MISSPECIFIED CRB ON PARAMETER ESTIMATION FOR A COUPLED MIXTURE OF POLYNOMIAL PHASE AND SINUSOIDAL FM SIGNALS

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ABSTRACT

This paper studies parameter estimation of a *coupled* mixture of polynomial phase signal (PPS) and sinusoidal frequency modulated (FM) signal, a newly introduced model motivated by industrial applications. Particularly, we analytically evaluate the estimation performance (or performance loss) via the misspecified Cramér-Rao bound (CRB) when system designers choose existing efficient estimation algorithms designed for an independent (decoupled) mixture model due to hardware limits. Our analysis provides an analytical tool to conveniently evaluate performance loss if the implemented system ignores the coupling effect. The achievability of the misspecified CRB is verified by numerical examples.

Index Terms— Parameter estimation, polynomial phase signal, frequency modulation, Cramér-Rao bounds.

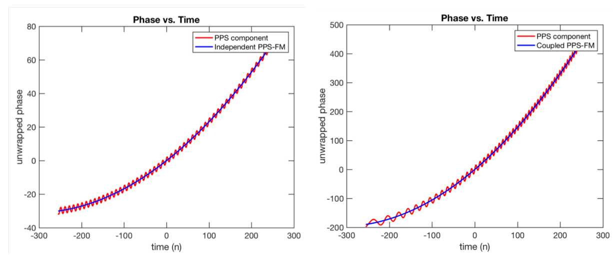
1. INTRODUCTION

The independent mixture of polynomial phase signal (PPS) and sinusoidal frequency modulated (FM) signal, also referred to the hybrid sinusoidal FM-PPS signal model [1], has been found in numerical applications such as electromagnetic sensing, acoustics and optics [1–8]. Particularly, the independent mixture signal assumes the following signal model

$$y(n) = x(n) + v(n), \quad n = 0, 1, \cdots, N-1, \qquad (1)$$
$$= Ae^{j2\pi\left[\sum_{p=0}^{P} \frac{a_p n^p}{p!} + b\sin(2\pi f_0 n + \phi_0)\right]} + v(n)$$

where A is the unknown amplitude, $b > 0$ is the sinusoidal FM modulation index, f_0 is the sinusoidal FM frequency, ϕ_0 is the initial phase, $\{a_p\}_{p=0}^{P}$ are the PPS phase parameters, P is the polynomial order, $v(n)$ is the white Gaussian noise with an unknown variance σ^2, N is the number of samples. We refer to this model as the independent mixture model simply due to the independence between f_0 and $\{a_p\}_{p=0}^{P}$. There are several methods for parameter estimation of the independent mixture signal such as the exact ML estimation method which yields to a multi-dimensional nonlinear optimization solution, the phase unwrapping least square (PULS) method, the HAF method [1], and a recently introduced local high-order phase function (LHPF) [7]. For the independent mixture model, the CRBs for any unbiased estimator were established in [1].

On the other hand, the *coupled* mixture model of the PPS and sinusoidal FM signal was recently introduced in [9, 10] for contactless electromagnetic positioning systems in industrial applications.



(a) The independent mixture (b) The coupled mixture

Fig. 1. The independent and coupled mixture models.

Specifically, the coupling is introduced to express the sinusoidal FM frequency as a function of the PPS parameters, i.e.,

$$x(n) = Ae^{j2\pi\left[\sum_{p=0}^{P} \frac{a_p n^p}{p!} + b\sin(2\pi c_0 \sum_{p=1}^{P} a_p n^p/p! + \phi_0)\right]}, \qquad (2)$$

where the fundamental sinusoidal FM frequency f_0 is now *coupled* with the PPS phase parameters $\mathbf{a} \triangleq [a_1, \cdots, a_P]^T$ as $f_0(n; \mathbf{a}) = c_0 \sum_{p=1}^{P} a_p n^{p-1}/p!$ where c_0 is a scaling factor. Fig. 1 shows the (unwrapped) phase functions for the independent and coupled mixture models. For the independent mixture model, the PPS component is indicated by the red line with a smooth up-going trend, while the ripples around the blue line indicate the sinusoidal FM component. It is clear to see that the ripple frequency stays constant, which is unlike the case of the coupled mixture where the ripple frequency increases as the PPS phase is larger. The corresponding CRBs for the coupled mixture model, referred to as the *coupled* CRB, were derived in [9]. It reveals that lower bounds for estimating the PPS parameters $\{a_p\}$ can be obtained as the coupled sinusoidal FM frequency provides additional information for the PPS parameters.

However, in practice, fully accounting for the coupling between the PPS and sinusoidal FM components often leads to computationally more expensive algorithms. For instance, [10] introduced a phase unwrapping approach followed by nonlinear least square which involves a high-dimensional search, while [11, 12] proposed a multi-stage short-time Fourier transform (STFT) approach followed by bias correction. Due to hardware limits (computational power and memory), practical applications may prefer to directly adopt existing algorithms for the independent mixture model that are computationally lighter, and simply ignore the coupling effect. In this case, it is our interest to understand the performance loss when one applies existing estimation algorithms for the independent mixture model to the coupled mixture signal. To this end, we derive the *misspecified* CRB on parameter estimates of the coupled mixture model when the independent mixture model is used as the assumed signal model. Our analysis reveals that, when the coupled sinusoidal FM component is weak, the performance loss for the PPS phase parameters is almost negligible, while other parameters such as the noise variance suffer from larger losses. Numerical examples verify the achievability of the misspecified CRB.

2. MISSPECIFIED CRB FOR COUPLED MIXTURE MODEL

In the following, we characterize performance tradeoff due to model misspecification. For this purpose, we adopt the misspecified CRB introduced in [13–16] for using the independent mixture model to replace the true coupled mixture model.

To be precise, signal observations are generated from a Gaussian probability density function (pdf) following the *true* coupled signal model of (2),

$$p_x(x_n) = \mathcal{CN}(\bar{\mu}_n, \bar{\sigma}^2), \tag{3}$$

where the mean is given as

$$\bar{\mu}_n = \bar{A} e^{j2\pi\left[\sum_{p=0}^{P} \frac{\bar{a}_p n^p}{p!} + \bar{b}\sin(2\pi \sum_{p=1}^{P} \frac{\bar{a}_p n^p}{p!} + \bar{\phi}_0)\right]}, \tag{4}$$

and the variance is $\bar{\sigma}^2$. Here we use $\bar{[\cdot]}$ to denote the true parameters in the true signal model.

Due to imperfect knowledge on the true signal model, or for the sake of a simpler implementation, the user assumes the signal observations are generated under a misspecified signal model, i.e., the independent mixture signal model of (1), $\mathcal{F}$,

$$f_x(x_n|\boldsymbol{\theta}) = \mathcal{CN}(\mu_n(\boldsymbol{\psi}), \theta), \tag{5}$$

where $\boldsymbol{\theta} = [\boldsymbol{\psi}^T, \sigma^2]^T$ with $\boldsymbol{\psi} = [a_0, a_1, \cdots, a_P, b, f_0, \phi_0, A]$, and $\mu_n(\boldsymbol{\psi}) = A\exp\{j2\pi[\sum_{p=0}^{P} \frac{a_p n^p}{p!} + b\sin(2\pi f_0 n + \phi_0)]\}$. As a result, for the user, the problem of interest is to estimate the signal parameters $\boldsymbol{\psi}$ and the noise variance σ^2 according to (5).

2.1. Pseudo-True Parameter

To derive the misspecified CRB, we need to first find a unique set of pseudo-true parameters for $\boldsymbol{\psi}$ and σ^2 in the assumed signal model of (5) which has the smallest Kullback-Leibler divergence (KLD) with respect to the true signal model of (3). Given (3) and (5), the KLD between the two signal models is given as

$$\begin{aligned} D(p_x\|f_x) &= \int \ln\left(\frac{p_x(\mathbf{x})}{f_x(\mathbf{x}|\boldsymbol{\theta})}\right) p_x(\mathbf{x}) d\mathbf{x} \\ &= N\ln\frac{\sigma^2}{\bar{\sigma}^2} - N + N\frac{\bar{\sigma}^2}{\sigma^2} + \frac{\|\boldsymbol{\mu}(\boldsymbol{\psi}) - \bar{\boldsymbol{\mu}}\|^2}{\sigma^2} \end{aligned} \tag{6}$$

Given the true $\bar{\boldsymbol{\mu}}$ and $\bar{\sigma}^2$, we can first minimize the KLD over the nonlinear parameter set $\boldsymbol{\psi}$, equivalent to minimizing $\|\boldsymbol{\mu}(\boldsymbol{\psi}) - \bar{\boldsymbol{\mu}}\|^2$, i.e.,

$$\begin{aligned} \boldsymbol{\psi}_0 &= \arg\min_{\boldsymbol{\psi}} \|\bar{\boldsymbol{\mu}} - \boldsymbol{\mu}(\boldsymbol{\psi})\|^2 \\ &= \arg\min_{\boldsymbol{\psi}} \sum_{n=0}^{N-1} \left| \bar{\mu}_n - A e^{j2\pi\left[\sum_{p=0}^{P} \frac{a_p n^p}{p!} + b\sin(2\pi f_0 n + \phi_0)\right]} \right|^2 \end{aligned} \tag{7}$$

which can be solved by a nonlinear least squared method or a noiseless phase unwrapping method. With $\boldsymbol{\psi}_0$ and define $|\mu_0|^2 = \|\bar{\boldsymbol{\mu}} - \boldsymbol{\mu}(\boldsymbol{\psi}_0)\|^2/N$, we can minimize the KLD over σ^2 as

$$\sigma_0^2 = \bar{\sigma}^2 + |\mu_0|^2. \tag{8}$$

In other words, in (7), the pseudo-true signal parameters $\boldsymbol{\psi}_0$ minimize the total distance over the two means $\bar{\mu}_n$ and $\mu_n(\boldsymbol{\psi})$. In (8), the pseudo-true nuisance parameter σ_0^2 is found to be the sum of the true noise variance $\bar{\sigma}^2$ and the squared residual term $|\mu_0|^2$. Overall, the pseudo-true parameter set is given as $\boldsymbol{\theta}_0 = [\boldsymbol{\psi}_0^T, \sigma_0^2]^T$.

2.2. The Matrix $\mathbf{A}_{\boldsymbol{\theta}_0}$

Like the conventional CRB, we need to compute FIM-like matrices. For the misspecified CRB, the first generalization of the FIM is the so-called matrix $\mathbf{A}_{\boldsymbol{\theta}_0}$ whose elements are obtained by taking the expectation of the second partial derivatives of the *assumed* log-likelihood function $\ln f_x(\mathbf{x}|\boldsymbol{\theta})$ of (5) over the *true* pdf $p_x(\mathbf{x})$ of (3), i.e.,

$$[\mathbf{A}_{\boldsymbol{\theta}_0}]_{ij} = \mathbb{E}_p\left\{\frac{\partial^2 \ln f_x(\mathbf{x}|\boldsymbol{\theta})}{\partial\boldsymbol{\theta}_i \partial\boldsymbol{\theta}_j}|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\right\}, \tag{9}$$

where $\mathbb{E}_p\{\cdot\}$ indicates the expectation operator with respect to the true pdf $p_x(\mathbf{x})$. From (5), the assumed log-likelihood function is given as

$$\ln f_x(\mathbf{x}|\boldsymbol{\theta}) = -N\ln\sigma^2 - \frac{\|\mathbf{x} - \boldsymbol{\mu}(\boldsymbol{\psi})\|^2}{\sigma^2}. \tag{10}$$

Plugging (10) into (9) yields

$$\begin{aligned}
\mathbf{A}_{a_p,a_q} &= -\frac{8\pi^2}{p!q!\sigma_0^2}\sum_n n^{p+q}\eta_{\Re}(n) \\
\mathbf{A}_{b,b} &= -\frac{8\pi^2}{\sigma_0^2}\sum_n s^2(n)\eta_{\Re}(n) \\
\mathbf{A}_{f_0,f_0} &= -\frac{16\pi^3 b_0}{\sigma_0^2}\sum_n n^2\left[2\pi b_0 c^2(n)\eta_{\Re}(n) - s(n)\eta_{\Im}(n)\right] \\
\mathbf{A}_{\phi_0,\phi_0} &= -\frac{4\pi b_0}{\sigma_0^2}\sum_n \left[2\pi b_0 c^2(n)\eta_{\Re}(n) - s(n)\eta_{\Im}(n)\right] \\
\mathbf{A}_{A,A} &= -2N\sigma_0^{-2}, \quad \mathbf{A}_{\sigma^2,\sigma^2} = -N\sigma_0^{-4} \\
\mathbf{A}_{a_p,b} &= -\frac{8\pi^2}{p!\sigma_0^2}\sum_n n^p s(n)\eta_{\Re}(n) \\
\mathbf{A}_{a_p,f_0} &= -\frac{16\pi^3 b_0}{p!\sigma_0^2}\sum_n n^{p+1}c(n)\eta_{\Re}(n) \\
\mathbf{A}_{a_p,\phi_0} &= -\frac{8\pi^2 b_0}{p!\sigma_0^2}\sum_n n^p c(n)\eta_{\Re}(n) \\
\mathbf{A}_{a_p,A} &= -\frac{4\pi}{p!A_0\sigma_0^2}\sum_n n^p\eta_{\Im}(n) \\
\mathbf{A}_{a_p,\sigma^2} &= \frac{4\pi}{p!\sigma_0^4}\sum_n n^p\eta_{\Im}(n) \\
\mathbf{A}_{b,f_0} &= -\frac{8\pi^2}{\sigma_0^2}\sum_n n\left[\pi b_0 s(n)\eta_{\Re}(n) + c(n)\eta_{\Im}(n)\right] \\
\mathbf{A}_{b,\phi_0} &= -\frac{4\pi}{\sigma_0^2}\sum_n \left[\pi b_0 s(n)\eta_{\Re}(n) + c(n)\eta_{\Im}(n)\right]
\end{aligned}$$

$$\mathbf{A}_{b,A} = -\frac{4\pi}{A_0\sigma_0^2}\sum_n s(n)\eta_{\Im}(n)$$

$$\mathbf{A}_{b,\sigma^2} = \frac{4\pi}{\sigma_0^4}\sum_n s(n)\eta_{\Im}(n)$$

$$\mathbf{A}_{f_0,\phi_0} = -\frac{8\pi^2 b_0}{\sigma_0^2}\sum_n n\left[2\pi b_0 c^2(n)\eta_{\Re}(n) - s(n)\eta_{\Im}(n)\right]$$

$$\mathbf{A}_{f_0,A} = -\frac{8\pi^2 b_0}{A_0\sigma_0^2}\sum_n nc(n)\eta_{\Im}(n)$$

$$\mathbf{A}_{f_0,\sigma^2} = \frac{8\pi^2 b_0}{\sigma_0^4}\sum_n nc(n)\eta_{\Im}(n)$$

$$\mathbf{A}_{\phi_0,A} = -\frac{4\pi b_0}{A_0\sigma_0^2}\sum_n c(n)\eta_{\Im}(n)$$

$$\mathbf{A}_{\phi_0,\sigma^2} = \frac{4\pi b_0}{\sigma_0^4}\sum_n c(n)\eta_{\Im}(n)$$

$$\mathbf{A}_{A,\sigma^2} = -\frac{2}{A_0\sigma_0^4}\sum_n \eta_{\Re}(n) + 2\sigma_0^{-4}NA_0,$$

where $s(n) = \sin(2\pi f_0 n+\phi_0)$, $c(n) = \cos(2\pi f_0 n+\phi_0)$, $\eta_{\Re}(n) = \Re\left\{\bar{\mu}^*(n)\mu_0(n)\right\}$ and $\eta_{\Im}(n) = \Im\left\{\bar{\mu}^*(n)\mu_0(n)\right\}$.

Remark: Convergence to the Independent FIM: If the assumed signal model is the true signal model, i.e., $\bar{\mu}(n) = \mu_0(n)$ and $\bar{\sigma}^2 = \sigma_0^2$, and notice that $\eta_{\Re}(n) = A^2$ and $\eta_{\Im}(n) = 0$, the above misspecified FIM matrix $\mathbf{A}_{\boldsymbol{\theta}_0}$ reduces to the FIM of [1] for the independent mixture model up to a sign change, i.e., $\mathbf{I}_{\boldsymbol{\theta}} = -\mathbf{A}_{\boldsymbol{\theta}_0}$. For instance,

$$\mathbf{I}_{a_p,a_q} = -\mathbf{A}_{a_p,a_q} = \frac{8\pi^2 A^2}{p!q!\sigma^2}\sum_n n^{p+q}$$

$$\mathbf{I}_{a_p,b} = -\mathbf{A}_{a_p,b} = \frac{8\pi^2 A^2}{p!\sigma^2}\sum_n n^p \sin(2\pi f_0 n + \phi_0)$$

converge to Eqs. (66) and (70) of [1].

2.3. The Matrix $\mathbf{B}_{\boldsymbol{\theta}_0}$

The second generalization of the FIM is the so-called matrix $\mathbf{B}_{\boldsymbol{\theta}_0}$ whose elements are obtained by taking the expectation of the cross product of first partial derivatives of the *assumed* log-likelihood function $\ln f_x(\mathbf{x}|\boldsymbol{\theta})$ over the *true* pdf $p_x(\mathbf{x})$, i.e.,

$$[\mathbf{B}_{\boldsymbol{\theta}_0}]_{ij} = \mathbb{E}_p\left\{\frac{\partial \ln f_x(\mathbf{x}|\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_i}|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\frac{\partial \ln f_x(\mathbf{x}|\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_j}|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\right\}, \quad (11)$$

where $\mathbb{E}_p\{\cdot\}$ indicates the expectation operator with respect to the true pdf $p_x(\mathbf{x})$. If there is no model mismatch, we have $-\mathbf{A}_{\boldsymbol{\theta}_0} = \mathbf{B}_{\boldsymbol{\theta}_0}$ which reduces to the conventional FIM. In the case of misspecified model, we have $-\mathbf{A}_{\boldsymbol{\theta}_0} \neq \mathbf{B}_{\boldsymbol{\theta}_0}$. Plugging (10) into (11) yields

$$\mathbf{B}_{a_p,a_q} = \frac{16\pi^2}{p!q!\sigma_0^4}\left(\sum_n n^p\zeta_{\Im}(n)\right)\left(\sum_n n^q\zeta_{\Im}(n)\right) + \frac{8\pi^2\bar{\sigma}^2A_0^2}{p!q!\sigma_0^4}\sum_n n^{p+q}$$

$$\mathbf{B}_{a_p,a_q} = \frac{8\pi^2}{p!q!\sigma_0^4}\left[2\sum_n n^p\eta_{\Im}(n)\sum_n n^q\eta_{\Im}(n) + \bar{\sigma}^2A_0^2\sum_n n^{p+q}\right]$$

$$\mathbf{B}_{b,b} = \frac{16\pi^2}{\sigma_0^4}\left[\sum_n s(n)\eta_{\Im}(n)\right]^2 + \frac{8\pi^2\bar{\sigma}^2A_0^2}{\sigma_0^4}\sum_n s^2(n)$$

$$\mathbf{B}_{f_0,f_0} = \frac{64\pi^4 b_0^2}{\sigma_0^4}\left[\sum_n nc(n)\eta_{\Im}(n)\right]^2 + \frac{32\pi^4 b_0^2\bar{\sigma}^2A_0^2}{\sigma_0^4}\sum_n n^2c^2(n)$$

$$\mathbf{B}_{\phi_0,\phi_0} = \frac{16\pi^2 b_0^2}{\sigma_0^4}\left[\sum_n c(n)\eta_{\Im}(n)\right]^2 + \frac{8\pi^2 b_0^2\bar{\sigma}^2A_0^2}{\sigma_0^4}\sum_n c^2(n)$$

$$\mathbf{B}_{A,A} = \frac{4}{\sigma_0^4A_0^2}\left[\sum_n \eta_{\Re}(n)\right]^2 + \frac{2N\bar{\sigma}^2 + 4N^2A_0^2}{\sigma_0^4} - \frac{8N}{\sigma_0^4}\sum_n \eta_{\Re}(n)$$

$$\mathbf{B}_{\sigma^2,\sigma^2} = \frac{N^2}{\sigma_0^4} - \frac{2N}{\sigma_0^6}(N\bar{\sigma}^2 + \|\bar{\boldsymbol{\mu}} - \boldsymbol{\mu}_0\|^2) + \frac{1}{\sigma_0^8}\left[(N+N^2)\bar{\sigma}^4 + 2(N+1)\bar{\sigma}^2\|\bar{\boldsymbol{\mu}} - \boldsymbol{\mu}_0\|^2 + \|\bar{\boldsymbol{\mu}} - \boldsymbol{\mu}_0\|^4\right]$$

$$\mathbf{B}_{a_p,b} = \frac{16\pi^2}{p!\sigma_0^4}\left[\sum_n n^p\eta_{\Im}(n)\right]\left[\sum_n s(n)\eta_{\Im}(n)\right] + \frac{8\pi^2\bar{\sigma}^2A_0^2}{p!\sigma_0^4}\sum_n n^p s(n)$$

$$\mathbf{B}_{a_p,f_0} = \frac{32\pi^3}{p!\sigma_0^4}\left[\sum_n n^p\eta_{\Im}(n)\right]\left[\sum_n nc(n)\eta_{\Im}(n)\right] + \frac{16\pi^3\bar{\sigma}^2A_0^2}{p!\sigma_0^4}\sum_n n^{p+1}c(n)$$

$$\mathbf{B}_{a_p,\phi_0} = \frac{16\pi^2 b}{p!\sigma_0^4}\left[\sum_n n^p\eta_{\Im}(n)\right]\left[\sum_n c(n)\eta_{\Im}(n)\right] + \frac{8\pi^2 b\bar{\sigma}^2A_0^2}{p!\sigma_0^4}\sum_n n^p c(n)$$

$$\mathbf{B}_{a_p,A} = -\frac{8\pi}{p!\sigma_0^4A_0}\left[\sum_n n^p\eta_{\Im}(n)\right]\left[\sum_n \eta_{\Re}(n)\right] + \frac{8\pi NA_0}{p!\sigma_0^4}\left[\sum_n n^p\eta_{\Im}(n)\right]$$

$$\mathbf{B}_{a_p,\sigma^2} = \frac{4\pi\left[N(\sigma_0^2 - \bar{\sigma}^2) - (\|\bar{\boldsymbol{\mu}} - \boldsymbol{\mu}_0\|^2 + \bar{\sigma}^2)\right]}{p!\sigma_0^6}\left[\sum_n n^p\eta_{\Im}(n)\right]$$

$$\mathbf{B}_{b,f_0} = \frac{32\pi^3 b_0}{\sigma_0^4}\left[\sum_n s(n)\eta_{\Im}(n)\right]\left[\sum_n nc(n)\eta_{\Im}(n)\right] + \frac{8\pi^3 b_0\bar{\sigma}^2A_0^2}{\sigma_0^4}\sum_n ns(n)$$

$$\mathbf{B}_{b,\phi_0} = \frac{16\pi^2 b_0}{\sigma_0^4}\left[\sum_n s(n)\eta_{\Im}(n)\right]\left[\sum_n c(n)\eta_{\Im}(n)\right] + \frac{4\pi^2 b_0\bar{\sigma}^2A_0^2}{\sigma_0^4}\sum_n s(n)$$

$$\mathbf{B}_{b,A} = -\frac{8\pi}{\sigma_0^4A_0}\left[\sum_n s(n)\eta_{\Im}(n)\right]\left[\sum_n \eta_{\Re}(n)\right] + \frac{8\pi NA_0}{\sigma_0^4}\sum_n s(n)\eta_{\Im}(n)$$

$$\mathbf{B}_{b,\sigma^2} = \frac{4\pi\left[N(\sigma_0^2 - \bar{\sigma}^2) - (\|\bar{\boldsymbol{\mu}} - \boldsymbol{\mu}_0\|^2 + \bar{\sigma}^2)\right]}{\sigma_0^6}\left[\sum_n s(n)\eta_{\Im}(n)\right]$$

$$\mathbf{B}_{f_0,\phi_0} = \frac{32\pi^3 b_0^2}{\sigma_0^4}\left[\sum_n nc(n)\eta_{\Im}(n)\right]\left[\sum_n c(n)\eta_{\Im}(n)\right] + \frac{16\pi^3 b_0^2\bar{\sigma}^2A_0^2}{\sigma_0^4}\sum_n nc^2(n)$$

$$\mathbf{B}_{f_0,A} = -\frac{16\pi^2 b_0}{\sigma_0^4A_0}\left[\sum_n nc(n)\eta_{\Im}(n)\right]\left[\sum_n \eta_{\Re}(n)\right] + \frac{16\pi^2 b_0NA_0}{\sigma_0^4}\left[\sum_n nc(n)\eta_{\Im}(n)\right]$$

$$\mathbf{B}_{f_0,\sigma^2} = \frac{8\pi^2 b_0\left[N(\sigma_0^2 - \bar{\sigma}^2) - (\|\bar{\boldsymbol{\mu}} - \boldsymbol{\mu}_0\|^2 + \bar{\sigma}^2)\right]}{\sigma_0^6}\left[\sum_n nc(n)\eta_{\Im}(n)\right]$$

$$\mathbf{B}_{\phi_0,A} = -\frac{8\pi b_0}{\sigma_0^4A_0}\left[\sum_n c(n)\eta_{\Im}(n)\right]\left[\sum_n \eta_{\Re}(n)\right] + \frac{8\pi b_0NA_0}{\sigma_0^4}\left[\sum_n c(n)\eta_{\Im}(n)\right]$$

$$\mathbf{B}_{\phi_0,\sigma^2} = \frac{4\pi b_0\left[N(\sigma_0^2 - \bar{\sigma}^2) - (\|\bar{\boldsymbol{\mu}} - \boldsymbol{\mu}_0\|^2 + \bar{\sigma}^2)\right]}{\sigma_0^6}\left[\sum_n c(n)\eta_{\Im}(n)\right]$$

$$\mathbf{B}_{A,\sigma^2} = \frac{2N(\bar{\sigma}^2 - \sigma_0^2)\Re\left\{\bar{\boldsymbol{\mu}}^H\boldsymbol{\mu}_0\right\} + 2\bar{\sigma}^2\left(\Re\left\{\bar{\boldsymbol{\mu}}^H\boldsymbol{\mu}_0\right\} - NA_0^2\right) + 2N^2A_0^2\left(\sigma_0^2 - \bar{\sigma}^2\right)}{\sigma_0^6A_0} + \frac{2(\Re\left\{\bar{\boldsymbol{\mu}}^H\boldsymbol{\mu}_0\right\} - NA_0^2)\|\bar{\boldsymbol{\mu}} - \boldsymbol{\mu}_0\|^2}{\sigma_0^6A_0}.$$

Remark: Convergence to the Independent FIM: Like in the previous case, we can show that the matrix $\mathbf{B}_{\boldsymbol{\theta}_0}$ converges exactly to the conventional FIM in [1] when the signal observations are generated from the independent mixture model,

$$\mathbf{I}_{\boldsymbol{\theta}} = \mathbf{B}_{\boldsymbol{\theta}_0}. \quad (12)$$

2.4. Misspecified CRB

With the derivations of both FIM-like matrices $\mathbf{A}_{\boldsymbol{\theta}_0}$ and $\mathbf{B}_{\boldsymbol{\theta}_0}$, we are ready to derive the misspecified CRB. For any MS-unbiased estimator $\hat{\boldsymbol{\theta}}(\mathbf{x})$, i.e., $\mathbb{E}_p\{\hat{\boldsymbol{\theta}}(\mathbf{x})\} = \boldsymbol{\theta}_0$ (the expectation of the estimator w.r.t. the true pdf $p_x(\mathbf{x})$ converges to the pseudo-true parameter $\boldsymbol{\theta}_0$ defined in Section 2.1), then the error covariance matrix of the mismatched estimator is given as

$$\mathbf{C}_p(\hat{\boldsymbol{\theta}}(\mathbf{x}), \boldsymbol{\theta}_0) = \mathbb{E}_p\left\{(\hat{\boldsymbol{\theta}}(\mathbf{x}) - \boldsymbol{\theta}_0)(\hat{\boldsymbol{\theta}}(\mathbf{x}) - \boldsymbol{\theta}_0)^H\right\} \quad (13)$$

where $\boldsymbol{\theta}_0$ is the pseudo-true parameter, is lower bounded by the misspecified CRB [16]

$$\mathbf{C}_p(\hat{\boldsymbol{\theta}}(\mathbf{x}), \boldsymbol{\theta}_0) \succeq \frac{1}{N}\mathbf{A}_{\boldsymbol{\theta}_0}^{-1}\mathbf{B}_{\boldsymbol{\theta}_0}\mathbf{A}_{\boldsymbol{\theta}_0}^{-1} = \mathrm{MCRB}(\boldsymbol{\theta}_0) \quad (14)$$

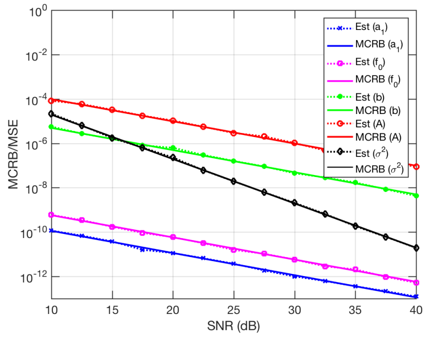


Fig. 2. Achievability of the derived misspecified CRB with observations generated according to a coupled mixture model by using a phase unwrapping method on the assumed independent mixture model.

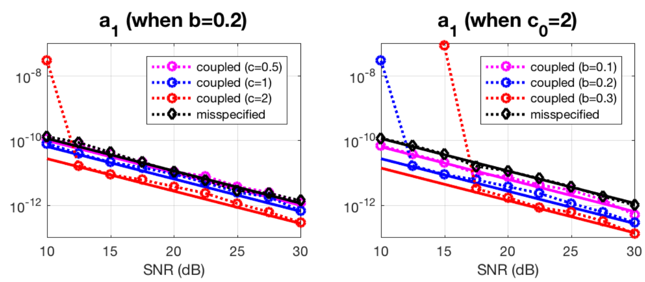


Fig. 3. Comparison of the coupled and misspecified CRBs as a function of c_0 and the sinusoidal FM index b.

3. NUMERICAL EXAMPLES

In the following, the observed samples are generated according to the coupled mixture model of (2) and we compare the derived misspecified CRB with

- the conventional phase unwrapping [6] for the independent mixture model to show the achievability of the misspecified CRB;
- the coupled CRB derived in [9];
- the coupled phase unwrapping method [10] for the coupled mixture model to show the achievability of the coupled CRB.

3.1. A Coupled Mixture of A Single-Tone Signal ($P = 1$) and A Sinusoidal FM Signal

We first consider a case of (2) with the PPS order $P = 1$ (hence a single-tone signal) and true parameters $\bar{A} = 1$, $\bar{a}_0 = 0$, $\bar{a}_1 = 0.1$, $\bar{b} = 0.1$, $\bar{\phi}_0 = 0$, $c_0 = 2$, and $N = 512$. In this case, we have the pseudo-true parameter $f_0 = c_0\bar{a}_1$ and other pseudo-true parameters are the same as their corresponding true parameters. Fig. 2 shows that, for all parameters including f_0 in the assumed independent model, the measured mean-squared errors (MSEs) of the conventional phase unwrapping achieve exactly their corresponding misspecified CRBs.

Next, we quantify the performance loss due to the model mismatch by comparing the coupled CRBs [9] with the misspecified ones. Fig. 3 shows CRB comparisons as a function of c_0 when b is fixed or b when c_0 is fixed. It shows that the performance loss of using the misspecified estimator increases as c_0 increases from $c_0 = 0.5$ to $c_0 = 2$ or b increases from $b = 0.1$ to $b = 0.3$. In other words, the estimator using the correct coupled mixture model can be more accurate with lower estimation variance when c_0 or b is large. This is intuitive as c_0 and b is large, the additional information extracting from the sinusoidal FM component for the estimation of a_1 increases and hence the coupled CRB is lower.

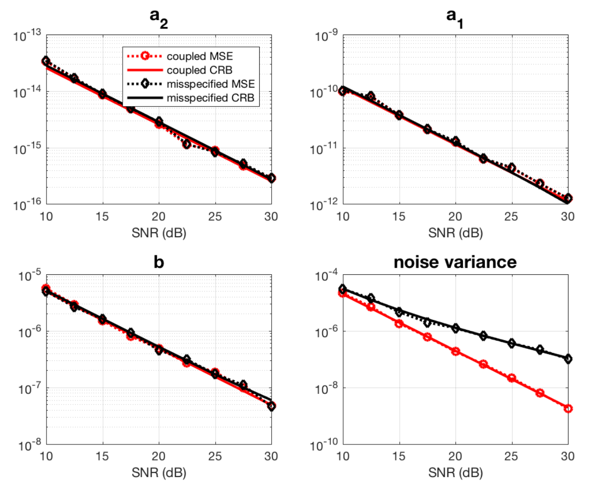


Fig. 4. Measured MSEs and corresponding CRBs for the coupled mixture of a chirp signal and a weak sinusoidal FM signal.

3.2. A Coupled Mixture of A Chirp Signal ($P = 2$) and A Weak Sinusoidal FM Signal ($b = 0.05$)

We then consider a coupled mixture signal of a second-order PPS component ($P = 2$, also referred to chirp signals) and a sinusoidal FM component with true parameters $\bar{A} = 1$, $\bar{a}_0 = 0.1$, $\bar{a}_1 = 0.15$, $\bar{a}_2 = 1.3889 \cdot 10^{-4}$, $b = 0.05$, $\phi_1 = 0$, $c_0 = 0.1$, and $N = 512$. In this case, we cannot find a single psuedo-true parameter f_0 converging to the true sinusoidal FM frequency $c_0(a_1 + 0.5a_2n)$ which is a time-varying function. In this case, the pseudotrue parameters $\{f_0, A, a_0, a_1, a_2, b, \phi_1\}$ are obtained from (7) while the pseudo-true parameter σ^2 is obtained from (8) by adding the squared residual to the true noise variance. Fig. 4 shows the measured MSE for the coupled and misspecified signal models and their corresponding CRBs for the signal parameters $\{a_2, a_1, b\}$ and the noise variance σ^2. It is shown that, for the simulated scenario, the performance loss due to the assumption of the independent signal model is negligible for estimating signal parameters, while the estimate of noise variance suffers from larger performance degradation.

4. CONCLUSIONS

This paper derives the misspecified CRB for a class of coupled mixture model between the PPS and sinusoidal FM components and provides an analytical tool for system designers to numerically evaluate performance loss if the implemented system ignores the coupling effect. With numerical validations, we have shown the achievability of the misspecified CRB at relatively high SNRs. The performance gap between the coupled and misspecified CRBs was shown to increase as the sinusoidal FM index b or the constant c_0 increases.

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