(\boldsymbol{x}) = \sum_j \frac{1}{4d_j^2}\left(\|\mathbf{x}-\mathbf{s}_j\|^2 - d_j^2\right)^2, \tag{6}$$

$$h_2(\boldsymbol{x}) = \sum_j e^{-2m_j}\left(\|\mathbf{x}-\mathbf{s}_j\|^2 - e^{m_j}\right)^2, \tag{7}$$

and for the multilateration cost function in Problem 3 we get

$$h_3(\boldsymbol{x}) = \sum_j \frac{1}{4(d_j+o)^2}\left(\|\mathbf{x}-\mathbf{s}_j\|^2 - (d_j+o)^2\right)^2. \tag{8}$$

Note that these approximate functions have very similar structure.

In this paper we present fast closed form solutions to the two non-convex optimization problems

$$\min_{\mathbf{x}} \sum_j w_j \left(\|\mathbf{x}-\mathbf{s}_j\|^2 - d_j^2\right)^2, \tag{w-TOA}$$

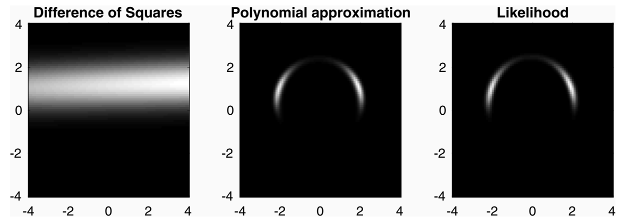

Fig. 1. Comparison of cost functions for a synthetic instance with two local minima. Left to right: Difference of squares as in (1), Weighted approximation (6), True likelihood.

$$\min_{\mathbf{x},\,o} \sum_j w_j \left(\|\mathbf{x}-\mathbf{s}_j\|^2 - (d_j+o)^2\right)^2, \tag{w-TDOA}$$

by deriving equivalent eigenvalue problems. In contrast to previous approaches we can enumerate all stationary points and guarantee global optimality. Our method works for arbitrary dimension. Since we enumerate all stationary points we are also able to identify situations where there are multiple good competing hypotheses. Additionally we further explore the IRLS-like scheme used in [19] to minimize different cost functions corresponding to ML estimates for different noise distributions, for example as in Problem 2.

2. OPTIMAL TRILATERATION

In this section we will show that the problem (w-TOA) is equivalent to an eigenvalue problem. The cost function in (w-TOA) can be written as

$$h(\boldsymbol{x}) = \sum_j w_j \left(\boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{s}_j + \boldsymbol{s}_j^T\boldsymbol{s}_j - d_j^2\right)^2. \tag{9}$$

Since the cost is differentiable everywhere, the globally optimal solution must lie at a stationary point of $h(\boldsymbol{x})$. The first order optimality conditions are $\nabla h(\boldsymbol{x}) =$

$$4\sum_j w_j \left(\boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{s}_j + \boldsymbol{s}_j^T\boldsymbol{s}_j - d_j^2\right)(\boldsymbol{x}-\boldsymbol{s}_j) = 0. \tag{10}$$

This gives us n polynomial equation system of degree 3 in n unknowns. Naive application of the Bezout bound [22] for this system yields that there are at most 3^n solutions (where n is dimension of the ambient space). However, we will show that due to the specific structure of the equations, there are in general only $2n+1$ stationary points.

2.1. Simplifying the Equations

Collecting the terms in (10) by degree we get

$$\frac{1}{4}\nabla h(\boldsymbol{x}) = (\sum_j w_j)(\boldsymbol{x}^T\boldsymbol{x})\boldsymbol{x} \tag{11}$$

$$- \left(\boldsymbol{x}^T\boldsymbol{x}I + 2\boldsymbol{x}\boldsymbol{x}^T\right)\left(\sum w_j\boldsymbol{s}_j\right) \tag{12}$$

$$+ \left(\sum_j w_j\left((\boldsymbol{s}_j^T\boldsymbol{s}_j - d_j^2)I + 2\boldsymbol{s}_j\boldsymbol{s}_j^T\right)\right)\boldsymbol{x} \tag{13}$$

$$+ \sum_j w_j\left(d_j^2 - \boldsymbol{s}_j^T\boldsymbol{s}_j\right)\boldsymbol{s}_j = 0. \tag{14}$$

Since the global coordinate system is arbitrary we can choose this to simplify the equations. Similarly to the approach in [18] we start by translating the senders $\boldsymbol{s}_j$ with

$$\boldsymbol{t} = -(\textstyle\sum_j w_j \boldsymbol{s}_j)/(\sum_j w_j). \tag{15}$$

This ensures that $\sum_j w_j \boldsymbol{s}_j = 0$ which cancels all the quadratic terms in (12). Similarly, since the cost function is homogeneous in the weights, we can w.l.o.g. assume that $\sum_j w_j = 1$, which make the coefficients for all third degree terms one. The equations are now of the form

$$(\boldsymbol{x}^T\boldsymbol{x})\boldsymbol{x} + A\boldsymbol{x} + \boldsymbol{b} = 0. \tag{16}$$

The matrix A is symmetric, and thus we can perform an orthogonal eigenvalue decomposition $A = UDU^T$. Performing the change of variables $\boldsymbol{x} \to U\boldsymbol{x}$ and $\boldsymbol{b} \to U\boldsymbol{b}$, the equations separate and we get (since $\boldsymbol{x}^T\boldsymbol{x} = (U\boldsymbol{x})^T U\boldsymbol{x}$)

$$(\boldsymbol{x}^T\boldsymbol{x})x_i + D_{ii}x_i + b_i = 0, \quad i = 1, 2, \ldots, n. \tag{17}$$

2.2. Deriving the Eigenvalue Problem

Multiplying each equation in (17) with x_i we get

$$(\boldsymbol{x}^T\boldsymbol{x})x_i^2 + D_{ii}x_i^2 + b_i x_i = 0, \quad i = 1, 2, \ldots, n. \tag{18}$$

With some abuse of notation we use $\boldsymbol{x}^2$ to denote the vector of pure squares, i.e. $\boldsymbol{x}^2 = (x_1^2, x_2^2, \ldots, x_n^2)^T$. The equations in (17) and (18) can then be written as

$$(\boldsymbol{x}^T\boldsymbol{x})\begin{pmatrix}\boldsymbol{x}^2\\ \boldsymbol{x}\\ 1\end{pmatrix} = \underbrace{\begin{bmatrix}-D & -\mathrm{diag}(\boldsymbol{b}) & \boldsymbol{0}\\ \boldsymbol{0} & -D & -\boldsymbol{b}\\ \mathbb{1}^T & \boldsymbol{0}^T & 0\end{bmatrix}}_{=:M}\begin{pmatrix}\boldsymbol{x}^2\\ \boldsymbol{x}\\ 1\end{pmatrix}. \tag{19}$$

where the last row is the trivial equation $(\boldsymbol{x}^T\boldsymbol{x}) = \mathbb{1}^T\boldsymbol{x}^2$. Note that the matrix in (19) is a constant square matrix and does not depend on the unknowns $\boldsymbol{x}$. Additionally the matrix M can easily be computed for a given problem instance.

Any solution to the original problem (10) must also satisfy (19) (after appropriate change of variables). This means that for any solution $\boldsymbol{x}$ we have that the vector $(\boldsymbol{x}^2, \boldsymbol{x}, 1)^T$ is an eigenvector to M with eigenvalue $\boldsymbol{x}^T\boldsymbol{x}$. So to enumerate all solutions to the original system we do the following:

1. Compute matrix M as described above.
2. Compute all eigenvectors.
3. Normalize eigenvectors such that the last element is one and extract $\boldsymbol{x}$ from the corresponding elements.
4. Evaluate original cost at each stationary point candidate and choose the one with smallest cost.

While it is possible that there are eigenvectors which do not correspond to a stationary point, all stationary points are among the eigenvectors, so by enumerating all of them we are guaranteed to find the global optimum.

2.3. Stationary Points and Degenerate Configurations

From the results in the previous section, it follows that there are at most $2n+1$ stationary points with different values for $\boldsymbol{x}^T\boldsymbol{x}$. In general each eigenvector will corresponds to a solution of the original system. However, it is possible to have eigenvalues with higher multiplicity. This happens for example in degenerate situations (e.g. all senders lie on a line). Note that in this case (19) still holds, and we still recover the correct eigenvalues (i.e. values of $\boldsymbol{x}^T\boldsymbol{x}$). However, since the eigenspaces are not necessarily one-dimensional extra care must be taken to recover the solutions.

2.4. Optimal Multilateration

In this section we consider the multilateration case where we also need to estimate an unknown offset between the senders and the receiver. Similarly to the trilateration problem the cost function can be expanded as $h(\boldsymbol{x}, o) =$

$$\textstyle\sum_j w_j\left(\boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{s}_j + \boldsymbol{s}_j^T\boldsymbol{s}_j - d_j^2 - 2d_j o - o^2\right)^2. \tag{20}$$

The first order optimality conditions are[1] $0 = \frac{1}{4}\nabla h(\boldsymbol{x}, o) =$

$$\sum_j w_j\left(\boldsymbol{x}^T\boldsymbol{x} - o^2 - 2\left(\boldsymbol{x}^T, o\right)\begin{pmatrix}\boldsymbol{s}_j\\ d_j\end{pmatrix} + \boldsymbol{s}_j^T\boldsymbol{s}_j - d_j^2\right)\begin{pmatrix}\boldsymbol{x} - \boldsymbol{s}_j\\ o + d_j\end{pmatrix}.$$

Again, shifting the coordinate systems (this time also in o) we can cancel the quadratic terms, and the equations become

$$(\boldsymbol{x}^T\boldsymbol{x} - o^2)\left(\boldsymbol{x}, o\right)^T + A\left(\boldsymbol{x}, o\right)^T + \boldsymbol{b} = 0. \tag{21}$$

Now the goal is to transform these equations into an eigenvalue problem. Unfortunately it is not possible to use the diagonalization trick here. The last $n+1$ rows are already given by (21) and we only need to determine the top n rows, i.e.

$$(\boldsymbol{x}^T\boldsymbol{x} - o^2)\begin{pmatrix}\boldsymbol{x}^2\\ o^2\\ \boldsymbol{x}\\ o\\ 1\end{pmatrix} = \begin{bmatrix}? & ? & ?\\ \boldsymbol{0}^T & -A & -b\\ (\mathbb{1}^T, -1) & \boldsymbol{0}^T & 0\end{bmatrix}\begin{pmatrix}\boldsymbol{x}^2\\ o^2\\ \boldsymbol{x}\\ o\\ 1\end{pmatrix}, \tag{22}$$

Let $\lambda = \boldsymbol{x}^T\boldsymbol{x} - o^2$. Multiplying (21) with $\mathrm{diag}(\boldsymbol{x}, o)$ we get

$$\lambda\left(\boldsymbol{x}^2, o^2\right)^T = -\mathrm{diag}(\boldsymbol{x}, o)\left(A\left(\boldsymbol{x}, o\right)^T + \boldsymbol{b}\right). \tag{23}$$

This almost yields the missing rows, except for a few quadratic mixed terms (e.g. x_1x_2) appearing in the RHS. To eliminate the mixed quadratic terms, the goal is to the express them in monomials appearing in the eigenvector, i.e. $(\boldsymbol{x}^2, o^2, \boldsymbol{x}, o, 1)$.

Multiplying the first equation in (21) with x_2 and subtracting the second equation multiplied with x_1, any terms containing λ cancel, and we are left with an equation containing only mixed quadratic terms and monomials which appear

[1] To simplify calculations the last equation has changed sign.

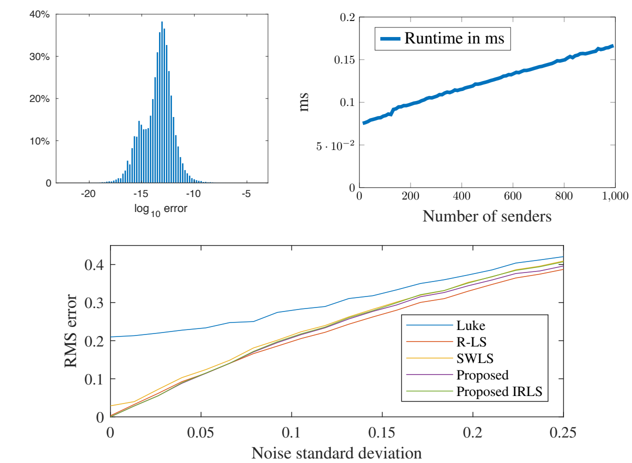

Fig. 2. *Top left:* Distribution of errors for synthetic noise-less instances. *Top right:* Runtime (in ms) for different number of senders. Bottom: RMS error in receiver position for various amounts of noise.

in the eigenvector in (22). Doing this for all pairs of equations in (21) yields a set of $\binom{n+1}{2}$ equations, on the form

$$C_0 \boldsymbol{m} + C_1 \left(\boldsymbol{x}^2, o^2, \boldsymbol{x}, o, 1\right)^T = 0 \tag{24}$$

where $\boldsymbol{m}$ is the vector of monomials containing the mixed quadratic terms. Inserting $\boldsymbol{m} = -C_0^{-1} C_1 \left(\boldsymbol{x}^2, o^2, \boldsymbol{x}, o, 1\right)^T$ into (23) we can eliminate all mixed terms and recover the missing rows in the eigenvalue problem (22).

3. EXPERIMENTS AND APPLICATIONS

3.1. Numerical Stability and Computational Cost

We evaluated our method on synthetically generated data. Figure 2 shows the equation residuals (left) and the runtime plotted against the number of senders (right).

3.2. Maximum Likelihood Estimation using IRLS

Since we can solve the weighted problem (w-TOA) optimally, we can use this to iteratively minimize the true ML cost functions. We iterate the following two steps until convergence:

1. $\boldsymbol{x}^t = \arg\min_{\mathbf{x}} \sum_j w_j^t \left(\|\mathbf{x} - \mathbf{s}_j\|^2 - d_j^2\right)^2$
2. $w_j^{t+1} = (2\|\mathbf{x}^t - \mathbf{s}_j\| \left(\|\mathbf{x}^t - \mathbf{s}_j\| + d_j\right))^{-1}$.

These weights are chosen such that the gradients of the ML and the weighted cost align in each iteration, i.e.

$$\nabla_{\mathbf{x}}\, w_j \left(\|\mathbf{x} - \mathbf{s}_j\|^2 - d_j^2\right)^2 = \nabla_{\mathbf{x}} \left(\|\mathbf{x} - \mathbf{s}_j\| - d_j\right)^2 \tag{25}$$

This ensures that any limit point is a stationary point of the original cost. Note that this weighting is different from the weighting used in [19].

Table 1. RMS errors (meters) and total execution time (seconds) when running eight real UWB datasets.

Data-set	Zhou [18]	Luke [16]	SR-LS [15]	R-LS [15]	ML	Prop.	Prop. IRLS
1	0.66	0.34	0.40	0.34	0.34	**0.32**	0.34
2	0.65	**0.52**	0.54	**0.52**	**0.52**	**0.52**	**0.52**
3	11.05	0.64	1.18	0.64	0.64	**0.42**	0.64
4	0.54	0.31	0.44	0.32	0.31	**0.30**	0.31
5	0.64	0.34	0.41	0.34	0.34	**0.33**	0.34
6	0.47	0.31	0.37	0.31	0.31	**0.28**	0.31
7	0.53	0.32	0.35	0.32	0.32	**0.31**	0.32
8	0.74	0.39	0.48	0.39	0.39	**0.36**	0.39
Time	**0.43**	6.49	3.63	93.00	33.41	0.50	2.13

Similarly, to solve Problem 2 we update the weights as

$$w_j^t = \frac{\log(\|\mathbf{x}^t - \mathbf{s}_j\|^2) - m_j}{\log(\|\mathbf{x}^t - \mathbf{s}_j\|^2)(\|\mathbf{x}^t - \mathbf{s}_j\|^2 - e^{m_j})}. \tag{26}$$

Figure 2 shows the RMS error in receiver positions for different amounts of noise in a setup of six senders and one receiver, all sampled uniformly from a unit cube. We compare the proposed method and its IRLS application as just presented with Luke [16], the R-LS solver from [15] and the SWLS solver from [19]. All methods perform similarly except for Luke which occasionally convergences to the wrong stationary point.

3.3. Real Data Experiment

We evaluate our method using TOA datasets gathered with an ultra-wideband (UWB) setup. Six senders were kept stationary as a single receiver was moved through the setup. Ground truth for sender and receiver positions was determined using an optical motion capture system. We compare the proposed method with Zhou [18], Luke [16], the SR-LS solver from [15] and the R-LS solver from [15] (see Table 1). For reference we include the ML estimate found by solving Problem 1 using standard iterative optimization methods initialized at the ground truth positions. All algorithms were implemented in MATLAB. In many cases the proposed solver without IRLS performs best. This is likely due to that the errors are not completely Gaussian.

4. CONCLUSION

In this paper we have introduced two eigenvalue solvers that give closed-form-solutions to two different non-linear weighted least squares problems. We have also shown how these solvers can be used to do optimal trilateration and multilateration.

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