AN IMPROVED METHOD FOR PARAMETRIC SPECTRAL ESTIMATION

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***Abstract*—One important class of problems within spectral estimation is when the signal can be well represented by a parametric model. These kind of problems can be found in many applications such as radar, sonar and wireless communication, and has therefore been extensively investigated. The main problem is to estimate frequencies and their corresponding amplitudes and damping factors from noisy measurements. One approach to this problem is to form a matrix of measurements, and then find an approximation to the range space of the matrix, with requirements that the approximation is of low rank and have a Hankel structure. From the approximation, the signal parameters can be extracted. In this work, we investigate three different methods which follows this methodology. The main contribution will be an illustration of how the problem formulation and rank constraint management affects the accuracy of the estimate. Numerical simulations indicates that a method which formulates a single convex envelope of a least squares fit to the measurement matrix and to the rank constraint jointly is more accurate than the other two investigated methods.**

***Index Terms*—Parametric spectral estimation, Hankel matrix, convex envelope, ADMM**

I. INTRODUCTION

Spectral estimation is a classical problem which appears in many signal processing applications. These applications include radar, sonar and wireless communication [1]. In this paper, we consider one class of problems within spectral estimation, when the signal can be well represented by a low dimensional parametric model. The main task is to estimate frequency components, and their corresponding amplitudes and damping factors, from noisy measurements.

A popular approach to this problem is to form the measurements in a matrix with Hankel structure [2]. If the measurements are not corrupted by noise, this matrix will have a rank which corresponds to the number of frequency components. If the noise free Hankel matrix is factorized into two full rank factors, these factors has a particular shift structure which can be utilized to calculate the signal parameters. Unfortunately, both these properties may be lost when the measurements are corrupted by noise.

The Estimation of Signal Parameters via Rotational Invariance Technique (ESPRIT) method is commonly used [3]–[7]. It finds an estimate to the range space to the matrix of measurements via a Singular Value Decomposition (SVD). Since the number of frequency components is assumed known, small singular values are set to zero to obtain a low rank approximation. However, truncating the estimated matrix by setting singular values to zero, the shift structure required for reconstruction is destroyed and the product of the two low rank factors is not a Hankel matrix. Therefore the signal components are estimated from a range space basis via a least squares fit. This is however sensitive to high noise levels.

Therefore, the problem can instead be formulated as a least squares optimization problem with requirements on the solution to both be of low rank and have a Hankel structure. Since a rank constraint is non-convex, such optimization problem is difficult to solve. A popular approach is therefore to relax the rank constraint by its convex envelope [8], [9], given by the nuclear norm function. The rationale behind this is to penalize the sum of the singular values making it likely that several of them are set to zero and thereby obtaining a low rank solution. However, since all singular values are penalized, and not just small ones, an unwanted bias is introduced in the solution.

To avoid a biased solution, [10], [11] propose to reformulate the rank constraint by an indicator function with threshold at the pre-determined rank value and include it in the cost function. As this reformulation does not change the non-convexity of the problem, Larsson *et. al.* [11] calculates the convex envelope to the least squares minimizer and the indicator function to make the formulation convex. By including the least squares minimizer into the convex envelope, small singular values can be penalized more than large ones.

In this work, we will compare the ESPRIT method, the method which relaxes the rank constraint by a nuclear norm function and the method introduced in [11], on the application of parametric spectral estimation. The main contribution of this paper will be an illustration of how the problem formulation and constraint management affects the accuracy of the estimate. The investigated methods mainly differs as follows; the ESPRIT method does not honor the Hankel structure of the solution. The method which relaxes the rank constraint by the nuclear norm, minimizes the sum of the least squares minimizer and the convex envelope to the rank function. While for the method introduced in [11] these quantities are both included in a single convex envelope before minimization. Moreover, because the method in [11] were recently introduced, it will be presented in more detail than the other two methods.

The paper is organized as follows. In Section II we introduce notations and preliminaries, together with brief introductions of the investigated methods. In Section III it is described how a single convex envelope is calculated to the least squares minimizer and the indicator function representing the rank constraint. In Section IV it is presented how the resulting

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convex optimization problem can be solved. Next, some illustrative simulations are shown in Section V. Finally, the paper is concluded in Section VI.

II. Preliminaries and notations

A. Notation and terminology

We use the notation $\{h_i\}_{i=1}^n$ for the sequence $[h(1), h(2), \dots, h(n)]^T$. Let $||\cdot||_F$ and $||\cdot||_*$ be the Frobenius norm and the nuclear norm, respectively. Further, let $\{\sigma_i(X)\}_{i=1}^M$ denote the singular values to X, where $\sigma_i(X) \geq \sigma_{i+1}(X)$ for all i. The nonnegative part operator is denoted $[x]_+$, which is zero for $x < 0$ and x otherwise.

For a matrix A, let $A_{a:b}$ denote the matrix with the ath to bth rows of A, where $a \leq b$.

Consider two closed proper convex functions $f, g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. Then, for the problem $\min_x f(x) + g(x)$, the iterative Alternating Direction Method of Multipliers (ADMM) algorithm provides a solution [12]:

$$x_{i+1} := \text{prox}_{\rho f}(z_i - u_i) \tag{1a}$$

$$z_{i+1} := \text{prox}_{\rho g}(x_{i+1} + u_i) \tag{1b}$$

$$u_{i+1} := u_i + x_{i+1} - z_{i+1} \tag{1c}$$

where *prox* is the proximal operator defined as:

$$\text{prox}_{\rho f}(v) = \arg\min_x \left(f(x) + \rho||x - v||_2^2\right) \tag{2}$$

where ρ is a constant [12].

The Fenchel conjugate f^* to f is defined as [13]:

$$f^*(X) = \sup_Y \langle Y, X\rangle - f(Y) \tag{3}$$

where $\langle Y, X\rangle = \text{tr}(X^T Y)$ is the inner product, trace function and $(\cdot)^T$ denotes the matrix transpose, respectively.

We denote the indicator function for the set χ as I_χ, i.e.

$$I_\chi = \begin{cases} 0, & x \in \chi \\ \infty, & \text{otherwise} \end{cases} \tag{4}$$

A Hankel matrix with m rows, for a sequence $\{y_i\}_{i=1}^N$, is defined as the matrix with equal elements on the anti-diagonals. Let $\mathcal{H}$ denote the set of Hankel matrices.

B. Background and problem description

Consider the discrete-time signal model

$$y(k) = \sum_{i=1}^n \alpha_i e^{(j\omega_i + \beta_i)k} + v(k) \tag{5}$$

where $\omega_i, \beta_i \in \mathbb{R}$ and $\alpha_i \in \mathbb{C}^n$ are unknown frequencies, damping factors and amplitudes respectively, and $v(k)$ is white noise. To make the model unique and real valued we assume that $\omega_i \in [-\pi, \pi]$ and $\alpha_i \neq 0$ for all i, $\omega_i \neq \omega_k$ for $i \neq k$ and for all i where $\omega_i \neq 0$ there exist an l such that $\alpha_i = \alpha_l^*$, $\omega_i = -\omega_l$ and $\beta_i = \beta_l$, where $(\cdot)^*$ denotes the complex conjugate.

Introducing the matrix $A = \text{diag}[e^{(j\omega_1+\beta_1)}, e^{(j\omega_2+\beta_2)}, \dots, e^{(j\omega_n+\beta_n)}]$, and the vectors $C = [\alpha_1, \alpha_2, \dots, \alpha_n]$ and $x_0 = [1, 1, \dots, 1]^T$, Equation (5) without the noise can be written on state space form as

$$x(k+1) = Ax(k), \quad x(0) = x_0, \tag{6a}$$

$$y(k) = Cx(k). \tag{6b}$$

Let Y_0 be a Hankel matrix, with $m > n$ and $N - m + 1 > n$, to a sequence of measurement $\{y_i\}_{i=1}^N$. Then, when the measurements are not corrupted by any noise, Y_0 can exactly be factorized as $Y_0 = \mathcal{O}\mathcal{C}$, where $\mathcal{O} = [C, CA, \dots, CA^{m-1}]^T$ and $\mathcal{C} = [x_0, Ax_0, \dots, A^{N-m}x_0]$ [14]. When Y_0 follows from measurements affected by noise, the factorization is invalid. In that case, an approximation of Y_0 has to be made.

C. ESPRIT

In the ESPRIT method [4], Y_0 is factorized by an SVD

$$Y_0 = U\Sigma V^T \tag{7}$$

where U and V are matrices containing the right and the left singular vectors, respectively. Σ is a matrix with the singular values on the diagonal. Since the number of frequency components, n, is known, the $n+1$ to M singular values are set to zero, where $M = \min(m, N - m + 1)$. This yields an approximative factorization of Y_0 with rank n,

$$Y = U_n \Sigma_n V_n^T \tag{8}$$

Since Y is estimated from a truncated SVD, the Hankel structure is generally lost. Therefore, by choosing $\mathcal{O} = U_n\Sigma_n^{1/2}$ and $\mathcal{C} = \Sigma_n^{1/2}V_n^T$, the state space representation are estimated from a least squares fit,

$$A = (\mathcal{O}_{1:m-1}^T \mathcal{O}_{1:m-1})^{-1} \mathcal{O}_{1:m-1}^T \mathcal{O}_{2:m}^T \tag{9a}$$

$$C = \mathcal{O}_{1:1}. \tag{9b}$$

D. Rank relaxation with a nuclear norm function

To find an approximation to Y_0 which honors the Hankel structure, consider the following optimization problem,

$$\min_Y ||Y - Y_0||_F^2 \tag{10a}$$

$$\text{s.t. } \text{rank}(Y) = n \tag{10b}$$

$$Y \in \mathcal{H}. \tag{10c}$$

As (10b) is non-convex, it can be relaxed by its convex envelope function [8]. The convex envelope to a rank function is the nuclear norm function. Hence, we have

$$\min_Y ||Y - Y_0||_F^2 + \gamma||Y||_* + I_{\mathcal{H}}. \tag{11}$$

where γ is a positive constant. This can be solved in multiple ways, in [8] semidefinite programming is used while in [12] the optimum is calculated with ADMM.

E. Method in [11]

Instead of using the convex envelope to the rank function, we will include the least squares minimizer together with the rank function in a single convex envelope. Hence, by reformulating the rank function by an indicator function with threshold at the pre-determined rank value, it can be included in the cost function. Thus, we obtain from (10)

$$\min_Y ||Y - Y_0||_F^2 + g(Y) + I_{\mathcal{H}} \tag{12}$$

where

$$g(Y) = \begin{cases} 0, & \text{if rank(Y)} \leq n \\ \infty, & \text{if rank(Y)} > n \end{cases} \tag{13}$$

For simplicity, we introduce the notations $f(Y) = ||Y - Y_0||_F^2 + g(Y)$ and $h(Y) = I_{\mathcal{H}}$.

III. Convexification of cost function

The derivation of the convex envelope in this section is based on calculations made in [11] but is shorter since we only consider the special case of the rank penalty function given in (13).

The convex envelope is given by the Fenchel bi-conjugate. Inserting $f(Y)$ into (3) we obtain the conjugate,

$$f^*(X) = \max_Y ||Z||_F^2 - ||Y_0||_F^2 - ||Y - Z||_F^2 - g(Y) \tag{14}$$

where we have used the notation $Z = \frac{1}{2}X + Y_0$. The first two terms are independent of Y and can therefore be considered as constants. Since $g(Y)$ is an indicator function with threshold at n, it will imply that the maximizing Y is of rank n. Hence, the optimal Y to (14) follows from the low rank approximation of Z obtained by the SVD. Thus, we have

$$f^*(X) = ||Z||_F^2 - ||Y_0||_F^2 - \sum_{i=n+1}^{M} \sigma_i^2(Z). \tag{15}$$

By inserting (15) in (3) we obtain the Fenchel bi-conjugate

$$f^{**}(Y) = \max_Z 2\langle Y, Z - Y_0\rangle - ||Z||_F^2 + ||Y_0||_F^2 + \sum_{i=n+1}^{M} \sigma_i^2(Z) \tag{16}$$

By completing squares the three first terms can be written as $||Y - Y_0||_F^2 - ||Z - Y||_F^2$, which yields

$$f^{**}(Y) = \max_Z \left(\sum_{i=n+1}^{M} \sigma_i^2(Z) - ||Z - Y||_F^2 \right) + ||Y - Y_0||_F^2 \tag{17}$$

IV. Alternating Direction Method of Multipliers

Now, we will implement the ADMM algorithm for the cost function which utilizes the corresponding convex envelope of f, derived in Section III. Hence, the convex cost function follows,

$$\min_Y f^{**}(Y) + h(Y). \tag{18}$$

First we note that the proximal operator to h is the Euclidian projection onto the space $\mathcal{H}$ [12].

For f^{**} we insert (17) into the definition of proximal operators, obtaining a convex-concave min-max problem.

$$\text{prox}_{\rho f^{**}}(U) = \arg\min_Y \max_Z \sum_{i=n+1}^{M} \sigma_i^2(Z) - ||Z - Y||_F^2 + \\ + ||Y - Y_0||_F^2 + \rho ||Y - U||_F^2 \tag{19}$$

To simplify calculations, the order of minimization and maximization can be switched according to Sion's minimax theorem [15]. Thus, the minimizing Y is,

$$Y = \frac{(\rho + 1)W - Z}{\rho} \tag{20}$$

where $W = \frac{Y_0 + \rho U}{1+\rho}$. Inserting Y into the cost function, we obtain,

$$\max_Z \sum_{i=n+1}^{M} \sigma_i^2(Z) - \frac{\rho + 1}{\rho} ||Z - W||_F^2. \tag{21}$$

Given that the first term is unitarly invariant, and the SVD $W = U\text{diag}(\sigma(W))V^T$, where U and V are matrices containing the right and the left singular vectors respectively, von Neumann's trace theorem gives that the maximizing Z has the form $Z = U\text{diag}(\sigma(Z))V^T$ [16], [17]. Hence, we obtain the following

$$\max_{\sigma(Z)} \sum_{i=n+1}^{M} \sigma_i^2(Z) - \frac{\rho+1}{\rho} \sum_{i=1}^{M} (\sigma_i(Z) - \sigma_i(W))^2 \tag{22a}$$

$$\text{s.t. } \sigma_i(Z) \geq \sigma_{i+1}(Z) \tag{22b}$$

Now, first, consider the unconstrained problem. The unconstrained optimal solution is

$$s_i = \begin{cases} \sigma_i(W), & i \leq n \\ (1+\rho)\sigma_i(W), & i > n. \end{cases} \tag{23}$$

If $\sigma_n(W) \geq (\rho+1)\sigma_{n+1}(W)$, then the unconstrained solution satisfies (22b), and hence it is the optimal solution to (22). If not, i.e. $\sigma_n(W) < (\rho+1)\sigma_{n+1}(W)$, then the unconstrained solution is not non-increasing, and hence violating (22b). In that case we reformulate the solution by consider $\max_s c(s) = \sum_{i=1}^{M} c_i(s)$, where

$$c_i(s) = \begin{cases} -\frac{\rho+1}{\rho}[s - \sigma_i(W)]_+^2, & i \leq n \\ -\frac{[s-\sigma_i(Y)]_+^2}{\rho} + (\rho+1)\sigma_i^2(W), & i > n \end{cases} \tag{24}$$

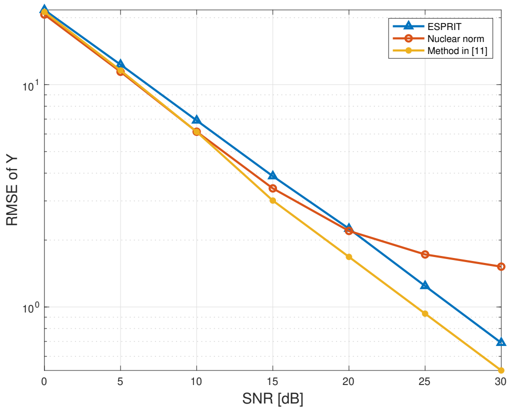
As c is concave and differentiable everywhere, and (22a) only increase on the interval $[\sigma_n(W), (\rho+1)\sigma_{n+1}(W)]$. Thus, the optimal solution s^* will either be on the boundary of $[\sigma_n(W), (\rho+1)\sigma_{n+1}(W)]$ or on a stationary point within the interval. The optimal solution to (22) is then given by

$$\sigma_i^*(Z) = \begin{cases} \max(\sigma_i(W), s^*), & i \leq n \\ \min((1+\rho)\sigma_i(W), s^*), & i > n \end{cases} \tag{25}$$

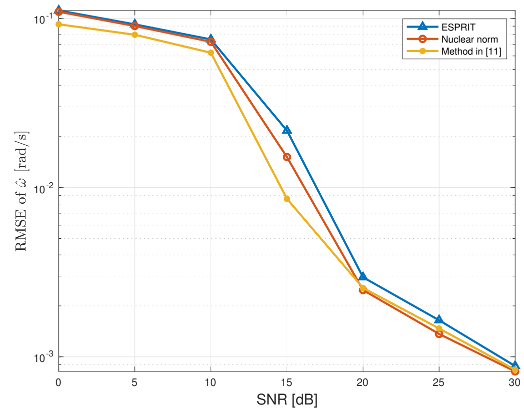
V. Results

In this section, we consider simulations to compare the performance between ESPRIT, and an ADMM implementation of (11), referred to as the nuclear norm method, and the method in [11].

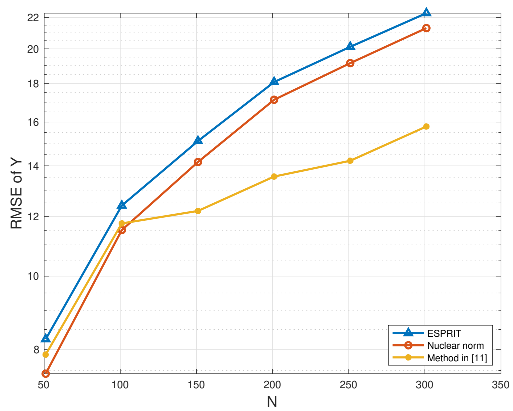
For the simulation setup, we consider a signal of six real valued sinusoidal signal components, with frequencies $\omega_i = \{0.2\pi, 0.22\pi, 0.24\pi, 0.46\pi, 0.6\pi, 0.8\pi\}$ rad/s and corresponding amplitudes $\alpha_i = \{1, 0.5, 1, 1, 1, 0.5\}$ with zero damping. The comparison is made by varying either the Signal



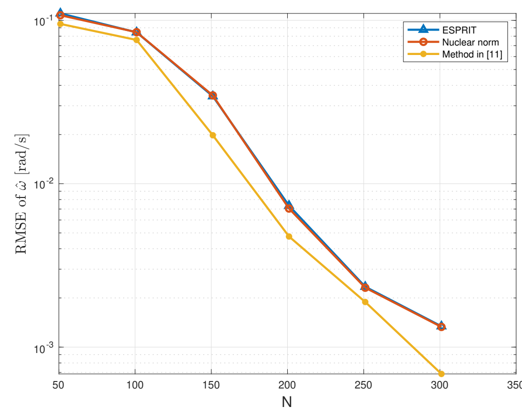
(a) The RMSE of Y versus SNR. $N = 100$.



(b) The RMSE of $\hat{\omega}$ versus SNR. $N = 100$.



(c) The RMSE of Y versus N. SNR is 5 dB.



(d) The RMSE of $\hat{\omega}$ versus N. SNR is 5 dB.

Fig. 1: Simulated RMSE for Y and $\hat{\omega}$ when either SNR or N is varied.

to Noise Ratio (SNR) or the number of signal samples, N, in the measurement sequence. When one parameter is varied, the other one is kept constant. The Hankel matrix is set to have $N/2$ rows, and $\gamma = 1$, in all simulations.

To measure performance, we consider the root mean square error (RMSE) obtained using a Monte-Carlo simulation with 250 trials.

In Fig. 1a the RMSE of Y is illustrated for different SNR-levels. The accuracy for the nuclear norm method and the method in [11] are similar for SNR-values between zero and ten, while for larger values the method in [11] are more accurate than the other two methods. Note that the behavior of the nuclear norm method for large SNR-values is due to the weighting factor γ which balances between minimizing the nuclear norm and being close to the measurement matrix.

Fig. 1b shows the RMSE of the estimated frequencies for different SNR-levels. As can be seen, the method [11] is more accurate for lower SNR than the two other methods.

In Fig. 1c it is visualized how the RMSE of Y vary with the number of samples N. For N small, the performance of the nuclear method is the most accurate, while for larger N the method [11] is significantly more accurate than the other methods. Note that the number of elements in the Hankel matrices increase with N, consequently the RMSE will also increase.

In Fig. 1d it is shown how the RMSE of the estimated frequencies varies with the sample size. From the figure it is visible that the method [11] is the most accurate one.

VI. Conclusion

In this paper we consider model based spectral estimation, and investigate three different methods of finding the estimate. All the methods find an estimate to the range space of a matrix of measurements. The investigated methods are ESPRIT, a nuclear norm relaxiation method and a method with formulates a single convex envelope of the least squares minimizer and the rank function. The main contribution of the paper is to illustrate how the problem formulation and management of constraint affects the accuracy of the estimate. Numerical simulations indicated that the method which formulates the single convex envelope is more accurate than the other two investigated methods.

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