INTRODUCING THE ORTHOGONAL PERIODIC SEQUENCES FOR THE IDENTIFICATION OF FUNCTIONAL LINK POLYNOMIAL FILTERS

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ABSTRACT

The paper introduces a novel family of deterministic signals, the orthogonal periodic sequences (OPSs), for the identification of functional link polynomial (FLiP) filters. The novel sequences share many of the characteristics of the perfect periodic sequences (PPSs). As the PPSs, they allow the perfect identification of a FLiP filter on a finite time interval with the cross-correlation method. In contrast to the PPSs, OPSs can identify also non-orthogonal FLiP filters, as the Volterra filters. With OPSs, the input sequence can have any persistently exciting distribution and can also be a quantized sequence. OPSs can often identify FLiP filters with a sequence period and a computational complexity much smaller than that of PPSs. Several results are reported to show the effectiveness of the proposed sequences identifying a real nonlinear audio system.

Index Terms— Nonlinear system identification, orthogonal periodic sequences, functional link polynomial filters.

1. INTRODUCTION

Functional Link Polynomial (FLiP) filters [1] are a subclass of linear-in-the-parameters (LIP) nonlinear filters. They consists of a linear combination of basis functions, which are product of nonlinear expansions of delayed input samples following the constructive rule of Volterra filters. The subclass of FLiP filters is very broad and it includes many families of polynomial filters of wide use in theory and practice [2–7], as the well known Volterra filters [8–10], the even mirror Fourier nonlinear (EMFN) [11], the Legendre nonlinear (LN) [12], Chebyshev nonlinear (CN) [13], and Wiener nonlinear (WN) [8, 14] filters. All families of FLiP nonlinear filters are universal approximators, since their basis functions form algebras that satisfy all requirements of the Stone-Weierstass theorem [15]. Some of the FLiP filters have orthogonal basis functions for some input distribution, e.g., EMFN, LN, CN, WN filters. For this reason the coefficients of orthogonal FLiP filter can be computed using the cross-correlation method, i.e., calculating the cross-correlation between the basis functions and the system output. Unfortunately, the cross-correlation method with stochastic inputs presents the drawback of requiring million of samples to accurately estimate the filter coefficients [9, 16]. As an alternative to random signals, appropriate deterministic input signals have been proposed for system identification. Among them, perfect periodic sequences (PPSs) [17, 18] are deterministic sequences with an ideal periodic impulsive autocorrelation function, which have been first used as inputs for linear system identification [19, 20]. In case of nonlinear filters, a periodic sequence is defined as a PPS if the cross-correlation between any two different basis functions, estimated over a period, is zero. This definition extends the linear case in which the input samples themselves can be considered as basis functions. PPSs have been proposed for the identification of Orthogonal FLiP filters for EMFN [21, 22], LN [12, 23], CN [24] and WN filters [25]. The PPSs have been obtained by imposing the orthogonality of the basis functions and solving a system of nonlinear equations with an iterative approach.

In this paper, a novel family of sequences called orthogonal periodic sequences (OPSs) is introduced. Their main purpose, as for the PPSs with whom they share many of the characteristics, is the identification of FLiP filters. As the PPSs, they allow the perfect estimation of a FLiP filter on a finite time interval with the cross-correlation method. In contrast to the PPSs, OPSs can identify also non-orthogonal FLiP filters, as the Volterra filters. With OPSs, the input sequence does not need to be perfect periodic, it can have any persistently exciting distribution and can also be a quantized sequence. Once defined the input sequence, a set of OPSs can be developed with each OPS designed to estimate by cross-correlation one of the so-called "diagonals" of the FLiP filter. OPSs can often identify FLiP filters with a sequence period and a computational complexity much smaller than that of PPSs. The proposed procedure differ from the classical methods of Lee-Schetzen [14, 26], which allow to identify WN filters using white noise, and of Korenberg [27], which determine the coefficients of a data dependent orthogonal representation. The last two methods are based on the Gram-Schmidt orthogonalization of the Volterra series and require filter conversions, which are not applied in the proposed approach.

The rest of the paper is organized as follows. In Section 2 FLiP filters are briefly reviewed. Section 3 introduces the novel family of OPSs and discusses their properties. Section 4 provides some experimental results about the identification of a real nonlinear device and compares OPSs with PPSs and least-square (LS) measurements.

The following notation is used throughout the paper: $\mathbb{R}_1$ is the interval $[-1, +1]$, $< a(n) >_L$ is the sum of $a(n)$ over a period of L consecutive samples, $E[\cdot]$ indicates expectation.

2. FLIP FILTERS

FLiP filters are a class of LIP nonlinear filters capable to arbitrarily well approximate any discrete-time, time invariant, finite memory, causal, continuous nonlinear system

$$y(n) = f[x(n), x(n-1), \ldots, x(n-N+1)], \tag{1}$$

where f is a continuous N-dimensional function from $\mathbb{R}_1^N$ to $\mathbb{R}$, and $x(n) \in \mathbb{R}_1$.

The basis functions of FLiP filters can be developed by considering an ordered set of univariate functions satisfying all requirements

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of Stone-Weierstass theorem [28] on $\mathbb{R}_1$,

$$\{g_0[\xi], g_1[\xi], g_2[\xi], ...\} \tag{2}$$

where $g_0[\xi]$ is a function of order 0, usually the constant 1, $g_{2i+1}[\xi]$ for any $i \in \mathbb{N}$ is an odd function of order $2i+1$, $g_{2i}[\xi]$ for any $i \in \mathbb{N}$ is an even function of order $2i$. Setting $\xi = x(n)$, the set of basis functions in (2) can arbitrarily well approximate the nonlinear system in (1) for $N = 1$. When $N > 1$, the functions in (2) are first written for $\xi = x(n), x(n-1), \ldots, x(n-N+1)$ and then the terms of different variable are multiplied in all possible manners, according to the constructive rule of Volterra filters, taking care of avoiding repetitions. The basis functions so developed form an algebra that satisfies all requirements of the Stone-Weierstass theorem and their linear combination can arbitrarily well approximate the system in (1) [1].

The order of a FLiP basis function is defined as the sum of the orders of the constituent factors $g_i[\xi]$. The diagonal number of a basis function is defined as the maximum time difference between the involved input samples. Accordingly, a FLiP filter of order K, memory N, diagonal number D is given by the linear combination of all FLiP basis functions, with order, memory and diagonal number up to K, N, D, respectively. A FLiP filter can be implemented in the form of a filter bank as follows:

$$y(n) = \sum_{p=0}^{R-1} \sum_{m=0}^{N_p-1} h_p(m) f_p(n-m) \tag{3}$$

where $f_p(n)$ are the zero lag basis functions, i.e., $f_0(n) = 1$, $f_1(n) = g_1[x(n)]$, $f_2(n) = g_2[x(n)]$, $f_3(n) = g_1[x(n)]g_1[x(n-1)]$, ..., $f_{2+D}(n) = g_1[x(n)]g_1[x(n-D)]$,$f_{3+D}(n) = g_3[x(n)]$, and so on, N_p is the memory length for the basis function $f_p(n)$, which is N minus the diagonal number of $f_p(n)$; R is the total number of zero lag basis functions, i.e.,

$$R = \binom{D+K}{D+1} + 1. \tag{4}$$

The FLiP filter in (3) has N_D coefficients with [1]

$$N_D = \binom{D+K+1}{D+1} + \binom{D+K}{D+1}(N-1-D). \tag{5}$$

Using the naming conventions of Volterra filters each sequence $h_p(m)$ with $0 \le p \le R-1$ is called a *diagonal* of the FLiP filter.

Any choice of the univariate functions $g_i(\xi)$ takes to a different family of nonlinear filters. In Section 4, Volterra, WN and LN filters are considered. For Volterra filters, $g_i(\xi) = \xi^i$. For LN filters, $g_i(\xi)$ are Legendre polynomials,

$$\{1, \xi, (3\xi^2-1)/2, \xi(5\xi^2-3)/2, \ldots\}. \tag{6}$$

For WN filters, $g_i(\xi)$ are Hermite polynomials of variance σ_x^2 according to the definition in [29],

$$\{1, \xi, \xi^2 - \sigma_x^2, \xi^3 - 3\sigma_x^2\xi, \ldots\}. \tag{7}$$

Orthogonal FLiP filters have basis functions that are orthogonal for a specific distribution of the input signal samples. For example, the basis functions of LN filters are orthogonal for a white uniform input in $\mathbb{R}_1$, those of WN filters are orthogonal for a zero mean white Gaussian input with variance σ_x^2. When the input distribution guarantees the orthogonality of the basis functions, the coefficients of the filter can be estimated with the cross-correlation method, computing the cross-correlation between the basis functions and the system output. Using stochastic inputs, millions of samples are often needed to accurately estimate the coefficients of the FLiP filter. Nevertheless, it has been shown that any orthogonal FLiP filter admits PPSs, i.e., deterministic periodic sequences that guarantee the orthogonality of the basis functions over a period. Using a PPS input, an orthogonal FLiP filter can still be identified with the cross-correlation method, with the coefficient $h_i(j)$ given by

$$h_i(j) = \frac{< y(n) f_i(n-j) >_L}{< f_i^2(n) >_L}. \tag{8}$$

3. ORTHOGONAL PERIODIC SEQUENCES

In this section OPSs are developed. Each OPS allows the identification of one of the diagonals of a FLiP filter using the cross-correlation method.

Let us consider a periodic input sequence $x(n)$ of period L. The input sequence is assumed to persistently excite the FLiP filter to guarantee the invertibility of the input data matrices introduced in the following. This condition is satisfied when the input samples of the fundamental period have a Gaussian distribution, a white uniform distribution in $\mathbb{R}_1$, or other random distribution. The sequence could also be a quantized sequence.

We want to develop an OPS $z_i(n)$ of period L such that the i-th diagonal of the FLiP filter in (3), $h_i(j)$ with $0 \le j \le N_i - 1$, can be estimated as follows

$$h_i(j) = < y(n) z_i(n-j) >_L \ . \tag{9}$$

Let us consider first $i = 0$ and $f_0(n) = 1$. Inserting (3) in (9) for $j = 0$, to be (9) true it must be

$$< f_0(n) z_0(n) >_L = < z_0(n) >_L = 1, \text{and} \tag{10}$$

$$< f_p(n-m) z_0(n) >_L = 0, \tag{11}$$

for all $0 \le m \le N_p - 1$, and $0 < p \le R-1$.

For $i > 0$, inserting (3) in (9), it can be proved the OPS $z_i(n)$ must satisfy the linear equations system

$$\begin{aligned} < z_i(n) >_L &= 0, & (12)\\ < f_i(n) z_i(n) >_L &= 1, & (13)\\ < f_i(n-m_i) z_i(n) >_L &= 0, & (14)\\ < f_p(n-m_p) z_i(n) >_L &= 0, & (15) \end{aligned}$$

for all $-(N_i-1) < m_i \le N_i - 1$ and $m_i \ne 0$, $-(N_i-1) \le m_p \le N_p - 1$ and $0 < p \le R-1$ with $p \ne i$. The system in (12)-(15) has Q_i equations and L variables (the samples of $z_i(n)$), with

$$Q_i = N_D + (R-1)(N_i - 1). \tag{16}$$

For $L \ge Q_i$, the system is critically determined or under-determined and, if the input is persistently exciting, it always admits a solution.

Let us write the system in matrix form,

$$\mathbf{S}\mathbf{z} = \mathbf{d} \tag{17}$$

with $\mathbf{z}$ a vector collecting the samples of $z_i(n)$, $\mathbf{d}$ a vector of all zeros apart from the element 1 corresponding to (13), and $\mathbf{S}$ a fat or square matrix, whose elements are basis function samples. The minimum norm solution of the system is

$$\mathbf{z} = \mathbf{S}(\mathbf{S}\mathbf{S}^T)^{-1}\mathbf{d}. \tag{18}$$

The elements of the matrix $\mathbf{S}\mathbf{S}^T$ are formed by cross-correlations between basis functions with different time delays. By properly sorting the rows of the matrix $\mathbf{S}$, $\mathbf{S}\mathbf{S}^T$ is block Toeplitz and admits efficient algorithms for its inversions. For example, we have found very useful the algorithm presented in [30]. Working with nonlinear basis functions, the matrix $\mathbf{S}\mathbf{S}^T$ could have a bad conditioning, but for sufficiently large L we have always been able to find a solution with sufficient accuracy working with a double precision arithmetic.

When $L \ge Q$ with $Q = N_D + (R-1)(N-1)$, i.e., $Q = \max_i Q_i$, it is possible to develop a set of OPSs $z_i(n)$ for $0 \le i \le$

$R-1$, which allows to estimate with the same input sequence all diagonals of the FLiP filter. The same input sequence could be used for estimating different types of FLiPs filters finding different sets of OPSs.

It should be noted that Q is in general much lower than the number of nonlinear equations $\overline{Q} \simeq R(N_D+1)$ that have to be solved for deriving a PPS for the same FLiP filter. Computing an entire set of OPSs requires less computations than a single iteration of the algorithm used to develop PPSs.

3.1. Nonlinear system identification with OPSs

In output noise absence, a set of OPSs sequences for a FLiP filter of order K, memory N, diagonal number D, allows the exact identification of any nonlinear system that can be modelled with the chosen FLiP filter. In presence of noise or in case of an under-estimation of the nonlinear system, the identification will be affected by an error.

In what follows, the effect of an output noise $\nu(n)$ on the coefficients identification with OPSs is studied. The mean square deviation (MSD) of the coefficients of $f_i(n-j)$ is defined as

$$\mathrm{MSD}_{i,j} = E[(h_i(j) - \tilde{h}_i(j))^2], \tag{19}$$

with $h_i(j)$ the estimated coefficient and $\tilde{h}_i(j)$ its true value. For OPSs, from (9) we have

$$\mathrm{MSD}_{i,j} = E[(<\nu(n) z_i(n-j)>_L)^2]. \tag{20}$$

It is evident that $\mathrm{MSD}_{i,j}$ is proportional to the noise power σ_ν^2. Moreover it is also inversely proportional to $<f_i^2(n)>_L$, because according to (13) $<z_i^2(n)>_L$ is inversely proportional to $<f_i^2(n)>_L$. To compare the different OPSs on equal terms, we introduce the noise gain G_ν, which is defined as the average value over all coefficients of

$$G_{\nu,i,j} = \frac{\mathrm{MSD}_{i,j}}{E[\nu^2(n)]} < f_i^2(n-j)>_L . \tag{21}$$

For PPSs, it can be proved that $G_{\nu,i,j}$ and G_ν are always 1, independently of the considered filter or the period L of the sequence. On the contrary, inserting (20) in (21) and computing the expectations, for OPSs we obtain

$$G_{\nu,i,j} = < z_i^2(n)>_L \cdot < f_i^2(n)>_L, \tag{22}$$

which is independent of the delay j of the basis function. It is shown in the next section that for OPSs G_ν changes with the chosen filter, the distribution of the input samples, and the period L. For a specific filter and input sample distribution, G_ν can greatly change with L, because the choice of L influences the power of the designed OPS $z_i(n)$. When $L = Q$, i.e., the minimum period of the OPS, we have found G_ν can assume very large values that makes the identification with OPSs useless. On the contrary, when $L \gg Q$, G_ν assumes reasonable values. In orthogonal FLiP filters, with the input sample distribution ideally guaranteeing the basis functions orthogonality (e.g., Guassian for Wiener filters, uniform for LN filters), for $L \to \infty$ $G_\nu \to 1$, the ideal value we have with PPSs, because the longer is L the closer the input sequence is to a PPS.

From (9), the computational cost of a filter identification with OPS is of LR multiplications and additions. This computational cost is much lower than that of a LS identification on the same data, which is order of LN_D^2 operations. Also for PPS sequences we have a computational cost of LR multiplications and additions if in (8) we neglect the cost of computing the basis functions and of the normalization. Nevertheless, we will show in the next section that OPS sequences can provide identification performance similar to the PPSs with much shorter periods.

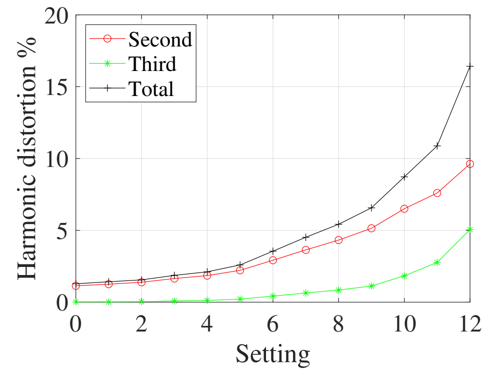


Fig. 1. Seconds, third, and total harmonic distortion.

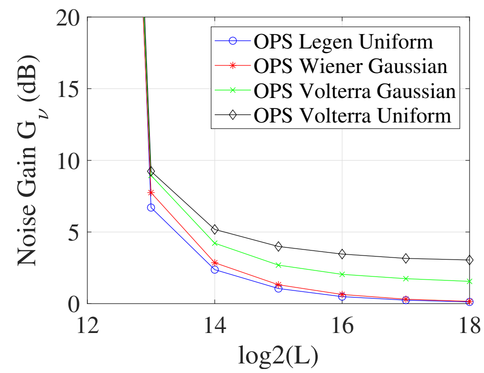


Fig. 2. Noise Gain of OPSs for LN, WN and Volterra filters.

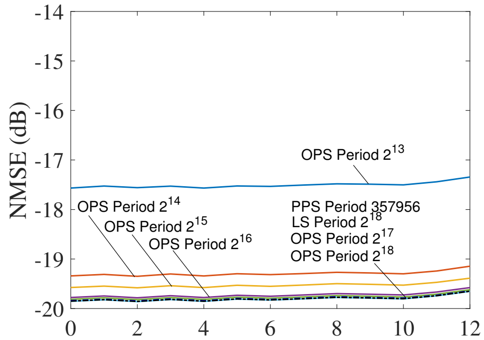
4. EXPERIMENTAL RESULTS

To assess the performance achievable with the OPSs, we have considered the identification of a real device, a Behringer Mic 100 Vacuum Tube Preamplifier. The device has a potentiometer that allows to introduce different nonlinear distortion levels on the output signal. Thirteen different settings have been considered and Fig. 1 shows the second, third and total harmonic distortion on a 200 Hz signal having signal power $1/12$. The signal to noise ratio was around 50 dB. Working at 8 kHz sampling frequency, the device has a memory lower than 20 samples, and different signals with power $1/12$ have been applied for its identification. Specifically, two PPSs for LN and WN filters, respectively, with order 3, memory 20, diagonal number 19, and period $L = 357\,956$, and eight periodic sequences with uniform and with Gaussian distributions, quantized with 10 bits, and with periods

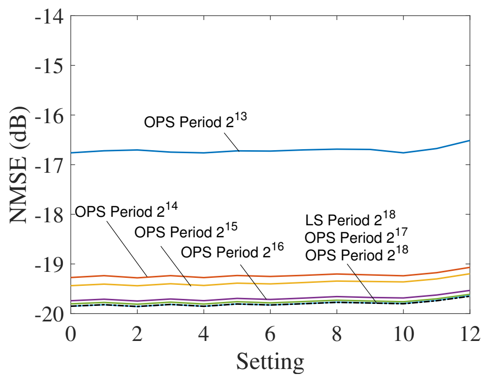
$$[6140, 2^{13}, 2^{14}, 2^{15}, 2^{16}, 2^{17}, 2^{18}],$$

have been considered. OPSs for LN, WN and Volterra filters with order 3, memory 20, diagonal number 19 have been derived for the periodic sequences and have been used to identify the preamplifier. The minimum period for deriving OPSs for these filters is $Q = 6140$.

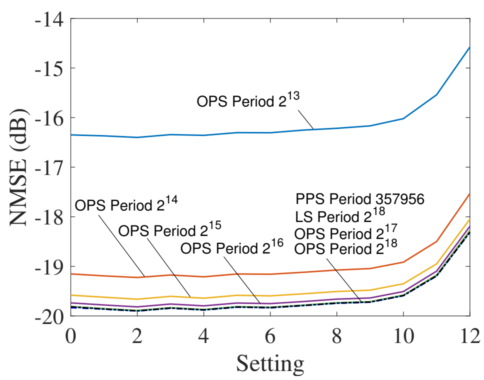
Fig. 2 provides the MSD noise gain G_ν in dB of the OPSs as a function of the base-2 logarithm of the period. The OPSs for LN filters have been determined on the sequences with uniform distribution, those for WN filters on the sequences with Gaussian distribution, while the OPSs for Volterra filters have been determined on both sets of periodic sequences. As we can notice, the G_ν has totally unacceptable values (of around 60 dB) when the period $L = 6140$. On the contrary, G_ν reduces when the period increases. For the orthogonal filters, i.e., LN and WN filters, G_ν approaches the optimal value of 0 dB we have with PPSs. On the contrary, for Volterra filters, which are not orthogonal and do not admit PPSs, G_ν does not



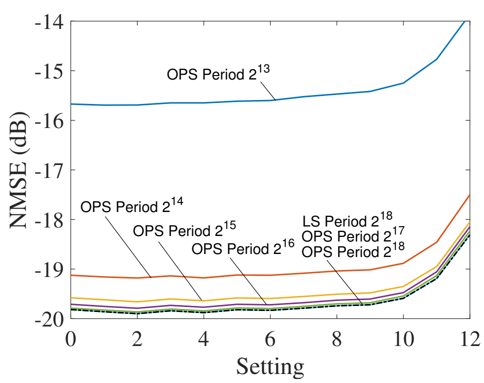
(a)



(b)



(c)



(d)

Fig. 3. NMSEs for (a) LN filter and (b) Volterra filter on uniform distribution input, and for (a) WN filter and (b) Volterra filter on Gaussian distribution input

converge to zero, but for large values of L assumes reasonable values, in the order of few dBs. By averaging the output signal over some periods, also the OPSs for Volterra filters can provide MSD values similar to those of the PPSs.

Fig. 3 compares the identification performance of the different filters in terms of normalized mean square error (NMSE). The MIC100 preamplifier has been identified at the different settings using one period of the PPS or OPS sequences and using $2^{18} = 262\,144$ samples with the least-squares algorithm. Then, the identified models have been tested on a different input sequence of 100 000 samples, matching the sample distribution of the periodic sequence used for the identification. The NMSE has been computed on the test sequence. The OPSs of period 6140 provided totally unacceptable results, which are not included in Fig. 3. For LN and WN filters, it is possible to notice that OPSs for period $L \geq 2^{16}$ provide results similar to the PPSs and to the LS method. In fact, the curves tend to overlap and are almost indistinguishable. For Volterra filters, the OPSs for $L \geq 2^{16}$ provide results similar to those of the LS method. At the highest settings, the WN and Volterra filters identified and tested on Gaussian noise provide larger MSEs than the LN and Volterra filters evaluated on uniform noise. This is consistent with the fact that for equal power a Gaussian noise excites larger amplitudes than a uniform noise. The experiment was also repeated modelling the filters on the same number of input samples for the different methods and the NMSE curves obtained are almost identical to those of Fig. 3. The results of Fig. 3 shows that the OPSs can identify nonlinear filters with performance similar to PPS or LS identification but with a reduced period and computational complexity. The computational complexity reduction in comparison with PPSs is directly proportional to the ratio between the OPS and PPS periods. For $L = 2^{16}$, the OPSs obtain a computation complexity reduction of a factor 0.18 in comparison with the PPSs of this experiment. The computational complexity of the LS method is orders of magnitude larger than that of PPS and OPS identification.

5. CONCLUSION

A novel family of deterministic signals, i.e., the orthogonal periodic sequences, has been presented. These sequences are useful for the identification of FLiP filters and share many of the characteristics of PPSs. As the PPSs they allow the perfect identification of a FLiP filter on a finite time interval with the cross-correlation method. Furthermore, OPSs present advantages in comparison to the PPSs. First of all, OPSs can identify also non-orthogonal FLiP filters, as the Volterra filters. The OPSs input can have any persistently exciting distribution and can also be a quantized sequence. Finally, OPSs can identify FLiP filters with a sequence period much smaller than that of PPSs providing a relevant advantage in terms of computational complexity. However, we have also shown that the OPS period L should be chosen sufficiently larger than the minimum possible period Q, because for $L \simeq Q$ the MSD noise gain assumes very large values that undermine the identification. These aspects have been underlined performing several experiments on a real nonlinear device. The obtained results show the effectiveness of the proposed approach in terms of final system identification and computational complexity reduction in comparison with PPS and LS approaches.

Future works will be oriented to the application of these sequences not only for the identification of nonlinear audio devices but also to the identification of room impulse responses, robust towards nonlinearities affecting the power amplifier or the loudspeaker of the measurement system.

Examples of OPSs can be downloaded from [31].

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