BLIND SUPER-RESOLUTION IN TWO-DIMENSIONAL PARAMETER SPACE

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ABSTRACT

In this paper, we provide a new mathematical framework for identifying the parameters of a linear system from its response to multiple *unknown waveforms*. We assume that the system response is given by an unknown number of scaled versions of time-delayed and frequency-shifted unknown waveforms. Then, we develop a blind two-dimensional super-resolution framework that is based on the convex atomic norm framework to recover the *continuous* time-frequency shifts as well as the unknown waveforms. We prove that under a minimum separation condition between the time-frequency shifts and with a certain lower bound on the total number of the observed samples, all the unknowns in the system can be recovered precisely and with high probability. Simulation results that confirm the theoretical findings in the paper are provided.

Index Terms— Super-resolution, atomic norm, blind deconvolution.

1. INTRODUCTION

Super-resolution techniques are concerned by obtaining high-resolution information from coarse-scale data. Moreover, they also known to break the natural limit achieved by standard compressed sensing algorithms. Such techniques have been applied in many applications such as radar imaging, medical imaging, and communication systems [1, 2, 3]. In this paper, we consider a linear system in which the observed signal $y(t)$ is a weighted sum of R different versions of time-delayed and frequency-shifted *unknown* signals $s_j(t)$, i.e.,

$$y(t) = \sum_{j=1}^{R} c_j s_j(t - \tilde{\tau}_j) e^{i2\pi \tilde{f}_j t}. \tag{1}$$

Here, the unknown $c_j \in \mathbb{C}$ while $\left(\tilde{\tau}_j, \tilde{f}_j\right)$ is the unknown *continuous* time-frequency shift. Finally, we assume that R is also unknown. Thus, the question here is that given $y(t)$ can we retrieve the unknown quintuple $(R, c_j, \tilde{\tau}_j, \tilde{f}_j, s_j(t))$?

The formulation in (1) appears in a variety of applications in signal processing and communication. In military radar application, a spying receiver might use the R unknown transmitted waveforms from the enemies transmitters to locate the position of a target. The target location can be obtained by estimating its distance and relative velocity from the spying receiver which are included in the continuous time-frequency shifts. These shifts can lie anywhere in the continuous domain and not necessarily on a discrete grid. Other applications include target detection using blind channel equalization [4, 5], blind super-resolution of a two-dimensional (2D) point source in microscopy [6], and image restoration in medical imaging [7]. Moreover, (1) is applied in passive indoor source localization to find the locations of R objects by obtaining $\left(\tilde{\tau}_j, \tilde{f}_j\right)$ and then using this information with anchors' location.

Prior Works: The recent approach for super-resolution is based on the atomic norm [8] which provides a framework for using convex optimization to recover a set of data in the continuous domain. This framework is applied to obtain shifts in the continuous domain $[0, 1]$ from low-frequency equally spaced sequent samples as in [9] or randomly selected samples as in [10]. Both works are based on 1D super-resolution, and they show that the exact recovery of the shifts is possible when they are well-separated. On the other hand, the work in [11] studies 2D super-resolution in radar where the problem is formulated using atomic norm. The received signal is modeled as a sum of time-delayed and Doppler-shifted versions of a known signal. The authors in [12] study super-resolving ensemble of Diracs on a sphere from their measurements.

On the other hand, the problem of blind deconvolution of signals from their convolution is an ill-posed problem that requires further constraints to be solved [13]. The authors in [14] develop an algorithm to blindly deconvolve two signals lying in known low-dimensional subspaces. This work is extended in [15] for sparse signals. A convex framework for estimating a single point function and a spike signal is introduced in [16]. Recently, the authors in [17] study the problem of estimating the parameters of complex exponentials from their modulations with unknown waveforms living in a known low-dimensional subspace. A 1D atomic norm minimization problem is formulated to obtain the shifts and the waveforms.

Contributions with Connections to Previous Works: In this paper, we provide a new mathematical framework for blind 2D super-resolution. The 2D term here is based on the fact that we are super-resolving two continuous unknowns ($\tilde{\tau}_j$ and $\tilde{f}_j$) simultaneously while the blindness of this framework is because $s_j(t)$ are unknown. Since the problem is severely ill-posed, we assume that $s_j(t)$ live in a common known low-dimensional subspace that satisfies certain randomness and concentration assumptions. Then, we show that the unknown

quintuple $(R, c_j, \tilde{\tau}_j, \tilde{f}_j, s_j(t))$ in (1) can be recovered precisely and with very high probability using $y(t)$ only. This exact recovery is granted under the assumption that $\left(\tilde{\tau}_j, \tilde{f}_j\right)$ are well-separated and that the number of the observed samples is linear up to a log-factor in specific parameters including R and the subspace dimension. The recovery problem is formulated as an atomic norm minimization and is then reformulated and solved via semidefinite programming (SDP).

The model in [11] has the same formula in (1); however, the waveforms in [11] are identical, *known*, and have a Gaussian distribution. The work in [17] is a special case of our approach by assuming that either $s_j(t)$ or $\tilde{\tau}_j$ is known. Considering $s_j(t-\tilde{\tau}_j)$ as a single unknown makes the approach in [17] fails to resolve $s_j(t)$ from $\tilde{\tau}_j$ in its final solution. The fact that there are two unknowns in $s_j(t-\tilde{\tau}_j)$ shifts the overall problem from 1D to 2D and makes most of the proof techniques and performance guarantee conditions in [17] invalid. Finally, [16] is a special case of [17] by assuming identical waveforms. Therefore, our approach is a generalization of [11, 16, 17]. However, this generalization comes with major differences. For example, to prove the existence of the solution of the problem in [17], a 1D polynomial is formulated using shifted versions of a *single* kernel. Such formulation fails in our case as our 2D vector polynomial has to satisfy certain constraints and thus, multiple kernels are used instead. Our proof technique allows us also to impose less restricted assumptions on the low-dimensional subspace than what in [16, 17]. The non-blindness with the Gaussianity assumption in [11] simplify the scaler polynomial formulation and make the proof technique in [11] inapplicable for our case.

2. SYSTEM MODEL AND PROBLEM SOLUTION

To start with, we assume that $s_j(t)$ are band-limited periodic signals with a bandwidth of W and a period of T and that $y(t)$ is observed over an interval of length T. Based on that, we can assume that $\left(\tilde{\tau}_j, \tilde{f}_j\right) \in ([-T/2, T/2], [-W/2, W/2])$. Now, according to the $2WT$-Theorem [18], we can characterize $y(t)$ by sampling it at a rate of $\frac{1}{W}$ samples-per-second to gather $L := WT$ samples. For simplicity, we will assume that L is an odd number. Upon sampling (1) and then applying the DFT and the inverse DFT (IDFT), we can show that

$$y(p) := y(p/W) = \frac{1}{L}\sum_{j=1}^{R} c_j \left(\sum_{k=-N}^{N}\left[\left(\sum_{l=-N}^{N} s_j(l)\, e^{\frac{-i2\pi kl}{L}}\right) e^{-i2\pi\tau_j k}\right] e^{\frac{i2\pi kp}{L}}\right) \times e^{i2\pi f_j p}, \qquad p=-N,\dots,N, \quad N := \frac{L-1}{2}, \tag{2}$$

where we set $\tau_j := \frac{\tilde{\tau}_j}{T}$ and $f_j := \frac{\tilde{f}_j}{W}$. Note that the samples $s_j(l)$ are L periodic and that $(\tau_j, f_j) \in [-1/2, 1/2]^2$. Due to the periodicity property, we can assume that $(\tau_j, f_j) \in [0,1]^2$. In this paper, we refer to (τ_j, f_j) by delay-Doppler shift pair.

Based on (2), the number of unknowns is $O(RL)$ which is much greater than the given samples L. Therefore, the recovery problem is severely ill-posed and cannot be solved without imposing extra assumptions. Inspired by [14, 16, 17], and by defining $\mathbf{s}_j := [s_j(-N), \dots, s_j(N)]$, we assume that all $\mathbf{s}_j$ belong to a common subspace that is spanned by the columns of a known matrix $\mathbf{D} \in \mathbb{C}^{L\times K}$ such that $\mathbf{s}_j = \mathbf{D}\mathbf{h}_j$ where $K \le L$. Here, $\mathbf{D} = \left[\mathbf{d}_{-N}, \dots, \mathbf{d}_N\right]^H, \mathbf{d}_l \in \mathbb{C}^{K\times 1}$, and thus $s_j(l) = \mathbf{d}_l^H \mathbf{h}_j$ while the unknown orientation vectors $\mathbf{h}_j \in \mathbb{C}^{K\times 1}$ are assumed to have $||\mathbf{h}_j||_2 = 1$ for all j. As a result, the number of unknowns reduces to $O(RK)$ which can be less than L when $R, K \ll L$. By substituting $s_j(l) = \mathbf{d}_l^H \mathbf{h}_j$ in (2) and then manipulating we obtain

$$y(p) = \sum_{j=1}^{R} c_j \sum_{k,l=-N}^{N} D_N\left(\frac{k}{L} - f_j\right) D_N\left(\frac{l}{L} - \tau_j\right) \times \mathbf{d}_{(p-l)}^H \mathbf{h}_j e^{\frac{i2\pi pk}{L}},\ p=-N,\dots,N, \tag{3}$$

where $D_N(t) := \frac{1}{L}\sum_{r=-N}^{N} e^{i2\pi tr}$ is Dirichlet kernel. Now, let us define the atoms $\mathbf{a}(\mathbf{r}_j) \in \mathbb{C}^{L^2\times 1}, \mathbf{r}_j := [\tau_j, f_j]^T$ with

$$[\mathbf{a}(\mathbf{r}_j)]_{((k,l),1)} = D_N\left(\frac{l}{L} - \tau_j\right) D_N\left(\frac{k}{L} - f_j\right),$$

and $\widetilde{\mathbf{D}}_p \in \mathbb{C}^{L^2\times K}$ such that $[\widetilde{\mathbf{D}}_p]_{((k,l),1\to K)} = e^{\frac{i2\pi pk}{L}} \mathbf{d}_{(p-l)}^H$, $p, k, l = -N, \dots, N$. Based on that, we can simplify (3) as

$$y(p) = \sum_{j=1}^{R} c_j \mathbf{a}(\mathbf{r}_j)^H \widetilde{\mathbf{D}}_p \mathbf{h}_j = \left\langle \mathbf{U}, \widetilde{\mathbf{D}}_p^H \right\rangle = \text{Trace}\left(\widetilde{\mathbf{D}}_p \mathbf{U}\right) \tag{4}$$

where $\mathbf{U} := \sum_{j=1}^{R} c_j \mathbf{h}_j \mathbf{a}(\mathbf{r}_j)^H$. In practical applications, $R \ll L$ and thus $\mathbf{U}$ is a sparse linear combination of different $\mathbf{a}(\mathbf{r}_j)$. By estimating $\mathbf{U}$, we can recover all the unknowns. Now, to force the sparsity when we estimate $\mathbf{U}$, we suggest applying the atomic norm [8]. To start with, we define the atomic set $\mathcal{A} := \left\{\mathbf{h}\mathbf{a}(\mathbf{r})^H : \mathbf{r} \in [0,1]^2, ||\mathbf{h}||_2 = 1, \mathbf{h} \in \mathbb{C}^K\right\}$. Based on that, the atomic norm of $\mathbf{U}$ can be defined as [8]

$$||\mathbf{U}||_{\mathcal{A}} = \inf_{\substack{c_j\in\mathbb{C}, ||\mathbf{h}_j||_2=1 \\ \mathbf{r}_j\in[0,1]^2}} \left\{\sum_{j=1}^{R} |c_j| : \mathbf{U} = \sum_{j=1}^{R} c_j \mathbf{h}_j \mathbf{a}(\mathbf{r}_j)^H\right\}$$

Now, we can formulate our blind super-resolution problem as

$$\begin{aligned} &\underset{\widetilde{\mathbf{U}}\in\mathbb{C}^{K\times L^2}}{\text{minimize}}\ ||\widetilde{\mathbf{U}}||_{\mathcal{A}} \\ &\text{subject to}: y(p) = \left\langle \widetilde{\mathbf{U}}, \widetilde{\mathbf{D}}_p^H \right\rangle,\ p=-N,\dots,N. \end{aligned} \tag{5}$$

Next, we discuss the optimality criteria of (5) and its solution.

2.1. Optimality Criteria and Problem Solution

Define $\mathcal{X} : \mathbb{C}^{K\times L^2} \to \mathbb{C}^L$ as $[\mathcal{X}(\mathbf{U})]_p = \text{Trace}\left(\widetilde{\mathbf{D}}_p \mathbf{U}\right)$. Then, we can relate $\mathbf{U}$ to $\mathbf{y} := [y(-N), \dots, y(N)]^T$ through $\mathbf{y} = \mathcal{X}(\mathbf{U})$. Now, the dual of (5) can be shown to be [19]

$$\underset{\mathbf{q}\in\mathbb{C}^{L\times 1}}{\text{maximize}}\ \langle \mathbf{q}, \mathbf{y}\rangle_{\mathbb{R}};\ \text{subject to}: ||\mathcal{X}^*(\mathbf{q})||_{\mathcal{A}}^* \le 1, \tag{6}$$

where $\mathcal{X}^*:\mathbb{C}^L\to\mathbb{C}^{K\times L^2}$ is the adjoint of $\mathcal{X}$, i.e., $\mathcal{X}^*(\mathbf{q})=\sum_{p=-N}^{N}[\mathbf{q}]_p\widetilde{\mathbf{D}}_p^H$ while $||\cdot||_{\mathcal{A}}^*$ is the dual of the atomic norm.

Since strong duality holds between (5) and (6) (Slater's condition is satisfied), and by denoting the solution of (5) by $\widehat{\mathbf{U}}$, then the solutions of (5) and (6) are equal if and only if $\widehat{\mathbf{U}}$ is the primal optimal and $\mathbf{q}$ is the dual optimal. Next, we show when $\widehat{\mathbf{U}}=\mathbf{U}$ based on (6). For that, we can first write the constraint in (6) based on the definition of $||\cdot||_{\mathcal{A}}^*$ in [8] as

$$||\mathcal{X}^*(\mathbf{q})||_{\mathcal{A}}^*=\sup_{\mathbf{r}\in[0,1]^2}||\mathcal{X}^*(\mathbf{q})\,\mathbf{a}(\mathbf{r})||_2=\sup_{\mathbf{r}\in[0,1]^2}||\mathbf{f}(\mathbf{r})||_2 \quad (7)$$

where $\mathbf{f}(\mathbf{r}):=\mathcal{X}^*(\mathbf{q})\,\mathbf{a}(\mathbf{r})=\sum_{p=-N}^{N}[\mathbf{q}]_p\widetilde{\mathbf{D}}_p^H\mathbf{a}(\mathbf{r})$.

Proposition 1. Define the set $\mathcal{R}:=\{\mathbf{r}_j\}_{j=1}^R$ and let $\text{sign}(c_j)=\frac{c_j}{|c_j|}$. Then, $\widehat{\mathbf{U}}=\mathbf{U}$ is the *unique optimal solution* to (5) if:

1. There exists a 2D vector polynomial $\mathbf{f}(\mathbf{r})\in\mathbb{C}^{K\times 1}$ satisfies
$$\mathbf{f}(\mathbf{r}_j)=\text{sign}(c_j)\,\mathbf{h}_j,\ \ \forall\mathbf{r}_j\in\mathcal{R} \quad (8)$$
$$||\mathbf{f}(\mathbf{r})||_2<1,\quad \forall\mathbf{r}\in[0,1]^2\setminus\mathcal{R}. \quad (9)$$
2. $\left\{\begin{bmatrix}\mathbf{a}(\mathbf{r}_j)^H\widetilde{\mathbf{D}}_{-N}\\ \vdots\\ \mathbf{a}(\mathbf{r}_j)^H\widetilde{\mathbf{D}}_{N}\end{bmatrix}\right\}_{j=1}^{R}$ is a linearly independent set.

The proof of Proposition 1 is in [20, Appendix B].

To solve (6), we obtain an SDP relaxation for it using the result in [21]. Upon defining the matrix $\widehat{\mathbf{Q}}\in\mathbb{C}^{K\times L^2}$ such that $\left[\widehat{\mathbf{Q}}\right]_{(i,(p,k))}=\left[\frac{1}{L}\mathbf{q}(p)\sum_{l=-N}^{N}\mathbf{d}_l e^{\frac{i2\pi k(p-l)}{L}}\right]_i$, $i=1,\ldots,K$, we can show after some algebraic manipulations that the equivalent SDP relaxation of (6) is given by [20]

$$\underset{\mathbf{q},\mathbf{Q}\succeq\mathbf{0}}{\text{maximize}}\ \langle\mathbf{q},\mathbf{y}\rangle_{\mathbb{R}}\ \ \text{subject to :}$$
$$\begin{bmatrix}\mathbf{Q}&\widehat{\mathbf{Q}}^H\\ \widehat{\mathbf{Q}}&\mathbf{I}_{\mathrm{K}\times\mathrm{K}}\end{bmatrix}\succeq\mathbf{0},\ \text{Trace}\left(\left(\widetilde{\mathbf{\Theta}}_{\tilde{l}}\otimes\widetilde{\mathbf{\Theta}}_{\tilde{k}}\right)\mathbf{Q}\right)=\delta_{\tilde{l},\tilde{k}}, \quad (10)$$

where $-(L-1)\le\tilde{l},\tilde{k}\le(L-1)$ while $\widetilde{\mathbf{\Theta}}_i$ is $L\times L$ Toeplitz matrix with ones on its i-th diagonal and zeros elsewhere. Finally, $\delta_{\tilde{l},\tilde{k}}$ is the Dirac function, i.e., $\delta_{\tilde{l},\tilde{k}}=0$ iff $\tilde{l}=\tilde{k}=0$. The problem in (10) can be solved using any SDP solver. Then, we formulate $\mathbf{f}(\mathbf{r})$ and a function of $\mathbf{r}$ and we recover $\mathcal{R}$ by either computing the roots of $1-||\mathbf{f}(\mathbf{r})||_2^2$ on the unit circle or by discretizing the domain $[0,1]^2$ on a fine grid and then locating $\mathbf{r}_j$ at which $||\mathbf{f}(\mathbf{r}_j)||_2=1$ (based on Proposition 1). In this paper, we use the later approach. Finally, we can formulate an overdetermined linear system based on (4) and solve it using the LS algorithm to estimate $c_j\mathbf{h}_j$.

3. MAIN RESULTS

In this section, we give the main theorem of the paper and its associated assumptions. We start with the assumptions first.

Assumption 1. We assume that the columns of $\mathbf{D}^H\in\mathbb{C}^{K\times L}$ are independent with their entries being independent and that they can be drawn from any distribution that satisfies

$$\mathbb{E}[\mathbf{d}_l]=\mathbf{0}_{\mathrm{K}\times 1},\ \ \mathbb{E}[\mathbf{d}_l\mathbf{d}_l^H]=\mathbf{I}_{\mathrm{K}\times\mathrm{K}},\ \ l=-N,\ldots,N.$$

Assumption 2. (*Concentration property*) We assume that the rows of $\mathbf{D}^H$, refer to its column form by $\hat{\mathbf{d}}_i\in\mathbb{C}^{L\times 1}$ such that $i=1,\ldots,K$, are $\widetilde{K}$-concentrated with $\widetilde{K}\ge 1$. That is, there exist two constants C_1^* and C_2^* such that for any 1-Lipschitz function $\varphi:\mathbb{C}^K\to\mathbb{R}$ and any $t>0$, it holds that

$$\Pr\left[\left|\varphi\left(\hat{\mathbf{d}}_i\right)-\mathbb{E}\left[\varphi\left(\hat{\mathbf{d}}_i\right)\right]\right|\ge t\right]\le C_1^*\exp\left(-C_2^*t^2/\widetilde{K}^2\right).$$

Assumption 3. The entries of $\mathbf{h}_j$ are i.i.d. from a uniform distribution on the complex unit sphere with $||\mathbf{h}_j||_2=1$.

Assumption 4. (*Minimum separation*) We assume that

$$\min_{j,j'=1,\ldots,R;\ j\neq j'}\max\left(|\tau_j-\tau_{j'}|,|f_j-f_{j'}|\right)\ge\frac{2.38}{N}, \quad (11)$$

where $|a-b|$ is the wrap-around distance on the unit circle.

Theorem 1. ***(Main result)*** *Let $y(p)\in\mathbb{C}$ be as in (4) with $p=-N,\ldots,N$ and $N\ge 512$ and assume that the vectors $\mathbf{s}_j$ can be written as $\mathbf{s}_j=\mathbf{D}\mathbf{h}_j$ with $\mathbf{D}^H$ satisfying Assumptions 1 and 2 and $\mathbf{h}_j$ following Assumption 3. Moreover, define the set $\mathcal{R}=\{\mathbf{r}_1,\ldots,\mathbf{r}_R\}$ such that its elements are satisfying Assumption 4. Then, there exist two numerical constants C_1 and C_2 such that when*

$$L\ge C_1RK\widetilde{K}^4\log^2\left(\frac{C_2R^2K^2L^3}{\delta}\right)\log^2\left(\frac{C_2(K+1)L^3}{\delta}\right) \quad (12)$$

is satisfied with $\delta>0$, $\mathbf{U}=\sum_{j=1}^{R}c_j\mathbf{h}_j\mathbf{a}(\mathbf{r}_j)^H$ is the optimal minimizer of (5) with probability at least $1-\delta$.

3.1. Remarks and Proof Outlines

Many random vectors in practice satisfy the concentration property in Assumption 2. For example, if the entries of $\hat{\mathbf{d}}_i$ are i.i.d. Gaussian, then $\hat{\mathbf{d}}_i$ is a 1-concentrated vector whereas if each entry in $\hat{\mathbf{d}}_i$ is upper bounded by a constant C, then $\hat{\mathbf{d}}_i$ is C-concentrated [22]. Thus, it is a more relaxed assumption than the incoherence property in [17, 16]. Moreover, the minimum separation has always been required in super-resolution theory with different forms to guarantee stable recovery and to ensure that the problem does not become ill-conditioned.

Theorem 1 gives sufficient condition for the exact recovery. For a given $\widetilde{K}$, (12) shows that $L=O(RK)$ is sufficient to recover the unknown shifts which coincides with the number of degrees of freedom in our problem. Finally, $N\ge 512$ is a technical assumption that is made to facilitate our proofs upon following what in [9]. This assumption can be discarded at the cost of using larger separation as [9] shows. Our simulations show that exact recovery still exists without this assumption being satisfied.

The detailed proof of Theorem 1 is given in [20]. The proof shows that $\mathbf{f}(\mathbf{r})$ can be formulated using random kernel matrices $\mathbf{M}_{(m,n)}(\mathbf{r},\mathbf{r}_j) \in \mathbb{C}^{K\times K}; m,n = 0,1$ and vector parameters $\boldsymbol{\alpha}_j, \boldsymbol{\beta}_j, \boldsymbol{\gamma}_j \in \mathbb{C}^{K\times 1}$ an that it takes the form

$$\mathbf{f}(\mathbf{r}) = \sum_{j=1}^{R} \mathbf{M}_{(0,0)}(\mathbf{r},\mathbf{r}_j)\,\boldsymbol{\alpha}_j + \mathbf{M}_{(1,0)}(\mathbf{r},\mathbf{r}_j)\,\boldsymbol{\beta}_j + \mathbf{M}_{(0,1)}(\mathbf{r},\mathbf{r}_j)\,\boldsymbol{\gamma}_j$$

4. SIMULATION RESULTS

In this section, we provide some simulation experiments to illustrate the performance of the proposed framework.

First, we set $L = 19, R = 2, K = 2$, and we let $[\mathbf{D}]_{(i,j)} \sim \mathcal{CN}(0,1)$. The elements of $\mathbf{h}_j$ are generated from an i.i.d. $\mathcal{CN}(0,1)$ and are then normalized to have $||\mathbf{h}_j||_2 = 1$. The locations of the shifts pairs (τ_j, f_j) are generated randomly from a uniform distribution in $[0,1]^2$ in accordance with (11) and found to be $(0.28, 0.53)$ and $(0.94, 0.42)$. Moreover, we set $|c_j| = 1$ with their real and imaginary parts being $\mathcal{N}(0,1)$.

To estimate the shifts, we discretize the domain $[0,1]^2$ with a step of 10^{-3} and then we locate $\mathbf{r}_j$ where $||\mathbf{f}(\mathbf{r}_j)||_2^2 = 1$. From Fig 1(a), we can see that the two shifts are recovered perfectly with $||\mathbf{f}(\mathbf{r})||_2^2 < 1, \forall \mathbf{r} \notin \mathcal{R}$. Looking at Fig 1(a), it may appear challenging to distinguish the value one from the values less than but very close to one. Fortunately, our extensive simulations in [20], as well as other foundational works in the literature such as [9, 10, 17], suggest that the numerical precision of standard numerical optimization tools such as CVX is sufficient to distinguish them. It remains an open problem to establish a uniform threshold to distinguish them. Finally, we point out that as indicated in the literature and Proposition 1, the duality analysis used here allows identifying the unknown shifts regardless of the power of the signals.

In Fig 1(b), we plot the magnitude of the recovered samples and we compare them with the true ones. From Fig 1(b), it is clear that we can retrieve precisely the signals samples. Finally, we find that $|\mathbf{h}_1^H \hat{\mathbf{h}}_1| = 1 - 10^{-8}$ and $|\mathbf{h}_2^H \hat{\mathbf{h}}_2| = 1.0$ which confirms the superiority of the proposed framework.

Though the focus of this paper is on the noise-free case, we follow the practice of the literature such as [17] by adding simulations for the noisy case to illustrate the stability of the framework to noise with the theoretical analysis being left to future work. Here, we set $L = 15, R = 1, K = 3$, $|c_1|$ to be fading, i.e., $0.5 + w^2$ with $w \sim \mathcal{N}(0,1)$, and we use the same settings in the previous scenario for $\mathbf{D}$ and $\mathbf{h}_1$ and the shifts with (τ_1, f_1) found to be $(0.74, 0.3)$. A white Gaussian noise vector $\mathbf{n}$ with SNR $= 10$ dB is added to $\mathbf{y}$. Then, we solve

$$\underset{\widetilde{\mathbf{U}}}{\text{minimize}} ||\widetilde{\mathbf{U}}||_{\mathcal{A}} \text{ subject to : } ||y(p) - \langle \widetilde{\mathbf{U}}, \widetilde{\mathbf{D}}_p^H \rangle|| \leq \zeta \quad (13)$$

where $||\mathbf{n}||_2 \leq \zeta$ (we set $\zeta = 3$). An SDP is obtained for (13) and then solved. From Figs 1(c)-(d) we can see that the estimated shift is $(0.7370, 0.2980)$ which is close to the original one while the recovered samples are close to the original ones with a tenuous error. Finally, we find that $|\mathbf{h}_1^H \hat{\mathbf{h}}_1| = 0.9674$.

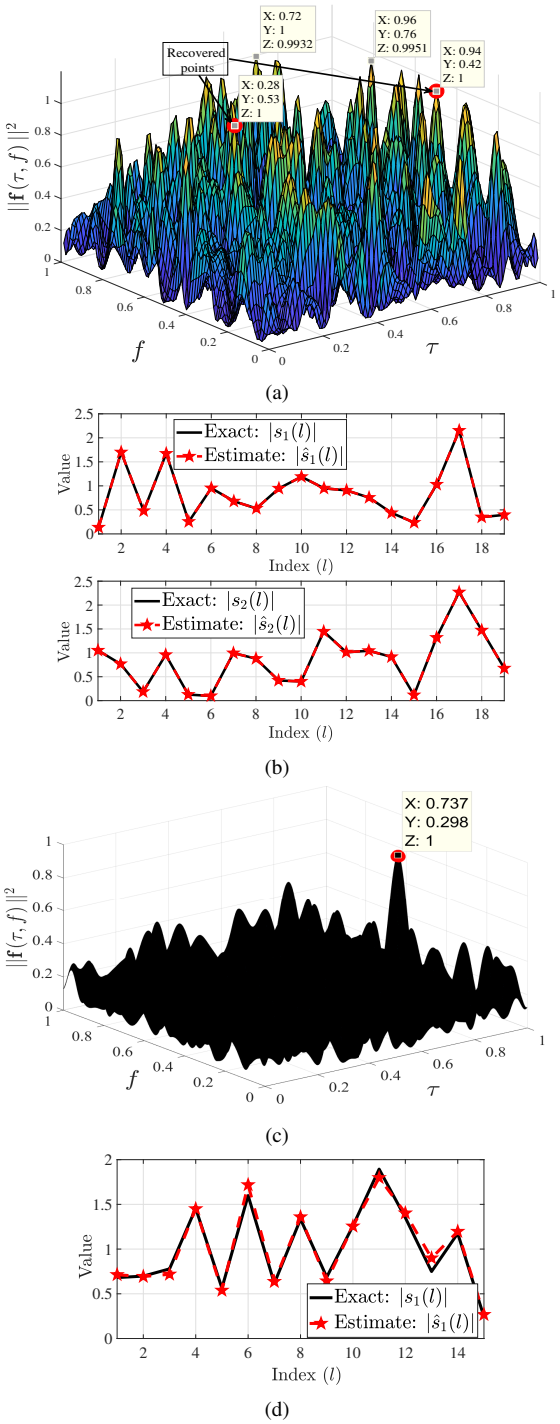

Fig. 1. (a),(c) The locations of the obtained shifts. (b),(d) Comparing the recovered samples with the true ones.

5. CONCLUSIONS

In this work, we developed a framework for blind 2D super-resolution. We showed that given a response of a linear system to multiple unknown time-delayed and frequency-shifted waveforms, we could obtain the time-frequency shifts precisely provided that the shifts are well-separated and that a bound on the number of the observed samples is satisfied.

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