MINIMAX MAGNITUDE RESPONSE APPROXIMATION OF POLE-RADIUS CONSTRAINED IIR DIGITAL FILTERS

Xiaoping Lai[*], *Jiuwen Cao*[*], *Zhiping Lin*[†]

[*] Institute of Information and Control, Hangzhou Dianzi University
[†] School of Electrical & Electronic Engineering, Nanyang Technological University

ABSTRACT

Design of infinite impulse response (IIR) digital filters to approximate some desired magnitude-frequency response is a classical research topic in signal processing. When a pole radius constraint is imposed, however, the problem becomes challenging and few solution methods are available. This paper converts the magnitude-response approximation problem into another problem that approximates the desired magnitude response and an accompanied phase response simultaneously. By iteratively updating the accompanied phase response, a solution to the original magnitude-response approximation problem can be obtained. A striking feature of the proposed method is that the pole radius constraint can be easily incorporated in the problem. Simulations and comparisons demonstrate the effectiveness of the method.

***Index Terms* —** IIR digital filter, approximation problem, magnitude response, pole radius, iterative algorithm

1. INTRODUCTION

Design of infinite impulse response (IIR) digital filters to approximate some desired magnitude responses is a classical research topic in signal processing [1] [2]. Several functions, e.g., butter(), cheby1(), cheby2(), ellip(), iirlpnorm(), and iirlpnormc(), are available in the MATLAB "signal processing" and "filter design" tool boxes, all approximating some desired magnitude response optimally in some senses. While some of these functions use analytic methods, the others utilize optimization methods to solve the design problem.

The magnitude-response approximation of a desired IIR digital filters is a highly nonconvex problem in terms of the filter coefficients. If no pole radius constraint is imposed, the problem can be converted into a magnitude square response approximation problem which is convex in terms of the coefficients of the magnitude square function [3]. The filter coefficients are then retrieved from the coefficients of the magnitude square function by some spectral factorization techniques [4] [5]. The stability of the filter is assured by choosing the filter poles as those of the magnitude square function inside the unit circle. However, the resultant maximum pole radius may be larger than the prescribed value.

Ref. [6] formulates the specifications and limitations on the magnitude square function as well as the stability conditions obtained via Rouché's theorem as semidefinite matrix constraints. It then casts the magnitude response approximation problem as a semi-definite programming problem. The resultant filter has a guaranteed maximum pole radius. The iirlpnorm() and iirlpnormc() functions in the MATLAB "filter design" toolbox directly minimize the L_p-norm of the magnitude-response approximation error formulated in terms of the second-order-section coefficients of the filter [2]. While the function iirlpnormc() imposes a constraint on the filter to limit its pole radius, the function iirlpnorm() doesn't allow such a constraint.

This paper formulates the magnitude-response approximation problem in terms of the filter coefficients. The problem is converted into some frequency-response approximation subproblems with the same desired magnitude response and some accompanied phase responses that are iteratively updated by the phase responses of the filters obtained in the previous iterations. It is shown in [7] that, the solution to the subproblems with such accompanied phase responses converges to the solution of the original magnitude-response approximation problem. For the design of IIR digital filters with simultaneous magnitude- and phase-response specifications, many algorithms [8] - [15] permit a constraint on the filter's pole radius and are applicable to those frequency-response approximation subproblems. The Gauss-Newton (GN) strategy [13] [15] incorporated with an elliptic-error constraint [7] is used in this paper. Simulation examples are provided to demonstrate the effectiveness of the proposed method and to compare with existing algorithms.

2. PROBLEM FORMULATION

Denote by $H(z,g,\boldsymbol{a},\boldsymbol{b})=gB(z,\boldsymbol{b})/A(z,\boldsymbol{a})$ the transfer function of an IIR filter, where g is the zero-frequency gain,

$$B(z,\boldsymbol{b}) = 1 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}, \tag{1a}$$

$$A(z,\boldsymbol{a}) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}, \tag{1b}$$

This work was supported by the National Natural Science Foundation of China under Grants 61573123, 61427808, and U1509205.

$\boldsymbol{b} = [b_1, b_2, \cdots, b_N]^T$ and $\boldsymbol{a} = [a_1, a_2, \cdots, a_N]^T$ are the real valued coefficient vectors of the filter's numerator and denominator, where the superscript $[\cdot]^T$ denotes the transpose of a vector. Denote by $D(\omega)$ the desired magnitude-frequency response defined on a dense grid Ω_0 of the frequency interval $[0, \pi]$, and by

$$E(\omega, g, \boldsymbol{a}, \boldsymbol{b}) = |H(e^{j\omega}, g, \boldsymbol{a}, \boldsymbol{b})| - D(\omega), \tag{2}$$

the approximation error between the actual and desired magnitude responses. We consider the minimax approximation in this paper, which is described as

$$\min_{g, \boldsymbol{a} \in S(\rho), \boldsymbol{b}} \max_{\omega \in \Omega} W(\omega) \left| E(\omega, g, \boldsymbol{a}, \boldsymbol{b}) \right|, \tag{3a}$$

where $W(\omega) > 0$ is a weight function,

$$S(\rho) = \left\{ \boldsymbol{a} \in \mathrm{R}^N \middle| \begin{array}{l} \text{all zeros of } A(z, \boldsymbol{a}) \text{ lie inside a circle} \\ \text{of radius } \rho \text{ centered at the origin} \end{array} \right\}, \tag{3b}$$

and $\Omega = \Omega_p \cup \Omega_s$ is a subset of Ω_0 with $\Omega_p \subset \Omega_0$ and $\Omega_s \subset \Omega_0$ denoting the passband and stopband frequency sets, respectively. By introducing $\delta = \max_{\omega \in \Omega} W(\omega) |E(\omega, g, \boldsymbol{a}, \boldsymbol{b})|$, the minimax approximation problem can be further rewritten as

$$\min_{g, \boldsymbol{a} \in S(\rho), \boldsymbol{b}, \delta} \delta, \tag{4a}$$

$$\text{s.t.: } W(\omega) \left| |H(e^{j\omega}, g, \boldsymbol{a}, \boldsymbol{b})| - D(\omega) \right| \le \delta, \ \omega \in \Omega. \tag{4b}$$

3. SOLUTION ALGORITHMS

Similarly as in [7], to solve the nonconvex mnimax magnitude-response approximation problem (4), we introduce a frequency-response approximation problem as follows:

$$\min_{g, \boldsymbol{a} \in S(\rho), \boldsymbol{b}, \delta} \delta, \tag{5a}$$

$$\text{s.t.: } W(\omega) \left| H(e^{j\omega}, g, \boldsymbol{a}, \boldsymbol{b}) - D(\omega) e^{j\phi(\omega)} \right| \le \delta, \ \omega \in \Omega, \tag{5b}$$

with $\phi(\omega)$ being some given phase function. As a corollary of Theorem 1 of [7], we have the theorem below.

Theorem 1. We assume $H(z, g^*, \boldsymbol{a}^*, \boldsymbol{b}^*)$ is a solution filter of the magnitude-response approximation problem (4), and denote by $\phi^*(\omega)$ the phase of $H(e^{j\omega}, g^*, \boldsymbol{a}^*, \boldsymbol{b}^*)$. Then the solution filter $H(z, \hat{g}, \hat{\boldsymbol{a}}, \hat{\boldsymbol{b}})$ of the frequency-response approximation problem (5) with $\phi(\omega) = \phi^*(\omega)$ is also a solution filter of the magnitude-response approximation problem (4).

Theorem 1 tells us that we can obtain the solution filter of the magnitude-response approximation problem (4) by solving the frequency-response approximation problem (5) if the phase function $\phi^*(\omega)$ is known. Unfortunately, since g^*, $\boldsymbol{a}^*$ and $\boldsymbol{b}^*$ are unknown, the phase function $\phi^*(\omega)$ is also unknown. However, with an initially guessed phase function $\phi_0(\omega)$, we can solve the frequency-response approximation problem (5) iteratively. That is, we let the phase function $\phi_{k+1}(\omega)$ in the $(k+1)$-th iteration to be the phase response of $H(z, g_k, \boldsymbol{a}_k, \boldsymbol{b}_k)$ obtained in the k-th iteration. In this manner, by iteratively solving problem (5), the solution of the magnitude-response approximation problem (4) can be obtained.

As in [7], in order to speed up the convergence of the iterative procedure, the frequency-response error constraint (4b) at a passband frequency is replaced by an elliptic-error constraint described as follows:

$$W(\omega) \left| \mathrm{Re}[\widetilde{E}_\phi(\omega, g, \boldsymbol{a}, \boldsymbol{b})] + \tfrac{j}{\lambda} \mathrm{Im}[\widetilde{E}_\phi(\omega, g, \boldsymbol{a}, \boldsymbol{b})] \right| \le \delta, \omega \in \Omega_p, \tag{6}$$

where $\widetilde{E}_\phi(\omega, g, \boldsymbol{a}, \boldsymbol{b}) = e^{-j\phi(\omega)} H(e^{j\omega}, g, \boldsymbol{a}, \boldsymbol{b}) - D(\omega)$ is a transformed frequency-response error function and $\lambda \ge 1$ is an algorithm parameter.

Then, the algorithm for solving the nonconvex minimax magnitude-response approximation problem (4) is described in Algorithm 1 below.

Algorithm 1

for the nonconvex minimax approximation problem (4)

Step 1. Given an initial phase function $\phi_0(\omega)$, and $\boldsymbol{a}_0 = \boldsymbol{0}$. Let $k = 0$.

Step 2. Solve for $\boldsymbol{a}_{k+1}$ and $\boldsymbol{b}_{k+1}$ the following nonconvex minimax complex approximation subproblem

$$(g_{k+1}, \boldsymbol{a}_{k+1}, \boldsymbol{b}_{k+1}, \delta_{k+1}) = \operatorname*{argmin}_{g, \boldsymbol{a} \in S(\rho), \boldsymbol{b}, \delta} \delta, \tag{7a}$$

$$\text{s.t.: } W(\omega) \left| H(e^{j\omega}, g, \boldsymbol{a}, \boldsymbol{b}) \right| \le \delta, \ \omega \in \Omega_s, \tag{7b}$$

$$W(\omega) \left| \mathrm{Re}[\widetilde{E}_{\phi_k}(\omega, g, \boldsymbol{a}, \boldsymbol{b})] + \tfrac{j}{\lambda} \mathrm{Im}[\widetilde{E}_{\phi_k}(\omega, g, \boldsymbol{a}, \boldsymbol{b})] \right| \le \delta, \omega \in \Omega_p. \tag{7c}$$

Step 3. Compute the phase response $\phi_{k+1}(\omega)$ of the filter $H(z, g_{k+1}, \boldsymbol{a}_{k+1}, \boldsymbol{b}_{k+1})$.

Step 4. If some stop condition is satisfied, terminate. Otherwise, let $k = k+1$ and go back to Step 2.

In step 2 of the Algorithm 1, the minimax subproblem (7) is still nonconvex. To solve the subproblem, we use the Gauss-Newton (GN) strategy [13] [15] to transform it into a convex minimax subproblem. That is, the frequency response $H(e^{j\omega}, \boldsymbol{x})$, where $\boldsymbol{x} = [g, \boldsymbol{a}^T, \boldsymbol{b}^T]^T$, is replaced by its first-order Taylor expansion near $\boldsymbol{x}(\ell)$ as

$$H(e^{j\omega}, \boldsymbol{x}) = H(e^{j\omega}, \boldsymbol{x}(\ell)) + \boldsymbol{\psi}(\omega, \boldsymbol{x}(\ell))^T [\boldsymbol{x} - \boldsymbol{x}(\ell)], \tag{8}$$

where $\boldsymbol{\psi}(\omega,\boldsymbol{x})=\partial H(e^{j\omega},\boldsymbol{x})/\partial \boldsymbol{x}$ is the gradient of the frequency response $H(e^{j\omega},\boldsymbol{x})$ with respect to the coefficient vector $\boldsymbol{x}$.

To assure the designed filter to be stable and have a prescribed maximum pole radius, a sufficient condition, i.e., the generalized positive realness condition [10], is used in this paper to constrain the filter's denominator coefficients. The condition is based on Lemma 2 below.

Lemma 2. All zeros of $A(z,\boldsymbol{a})$ will be inside a circle of radius ρ centered at the origin if $A(z,\boldsymbol{a}(\ell))$ has all zeros inside the same circle and $\mathrm{Re}[A(\rho e^{j\omega},\boldsymbol{a})/A(\rho e^{j\omega},\boldsymbol{a}(\ell))]>0$ for all $\omega\in[0,\pi]$. (see [8] for a proof)

From Lemma 2 we may obtain the following stability constraint:

$$-\mathrm{Re}[e^{-j\theta_\ell(\omega)}A(\rho e^{j\omega},\boldsymbol{a})]<0,\ \ \omega\in\Omega_0, \tag{9}$$

where $\theta_\ell(\omega)$ is the phase of $A(\rho e^{j\omega},\boldsymbol{a}(\ell))$. For any given ω, constraint (9) is linear with respect to the coefficient vector $\boldsymbol{a}$ of the filter denominator $A(z,\boldsymbol{a})$. It can be easily incorporated in the design problem (7) to replace the stability constraint $\boldsymbol{a}\in S(\rho)$.

Algorithm 2

for the nonconvex minimax subproblem (7)

Step 1. Given an initial solution $\boldsymbol{x}(0)=[g(0),\boldsymbol{a}(0)^T,\boldsymbol{b}(0)^T]^T$, with $\boldsymbol{a}(0)=\boldsymbol{a}_k\in S(\rho)$. Let $\ell=0$.

Step 2. Compute $H(e^{j\omega},\boldsymbol{x}(\ell))$, $\boldsymbol{\psi}(\omega,\boldsymbol{x}(\ell))$, and the phase $\theta_\ell(\omega)$ of $A(\rho e^{j\omega},\boldsymbol{a}(\ell))$.

Step 3. Solve for $\boldsymbol{x}(\ell+1)=[g(\ell+1),\boldsymbol{a}(\ell+1)^T,\boldsymbol{b}(\ell+1)^T]^T$ the following convex minimax subproblem

$$(\boldsymbol{x}(\ell+1),\delta(\ell+1))=\arg\min_{x,\delta}\ \delta, \tag{10a}$$

$$\text{s.t.: } W(\omega)\left|\begin{array}{l}\widetilde{H}_k(\omega,\boldsymbol{x}(\ell),\lambda)]-D(\omega)\\ +\widetilde{\boldsymbol{\psi}}_k(\omega,\boldsymbol{x}(\ell),\lambda)^T[\boldsymbol{x}-\boldsymbol{x}(\ell)]\end{array}\right|\le\delta,\ \ \omega\in\Omega_p, \tag{10b}$$

$$W(\omega)\left|H(e^{j\omega},\boldsymbol{x}(\ell))+\boldsymbol{\psi}(\omega,\boldsymbol{x}(\ell))^T[\boldsymbol{x}-\boldsymbol{x}(\ell)]\right|\le\delta,\omega\in\Omega_s, \tag{10c}$$

$$-\mathrm{Re}[e^{-j\theta_\ell(\omega)}A(\rho e^{j\omega},\boldsymbol{a})]\le-\varepsilon,\quad \omega\in\Omega_0, \tag{10d}$$

$$|g-g(\ell)|\le\nu,\ \|\boldsymbol{b}-\boldsymbol{b}(\ell)\|_\infty\le\nu,\ \|\boldsymbol{a}-\boldsymbol{a}(\ell)\|_\infty\le\nu, \tag{10e}$$

where $\varepsilon>0$ is a sufficiently small number, say 10^{-6}, and $\nu>0$ is a step size parameter.

Step 4. If $\|\boldsymbol{x}(\ell+1)-\boldsymbol{x}(\ell)\|\le10^{-4}\|\boldsymbol{x}(\ell)\|$, let $g_{k+1}=g(\ell+1)$, $\boldsymbol{a}_{k+1}=\boldsymbol{a}(\ell+1)$, $\boldsymbol{b}_{k+1}=\boldsymbol{b}(\ell+1)$, and terminate the algorithm. Otherwise, let $\ell=\ell+1$ and go back to Step 2.

By incorporating with the first-order Taylor expansion (8) and the stability constraint (9), the solution algorithm for the nonconvex minimax subproblem (7) is described in Algorithm 2. In Step 3 of the algorithm,

$$\widetilde{H}_k(\omega,\boldsymbol{x}(\ell),\lambda)=\mathrm{Re}[\widetilde{H}_k(\omega,\boldsymbol{x}(\ell))]+\tfrac{j}{\lambda}\mathrm{Im}[\widetilde{H}_k(\omega,\boldsymbol{x}(\ell))]$$

with $\widetilde{H}_k(\omega,\boldsymbol{x}(\ell))=e^{-j\phi_k(\omega)}H(\omega,\boldsymbol{x}(\ell))$,

and $\widetilde{\boldsymbol{\psi}}_k(\omega,\boldsymbol{x}(\ell),\lambda)=\mathrm{Re}[\widetilde{\boldsymbol{\psi}}_k(\omega,\boldsymbol{x}(\ell))]+\tfrac{j}{\lambda}\mathrm{Im}[\widetilde{\boldsymbol{\psi}}_k(\omega,\boldsymbol{x}(\ell))]$

with $\widetilde{\boldsymbol{\psi}}_k(\omega,\boldsymbol{x}(\ell))=e^{-j\phi_k(\omega)}\boldsymbol{\psi}(\omega,\boldsymbol{x}(\ell))$.

If the maximum zero radius is also needed to be smaller than some prescribed value r, we could add in the convex minimax subproblem (10) another constraint described by

$$-\mathrm{Re}[e^{-j\varphi_\ell(\omega)}B(r\,e^{j\omega},\boldsymbol{b})]\le-\varepsilon,\quad \omega\in\Omega_0, \tag{11}$$

where $\varphi_\ell(\omega)$ is the phase of $B(re^{j\omega},\boldsymbol{b}(\ell))$.

4. SIMULATION AND COMPARISONS

Two examples of lowpass IIR filter design are provided to show the efficacy of the proposed method and to compare with the ellip() and iirlpnromc() functions and the recent method in [6]. In the design, the dense grid $\Omega_0=\{\omega=k\pi/400, k=0, 1, \cdots, 400\}$, the algorithm parameter $\lambda=100$ and the step-size parameter $\nu=\infty$. Letting $\nu=\infty$ corresponds to the case that the constraint (10e) is not imposed in the problem (10), which is often fine when the filter order is not large.

Example 1. Design of a 4th-order low-pass filter with a passband $[0, 0.2\pi]$ and a stopband $[0.45\pi,\pi]$. The weight function $W(\omega)$ is taken as 1.0 in the passband and 188.5 in the stopband. As the first design, the maximum pole radius parameter is set to be a value close to 1.0, ρ=0.999. The proposed method converges fast. Algorithm 1 obtains a δ=0.1081 after 21 iterations. Fig. 1 shows the curve of δ_k versus the iteration number k. Fig. 2 draws the magnitude response of the resultant filter. It has a maximum passband ripple Rp=1.88463 dB and a minimum stopband attenuation Rs=65.724 dB. With the Rp and Rs values, the MATLAB function ellip(4,Rp,Rs,0.2) obtains almost the same filter, whose magnitude response is also shown in Fig. 2.

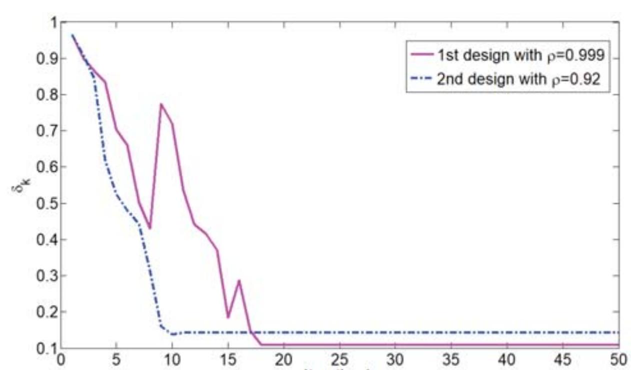

Fig. 1 Convergence of δ_k by the proposed Algorithm 1 in the first two designs of Example 1

The filters obtained by the ellip() function and the proposed method have a maximum pole radius of 0.9422 and 0.9423, respectively. If we require a smaller maximum pole radius, say, ρ=0.92, the ellip() function has no way to design the filter. The proposed method obtains a filter with a maximum passband ripple Rp=2.48335 dB and a minimum stopband attenuation Rs=63.6146 dB after 12 iterations. The convergence curve for this design and the magnitude response of the resultant filter are shown in Figs. 1 and 2, respectively. The maximum pole radius of this filter is 0.9178.

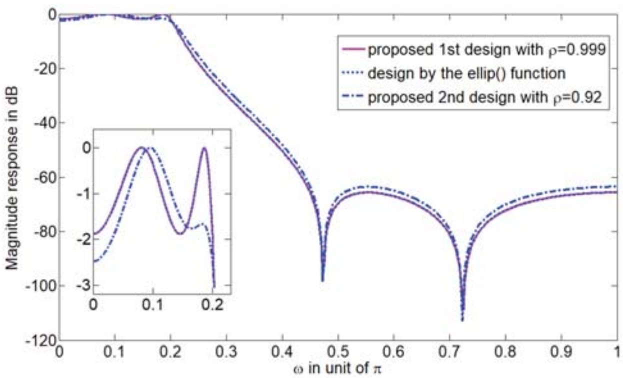

Fig. 2 Magnitude responses of the two filters by the proposed method and the filter by the ellip() function

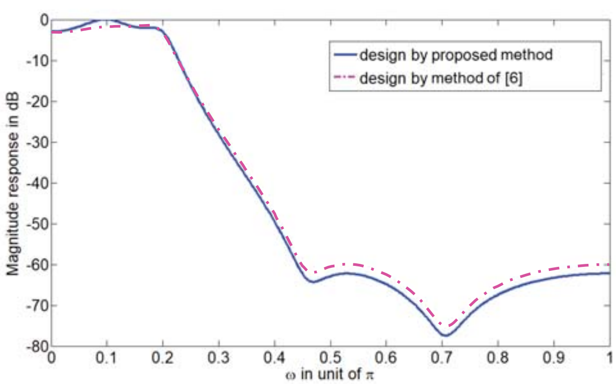

Fig. 3 Magnitude responses of the two filters with pole/zero radiuses smaller than 0.92

To compare with ref. [6], where both the maximum pole and zero radiuses are required to be smaller than 0.92, another filter is designed with our proposed method. The design is conducted under the same passband to stopband ripple ratio and the same maximum pole/zero radius parameter as in [6]. The resultant filter has a maximum passband ripple Rp=2.9866 dB, a minimum stopband attenuation Rs=62.2503 dB, which are better than the corresponding values in [6], i.e., Rp=3 dB and Rs=60 dB. The maximum pole/zero radiuses of the designed filter and the filter in [6] are 0.9177/0.92 and 0.9061/0.9192, respectively. Fig. 3 depicts the magnitude responses of the two filters.

Example 2. Design of a 12th-order low-pass filter with a pass-band $[0, 0.5\pi]$ and a stopband $[0.55\pi, \pi]$. The weight function $W(\omega)$ is taken as 1.0 both in the passband and the stopband. Again, we first design a filter using the proposed method with a maximum pole radius parameter value close to 1.0, ρ=0.999. The resultant filter has a maximum passband ripple Rp=0.00115773 dB, a minimum stopband attenuation Rs=83.5253 dB, and a maximum pole radius 0.9781. With the resultant Rp and Rs values, the function ellip(12, Rp,Rs,0.5) obtains an elliptic filter almost the same as that by our proposed method. We now reduce the maximum pole radius by setting ρ=0.94 and redesign the filter using the proposed method, resulting in a constrained filter with a maximum passband ripple Rp=0.0312542 dB, a minimum stopband attenuation Rs=54.9143 dB, and a maximum pole radius 0.9377. We do not compare with [6] in this example since it is not considered in [6]. Instead, we design another constrained filter by iirlpnormc(12,12,[0 0.5 0.55 1],[0 0.5 0.55 1],[1 1 0 0],[1 1 1 1],0.94). The resultant maximum passband ripple is 0.045323 dB, minimum stopband attenuation is 51.464 dB, and maximum pole radius is 0.94.

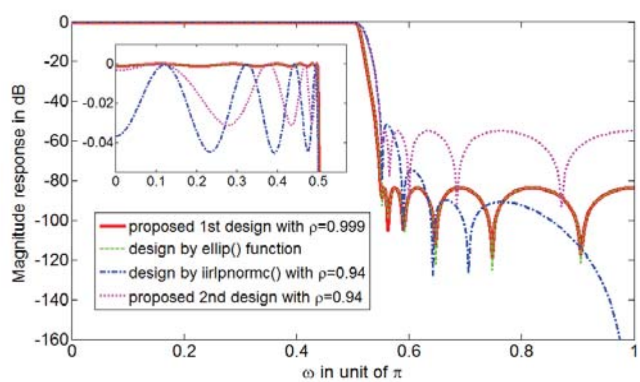

Fig. 4 Magnitude responses of the two filters by the proposed method and the filters by the ellip() and iirlpnormc() functions

Fig. 4 shows the magnitude responses of the four filters, two with a large pole radius about 0.978, and two with a smaller pole radius about 0.94. It is seen that, when ρ<1 is close to 1.0, the proposed method obtains the same stable filter as the ellip() function. When ρ=0.94, the proposed method obtains a better filter than the iirlpnormc() function.

5. CONCLUSION

The proposed method has incorporated a maximum pole radius constraint in the magnitude-response approximation of IIR digital filters. When the maximum pole radius is close to 1.0, the method has obtained the same solution as that by the ellip() function. With a maximum pole radius obviously smaller than 1.0, the proposed method has obtained better filters than the iirlpnormc() function and the method of [6], but the ellip() function may not be able to design such filter.

REFERENCES

[1] A. Antoniou, *Digital Filters: Analysis, Design, and Applications, 2nd ed.*, McGraw-Hill, New York, 1993.

[2] A. Antoniou, *Digital Signal Processing: Signals, Systems, and Filters*, McGraw-Hill, New York, 2005.

[3] L. R. Rabiner, N. Y. Graham, and H. D. Helms, “Linear programming design of IIR digital filters with arbitrary magnitude function,” *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-22, no. 2, pp. 117-123, 1974.

[4] S. C. Pei and S. T. Lu, “Design of minimum-phase FIR filters by differential cepstrum,” *IEEE Transactions on Circuits and Systems*, vol. CAS-33, no. 5, pp. 570-576, 1986.

[5] H. J. Orchard and A. N. Willson, “On the computation of a minimum phase spectral factor,” *IEEE Trans. Circuits and Systems – I: Fundamental Theory and Applications*, vol. 50, no. 3, pp. 365-375, 2003.

[6] B. Limketkai, W. Ng, and T. Rockwood, “Optimization-based synthesis of photonic IIR filters accounting for internal losses in microresonators,” *Journal of Lightwave Technology*, vol. 35, no. 20, pp. 4459-4467, 2017.

[7] X. Lai and Z. Lin, “Optimal Design of Constrained FIR Filters without Phase Response Specifications,” *IEEE Transactions on Signal Processing*, vol. 62, no. 17, pp. 4532-4546, 2014.

[8] X. Lai and Z. Lin, “Minimax design of IIR digital filters using a sequential constrained least-squares method,” *IEEE Transactions on Signal Processing*, vol. 58, no. 7, pp. 3901–3906, 2010.

[9] X. Lai, Z. Lin, and H. K. Kwan, “A sequential minimization procedure for minimax design of IIR filters based on second-order factor updates,” *IEEE Transactions on Circuits Systems - II: Express Briefs*, vol. 58, no. 1, pp. 51–55, 2011.

[10] B. Dumitrescu and R. Niemisto, “Multistage IIR filter design using convex stability domains defined by positive realness,” *IEEE Trans. Signal Process.*, vol. 52, no. 4, pp. 962-974, Apr. 2004.

[11] A. Jiang and H. K. Kwan, “Minimax design of IIR digital filters using iterative SOCP,” *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 57, no. 6, pp. 1326-1337, Jun. 2010.

[12] A. Jiang and H. K. Kwan, “Minimax design of IIR digital filters using SDP relaxation techniques,” *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 57, no. 2, pp. 378-390, Feb. 2010.

[13] M. C. Lang, “Least-squares design of IIR filters with prescribed magnitude and phase responses and a pole radius constraint,” *IEEE Trans. Signal Process.*, vol. 48, no. 11, pp. 3109-3121, Nov. 2000.

[14] W. S. Lu, “Design of recursive digital filters with prescribed stability margin: A parameterization approach,” *IEEE Trans. Circuits Syst. II*, vol. 45, no. 9, pp. 1289-1298, Sept. 1998.

[15] W. S. Lu and T. Hinamoto, “Optimal design of IIR digital filters with robust stability using conic quadratic-programming updates,” *IEEE Trans. Signal Process.*, vol. 51, no. 6, pp. 1581-1592, Jun. 2003.