INFORMATION THEORETIC LOWER BOUND OF RESTRICTED ISOMETRY PROPERTY CONSTANT

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ABSTRACT

Compressed sensing seeks to recover an unknown sparse vector from undersampled rate measurements. Since its introduction, there have been enormous works on compressed sensing that develop efficient algorithms for sparse signal recovery. The restricted isometry property (RIP) has become the dominant tool used for the analysis of exact reconstruction from seemingly undersampled measurements. Although the upper bound of the RIP constant has been studied extensively, as far as we know, the result is missing for the lower bound. In this work, we first present a tight lower bound for the RIP constant, filling the gap there. The lower bound is at the same order as the upper bound for the RIP constant. Moreover, we also show that our lower bound is close to the upper bound by numerical simulations. Our bound on the RIP constant provides an information-theoretic lower bound about the sampling rate for the first time, which is the essential question for practitioners.

Index Terms— Compressed sensing, sparse recovery, restricted isometry property, Gaussian random matrix, upper and lower bound

1. INTRODUCTION

In the field of compressed sensing, sparsity of data and structure can be harnessed for signal reconstruction with samples much fewer than Nyquist sampling [1, 2, 3, 4]. This has opened up a new era of signal processing, offering solutions to real-world problems where there is very limited accessible data [3, 5], including sensor networks [6], medical imaging [7], camera design [8], etc.

Candes et al [1] proved that the compressed sensing problem can be reformulated as an l_0-norm minimization. Since l_0-norm optimization is known to be NP-hard, it is shown [1] that when the sensing matrix possesses certain favorable properties, the solution to the l_0-norm optimization is equal to the corresponding l_1-norm optimization. This property, known as the restricted isometry property (RIP), has been the core of compressed sensing, with many researchers working towards extending the conditions [9, 10] as well as improving the results [11].

Since the proposal of compressed sensing, many efficient algorithms have been developed for sparse signal recovery [12, 13, 14, 15], and the RIP is considered as the main tool for analyzing the algorithms [1, 16, 17, 18].

We first recall the definition of RIP. Let $\mathbf{\Phi}$ be a matrix in $\mathbb{R}^{n\times N}$. We say $\mathbf{\Phi}$ satisfies the RIP with constant δ_k for k-sparse vectors, if δ_k is the minimum constant to satisfy the following condition:

$$1-\delta_k \le \frac{\|\mathbf{\Phi v}\|}{\|\mathbf{v}\|} \le 1+\delta_k, \quad \forall \mathbf{v}\in\mathbb{R}^N\cap\Psi_k\backslash\{\mathbf{0}\}, \qquad (1)$$

where Ψ_k is the set of k-sparse vectors, i.e. vectors which have at most k nonzero entries. $\|\cdot\|$ denotes the vector l_2-norm unless otherwise stated.

However, as far as we are concerned, the currently existing results of RIP constant for compression matrices $\mathbf{\Phi}$ are all about its upper bound, i.e,

$$\left|\frac{\|\mathbf{\Phi v}\|-\|\mathbf{v}\|}{\|\mathbf{v}\|}\right| \le \delta_k^u, \quad \forall \mathbf{v}\in\mathbb{R}^N\cap\Psi_k\backslash\{\mathbf{0}\}.$$

However, the information-theoretic lower bound is also of great importance. In this work, we derive a tight lower bound of RIP constant for Gaussian random matrices by the construction method, i.e.

$$\left|\frac{\|\mathbf{\Phi v}\|-\|\mathbf{v}\|}{\|\mathbf{v}\|}\right| \ge \delta_k^l, \quad \exists \mathbf{v}\in\mathbb{R}^N\cap\Psi_k\backslash\{\mathbf{0}\},$$

which is the main contribution of this work.

Besides, to show the tightness of the lower bound, we provide a new upper bound for RIP constant, which is of the same order as existing results, improved by a constant. We also show that the lower and upper bounds are of the same order rigorously.

The rest of this paper is organized as follows. In Section 2, we state some backgrounds about the RIP and sparse recovery. In Section 3, we first present the main results of this paper regarding the lower and upper bounds of RIP constant of Gaussian random matrices, and then provide the sketch of the proof. In Section 5, numerical simulations are implemented to compare the lower bound, the new upper bound and existing work for the RIP constant. We conclude this paper in Section 6. The details of the proof of the main results, and more detailed simulations can be found in [19].

2. BACKGROUND

In this section, we state some background about the RIP and sparse recovery.

2.1. The Restricted Isometry Property

The compressed sensing setting poses the k-sparsity constraint on the vector $\mathbf{v}$. An alternate view of this is to convert the sparsity constraint onto $\mathbf{\Phi}$. Let $\mathcal{S}$ be the index set for which the entries $\mathbf{v}$ is nonzero. Now we set all columns with indices not in $\mathcal{S}$ as zero, such that $\mathbf{\Phi}$ is k-sparse column-wise. Denote the resulting matrix as $\mathbf{\Phi}_{\mathcal{S}}$.

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Therefore, the condition in (1) can be rewritten as

$$1-\delta_k \le \sqrt{\lambda_{\min}(\mathbf{\Phi}_{\mathcal{S}}^{\mathrm{T}}\mathbf{\Phi}_{\mathcal{S}})} \le \sqrt{\lambda_{\max}(\mathbf{\Phi}_{\mathcal{S}}^{\mathrm{T}}\mathbf{\Phi}_{\mathcal{S}})} \le 1+\delta_k, \quad \forall\, |\mathcal{S}| \le k,$$

or equivalently

$$1-\delta_k \le s_{\min}(\mathbf{\Phi}_{\mathcal{S}}) \le s_{\max}(\mathbf{\Phi}_{\mathcal{S}}) \le 1+\delta_k, \quad \forall\, |\mathcal{S}| \le k.$$

Here $\lambda_{\max}$ and $s_{\max}$ denote respectively the maximum eigenvalue and the maximum singular value of a matrix. $\lambda_{\min}$ and $s_{\min}$ are defined likewise.

To simplify the notations, we define

$$s_{\max}^k(\mathbf{\Phi}) := \max_{\mathbf{u}\in\mathbb{S}^{n-1}} \max_{\substack{\mathbf{v}\in\mathbb{S}^{N-1}\\ \mathbf{v}\in\Psi_k}} \mathbf{u}^{\mathrm{T}}\mathbf{\Phi}\mathbf{v},$$
$$s_{\min}^k(\mathbf{\Phi}) := \max_{\mathbf{u}\in\mathbb{S}^{n-1}} \min_{\substack{\mathbf{v}\in\mathbb{S}^{N-1}\\ \mathbf{v}\in\Psi_k}} \mathbf{u}^{\mathrm{T}}\mathbf{\Phi}\mathbf{v}.$$

Therefore we have

$$\delta_k = \max\left\{\delta_k^+, \delta_k^-\right\},$$

where

$$\delta_k^+ = s_{\max}^k(\mathbf{\Phi}) - 1,$$
$$\delta_k^- = 1 - s_{\min}^k(\mathbf{\Phi}).$$

$s_{\max}^k(\mathbf{\Phi})$ and $s_{\min}^k(\mathbf{\Phi})$ can also be understood as the maximum and minimum singular values of all possible $\mathbf{\Phi}_{\mathcal{S}}$ matrices. Then the upper and lower bound of the RIP constant δ_k satisfies

$$\delta_k^u \ge \max\left\{\delta_k^+, \delta_k^-\right\},$$
$$\delta_k^l \le \max\left\{\delta_k^+, \delta_k^-\right\}.$$

Hence, δ_k^u is related to estimation of upper bound of $s_{\max}^k(\mathbf{A})$ and lower bound of $s_{\min}^k(\mathbf{A})$, and δ_k^l is related to estimation of lower bound of $s_{\max}^k(\mathbf{A})$ and upper bound of $s_{\min}^k(\mathbf{A})$.

2.2. Previous Results of Gaussian RIP

One of the most studied families of matrices with favorable RIP property is Gaussian random matrices, in the form of

$$\mathbf{\Phi} = \frac{1}{\sqrt{n}}\mathbf{A}, \tag{2}$$

where each entry in $\mathbf{A} \in \mathbb{R}^{n\times N}$ is $i.i.d.$ standard Gaussian, namely $\mathcal{N}(0,1)$. In the following we recall a benchmark result regarding Gaussian RIP, which contrasts sharply with our result.

The following result is obtained from covering numbers:

Theorem 1 (Theorem 5.2 in [20]). *Let $\mathbf{\Phi} \in \mathbb{R}^{n\times N}$ be a random matrix in the form of (2). Then for any $0<\delta<1$, we have*

$$(1-\delta)\|\mathbf{x}\| \le \|\mathbf{\Phi}\mathbf{x}\| \le (1+\delta)\|\mathbf{x}\|, \quad \forall \mathbf{x}\in\mathbb{R}^N \cap \Psi_k,$$

with probability at least

$$1 - \binom{N}{k} 2\left(\frac{12}{\delta}\right)^k \mathrm{e}^{-c_0 n}, \tag{3}$$

where $c_0 = \frac{\delta^3}{32} - \frac{\delta^4}{96}$.

Remark 1. *Using the approximation $\binom{N}{k} \approx \left(\frac{\mathrm{e}N}{k}\right)^k$, we can rewrite (3) as*

$$1 - \mathrm{e}^{-c_1 n},$$

where

$$c_1 = c_0 - \frac{k}{n}\ln\frac{\mathrm{e}N}{k} - \frac{\ln 2}{n} - \frac{k}{n}\ln\frac{12}{\delta}.$$

In order to justifiably consider the asymptotic case of $n\to\infty$, an immediate requirement is $c_1>0$, which is equivalent to

$$n > \frac{1}{c_0}\left(k\ln\frac{\mathrm{e}N}{k} + \ln 2 + k\ln\frac{12}{\delta}\right).$$

Therefore a really large n is required to make the asymptotic analysis valid.

2.3. Sparse Recovery

Sparse recovery concerns recovering a k-sparse vector $\mathbf{x}$ from linear measurements $\mathbf{y} = \mathbf{\Phi}\mathbf{x}$, where the following results are provided in literature [1, 2, 4].

Theorem 2. *Suppose $\mathbf{x}$ is k-sparse. If $\delta_{2k} < 1$, then l_0-minimization*

$$\min_{\mathbf{x}} \|\mathbf{x}\|_0 \quad \text{s.t.} \quad \mathbf{y} = \mathbf{\Phi}\mathbf{x}$$

is exact and unique.

Theorem 3. *Suppose $\mathbf{x}$ is k-sparse. If $\delta_{2k} < \sqrt{2}-1$, then l_1-minimization*

$$\min_{\mathbf{x}} \|\mathbf{x}\|_1 \quad \text{s.t.} \quad \mathbf{y} = \mathbf{\Phi}\mathbf{x}$$

is exact and unique.

Remark 2. *If the RIP constant of $\mathbf{\Phi}$ is greater than 1, l_0-minimization may fail. If the RIP constant of $\mathbf{\Phi}$ is greater than $\sqrt{2}-1$, l_1-minimization may fail. However, evaluating the RIP constant for a given matrix is NP-hard [21]. Hence, it's very important to derive tight information-theoretic bounds for the analysis of the reconstruction algorithms.*

3. MAIN RESULTS

In this section, we first present the main results of this paper regarding the lower bound of RIP constant of Gaussian random matrices, then derive a new upper bound of RIP constant for comparison, and finally provide the proof.

While Gaussian matrices $\mathbf{\Phi}$ of the form (2) are adopted as sensing matrices, in the proof we shall first focus on standard Gaussian $\mathbf{A}$ for simplicity on expression, before switching back to properties of $\mathbf{\Phi}$ using properties derived for $\mathbf{A}$. In the proof, we will consider $s_{\max}^k(\mathbf{A})$ and $s_{\min}^k(\mathbf{A})$ directly, instead of considering δ_k^+ and δ_k^-.

Theorem 4. *Let $\mathbf{A}$ be a Gaussian matrix in $\mathbb{R}^{n\times N}$. Then with probability at least $1-6\mathrm{e}^{-\frac{n\varepsilon^2}{2}}$, the RIP constant of $\mathbf{\Phi} = \frac{1}{\sqrt{n}}\mathbf{A}$ satisfies*

$$\begin{aligned}\delta_k^+ \ge& \frac{1}{\sqrt{2}}\left(2+pT^2+\sqrt{(pT^2)^2+4pT^2}\right)^{1/2} - 1\\ &+ \mathcal{O}\left(\varepsilon+\frac{1}{\sqrt{n}}\right),\\ \delta_k^- \ge& 1-\frac{1}{\sqrt{2}}\left(2+pT^2-\sqrt{(pT^2)^2+4pT^2}\right)^{1/2}\\ &+ \mathcal{O}\left(\varepsilon+\frac{1}{\sqrt{n}}\right),\end{aligned}$$

where $p = \frac{k}{n}$ *and* $T = \sqrt{\mathbb{E}X^2 \Big| X^2 > t}$ *with* $\mathbb{P}\left(X^2 > t\right) = \frac{k-1}{N-1}$.

Proof. The proof is postponed to Section 4. The details about $\mathcal{O}(\varepsilon + \frac{1}{\sqrt{n}})$, which will be verified in the numerical simulations, are explained in [19]. $\square$

Theorem 5. *Let* $\mathbf{A}$ *be a Gaussian matrix in* $\mathbb{R}^{n\times N}$. *Then* $\mathbf{\Phi} = \frac{1}{\sqrt{n}}\mathbf{A}$ *satisfies the RIP with constant*

$$\delta_k := \sqrt{p}T + \frac{2\sqrt{2\pi}}{T\sqrt{n}} + \varepsilon$$

for k-sparse vectors with probability at least $1 - 2\mathrm{e}^{-\frac{n\varepsilon^2}{2}}$, *where* $p = \frac{k}{n}$ *and* $T = \sqrt{\mathbb{E}X^2 \Big| X^2 > t}$ *with* $\mathbb{P}\left(X^2 > t\right) = \frac{k}{N}$.

Proof. We briefly introduce the derivation of the upper bound of $s^k_{\max}(\mathbf{A})$ here, and the lower bound of $s^k_{\min}(\mathbf{A})$ can be obtained similarly. In order to derive the upper bound, we first use a result regarding Lipschitz functions that converts analysis on $s^k_{\max}(\mathbf{A})$ to that on $\mathbb{E}s^k_{\max}(\mathbf{A})$. Then we construct two Gaussian processes which allow the comparison lemma to relax $s^k_{\max}(\mathbf{A})$ to another more tractable form. Finally, we combine the two steps to obtain the result. The details of the proof are included in [19]. $\square$

Remark 3. *Since* $t = \mathcal{O}\left(\log\frac{N}{k}\right)$ *and* $T = \mathcal{O}\left(\sqrt{\log\frac{N}{k}}\right)$, *the results above show that* $\mathbf{\Phi}$ *satisfies the RIP if and only if* $n = \mathcal{O}\left(k\log\frac{N}{k}\right)$. *This reveals that the lower and upper bounds match in order.*

4. PROOF OF THEOREM 4

We look for the lower bound of the RIP constant, i.e. the lower bound of $s^k_{\max}(\mathbf{A})$ and the upper bound of $s^k_{\min}(\mathbf{A})$. We do this by construction.

Since in the expression $\mathbf{u}^{\mathrm{T}}\mathbf{A}\mathbf{v}$ there are no constraints on $\mathbf{u}$ other than $\mathbf{u} \in \mathbb{S}^{n-1}$, we can always find some orthogonal matrix $\mathbf{U}$ such that

$$\mathbf{u}^{\mathrm{T}}\mathbf{A}\mathbf{v} = (\mathbf{U}\mathbf{u})^{\mathrm{T}}(\mathbf{U}\mathbf{A})\mathbf{v} := \tilde{\mathbf{u}}^{\mathrm{T}}\tilde{\mathbf{A}}\mathbf{v}.$$

Here $\tilde{\mathbf{u}} = \mathbf{U}\mathbf{u}$ can still represent any arbitrary vector in $\mathbb{S}^{n-1}$, and $\tilde{\mathbf{A}} := \mathbf{U}\mathbf{A}$ satisfies the following form:

$$\tilde{\mathbf{A}} = \begin{bmatrix} \|\mathbf{a}_1\| & \mathbf{c}^{\mathrm{T}} \\ \mathbf{0} & \mathbf{B} \end{bmatrix},$$

where $\mathbf{a}_1$ denotes the first column of the original matrix $\mathbf{A}$, $\mathbf{c}^{\mathrm{T}} = [c_1, \cdots, c_{N-1}]$ is a row vector in $\mathbb{R}^{N-1}$ with its entries' absolute values in decreasing order, and $\mathbf{B} \in \mathbb{R}^{(n-1)\times(N-1)}$. This form of $\tilde{\mathbf{A}}$ is achievable by choosing the first column of $\mathbf{U}$ parallel to the first column of $\mathbf{A}$ and applying certain permutations to the latter $N-1$ columns.

Observing the fact that

$$s^k_{\max}(\mathbf{A}) = \max_{\substack{\mathbf{v}\in\mathbb{S}^{N-1}\\ \mathbf{v}\in\Psi_k}} \left\|\tilde{\mathbf{A}}\mathbf{v}\right\|$$

and similarly

$$s^k_{\min}(\mathbf{A}) = \min_{\substack{\mathbf{v}\in\mathbb{S}^{N-1}\\ \mathbf{v}\in\Psi_k}} \left\|\tilde{\mathbf{A}}\mathbf{v}\right\|,$$

we only need to construct two extreme numbers of $\left\|\tilde{\mathbf{A}}\mathbf{v}\right\|$ in order to derive the required bounds of $s^k_{\max}(\mathbf{A})$ and $s^k_{\min}(\mathbf{A})$ with high probability.

With the constraints $\mathbf{v} \in \mathbb{S}^{N-1}$ and $\mathbf{v} \in \Psi_k$, we consider in particular $\mathbf{v}$ that satisfies the following form:

$$\mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cos\theta + \begin{bmatrix} 0 \\ c_1 \\ \vdots \\ c_{k-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \frac{\sin\theta}{\|\mathbf{c}^{(k-1)}\|},$$

where $\theta \in \mathbb{R}$, and $\mathbf{c}^{(k-1)} = [c_1, \cdots, c_{k-1}]^{\mathrm{T}}$ denotes the sub-vector containing the first $k-1$ entries of $\mathbf{c}$, i.e. the ones with the $k-1$ largest absolute values.

For $\mathbf{v}_0$ in such form, we have

$$\begin{aligned} &\left\|\tilde{\mathbf{A}}\mathbf{v}_0\right\|^2 \\ &= \left\| \begin{bmatrix} \|\mathbf{a}_1\| \\ 0 \end{bmatrix} \cos\theta + \begin{bmatrix} \left\|\mathbf{c}^{(k-1)}\right\| \\ \mathbf{b} \end{bmatrix} \sin\theta \right\|^2 \\ &= \left(\|\mathbf{a}_1\|\cos\theta + \left\|\mathbf{c}^{(k-1)}\right\|\sin\theta \right)^2 + \|\mathbf{b}\|^2 \sin^2(\theta) \\ &=: g(\theta), \end{aligned}$$

where

$$\mathbf{b} = \mathbf{B}\left([0, c_1, \cdots, c_{k-1}, 0, \cdots, 0]^{\mathrm{T}}, \frac{1}{\|\mathbf{c}^{(k-1)}\|} \right)$$

is the product of a standard Gaussian matrix $\mathbf{B}$ with a unit vector independent of $\mathbf{B}$, and is therefore an standard Gaussian vector. Therefore $\mathbf{a}_1$, $\mathbf{c}^{(k-1)}$ and $\mathbf{b}$ are mutually independent, with $\mathbf{a}_1 \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^{n-1}$ being standard Gaussian vectors, and $\mathbf{c}^{(k-1)}$ containing the elements with the $(k-1)$-largest absolute values of $N-1$ random variables of standard Gaussian, as previously defined.

We rewrite $g(\theta)$ as the following:

$$g(\theta) = A\cos^2\theta + B\sin^2\theta + 2C\sin\theta\cos\theta,$$

where

$$\begin{aligned} A &= \|\mathbf{a}_1\|^2, \\ B &= \|\mathbf{b}\|^2 + \left\|\mathbf{c}^{(k-1)}\right\|^2, \\ C &= \|\mathbf{a}_1\| \cdot \left\|\mathbf{c}^{(k-1)}\right\|. \end{aligned}$$

Letting $\frac{\mathrm{d}g(\theta)}{\mathrm{d}\theta} = 0$, we have

$$\theta = \frac{1}{2}\arctan\frac{2C}{A-B} + \frac{l\pi}{2} =: \theta_0 + \frac{l\pi}{2}, \quad l \in \mathbb{Z}.$$

Plugging the extrema of θ into $g(\theta)$, we have

$$\begin{aligned} s^k_{\max}(\mathbf{A}) &\geq \frac{1}{\sqrt{2}}\left(A + B + \sqrt{(A-B)^2 + 4C^2}\right)^{1/2}, \\ s^k_{\min}(\mathbf{A}) &\leq \frac{1}{\sqrt{2}}\left(A + B - \sqrt{(A-B)^2 + 4C^2}\right)^{1/2}. \end{aligned}$$

Let's first calculate the values of A, B, and C asymptotically to have a rough idea of this, i.e.

$$
\begin{aligned}
&\lim_{n\to\infty} A/n = 1, \\
&\lim_{n\to\infty} B/n = 1 + \frac{kT^2}{n}, \\
&\lim_{n\to\infty} C/n = \sqrt{\frac{kT^2}{n}},
\end{aligned}
$$

Summing up, we have

$$
\begin{aligned}
\lim_{n\to\infty} s^k_{\max}(\mathbf{\Phi}) &= \lim_{n\to\infty} \frac{1}{\sqrt{n}} s^k_{\max}(\mathbf{A}) \\
&\geq \frac{1}{\sqrt{2}} \left(2 + pT^2 + \sqrt{(pT^2)^2 + 4pT^2}\right)^{1/2}, \\
\lim_{n\to\infty} s^k_{\min}(\mathbf{\Phi}) &= \lim_{n\to\infty} \frac{1}{\sqrt{n}} s^k_{\min}(\mathbf{A}) \\
&\leq \frac{1}{\sqrt{2}} \left(2 + pT^2 - \sqrt{(pT^2)^2 + 4pT^2}\right)^{1/2},
\end{aligned}
$$

where $p = \lim_{n\to\infty} k/n$, i.e.

$$
\begin{aligned}
&\lim_{n\to\infty} \delta^+_k \geq \frac{1}{\sqrt{2}} \left(2 + pT^2 + \sqrt{(pT^2)^2 + 4pT^2}\right)^{1/2} - 1, \\
&\lim_{n\to\infty} \delta^-_k \geq 1 - \frac{1}{\sqrt{2}} \left(2 + pT^2 - \sqrt{(pT^2)^2 + 4pT^2}\right)^{1/2}.
\end{aligned}
$$

The above results are established asymptotically. The cases for finite n can be achieved by merely estimating A, B, and C by using Lemma 1, which is proved in detail in [19].

Lemma 1. *[19] Let $X_1, \dots, X_n$ denote n i. i. d. Gaussian random variables, and $X_{(i)}$ denotes the one with the i-th largest absolute value among them. Define*

$$
T_k := \sqrt{\frac{1}{k}\sum_{i=1}^{k} X^2_{(i)}},
$$

and

$$
T = \sqrt{\mathbb{E}X^2 \Big| X^2 > t},
$$

where t satisfies $\mathbb{P}\left(X^2 > t\right) = \frac{k}{n}$, then

$$
|\mathbb{E}T_k - T| < \frac{2\sqrt{2\pi}}{T\sqrt{k}},
$$

and

$$
\mathbb{P}\left(|T_k - T| > \frac{2\sqrt{2\pi}}{T\sqrt{k}} + \varepsilon\right) < 2\mathrm{e}^{-\frac{k\varepsilon^2}{2}}, \quad \forall \varepsilon > 0.
$$

5. NUMERICAL SIMULATIONS

In this section, we compare the lower and upper bounds for the RIP constant by numerical calculations. The bounds for the RIP constant are calculated according to Theorem 1, 4, 5, respectively. Here, we fix sparsity levels $k/N = (0.1, 0.01, 0.001), P = 0.99, N = 10000$, and calculate the bound versus compression rate. Moreover, to facilitate plotting, we truncate the bounds by 2, i.e., any bound with value greater than 2 is set to 2. From Fig. 1, we can find that when compression rate N/n is not too large, the lower bound and new upper bound very close to each other.

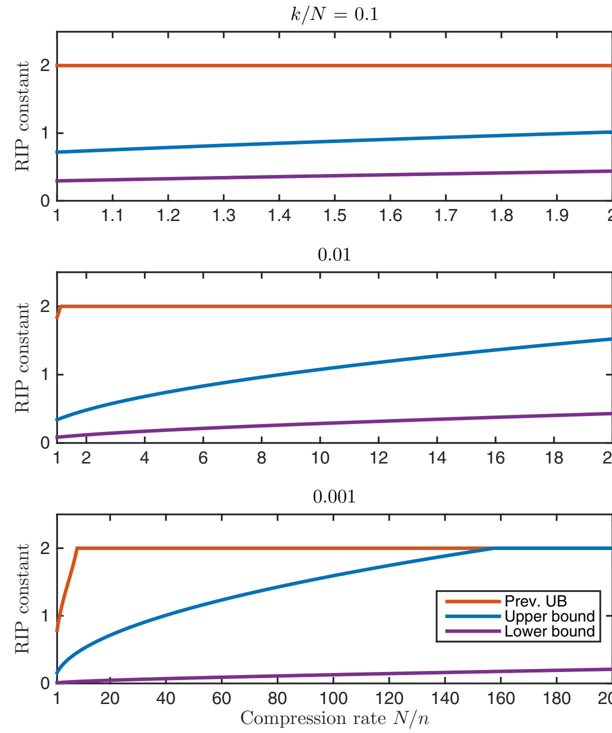

Fig. 1. Numerical calculation of the lower bound, the upper bound, and the previous work for the RIP constant (Prev. UB) about compression rate, when fixing sparsity $k/N, P = 0.99, N = 10000$. All curves are truncated by 2.

6. CONCLUSION

In this paper, we propose a lower bound for the RIP constant for the first time, which fills the gap that only upper bound is provided for the RIP constant. Moreover, we prove that the lower and upper bounds have the same order, and hence match in order. Finally, numerical calculations validate the tightness of the lower and new upper bound for the RIP constant. Future works may include the performance analysis of sparse recovery algorithms with both lower and upper bounds.

References

[1] Emmanuel J Candès and Terence Tao, "Decoding by linear programming," *IEEE transactions on information theory*, vol. 51, no. 12, pp. 4203–4215, 2005.

[2] Emmanuel J Candès and Terence Tao, "Near-optimal signal recovery from random projections: Universal encoding strategies?," *IEEE transactions on information theory*, vol. 52, no. 12, pp. 5406–5425, 2006.

[3] David L Donoho, "Compressed sensing," *IEEE Transactions on information theory*, vol. 52, no. 4, pp. 1289–1306, 2006.

[4] Emmanuel J Candès and Michael B Wakin, "An introduction

to compressive sampling," *IEEE signal processing magazine*, vol. 25, no. 2, pp. 21–30, 2008.

[5] Yonina C Eldar and Gitta Kutyniok, *Compressed sensing: theory and applications*, Cambridge University Press, 2012.

[6] Jarvis Haupt, Waheed U Bajwa, Michael Rabbat, and Robert D Nowak, "Compressed sensing for networked data," *Signal Processing Magazine IEEE*, vol. 25, no. 2, pp. 92–101, 2008.

[7] Michael Lustig, David Donoho, and John M Dpauly, "Sparse mri: The application of compressed sensing for rapid mr imaging," *Journal of the Chemical Fertilizer Industry*, vol. 58, no. 6, pp. 1182, 2007.

[8] Marco F Duarte, Mark A Davenport, Dharmpal Takhar, Jason N Laska, Ting Sun, Kevin F Kelly, and Richard G Baraniuk, "Single-pixel imaging via compressive sampling," pp. 83–91, 2008.

[9] Junhong Lin, Song Li, and Yi Shen, "New bounds for restricted isometry constants with coherent tight frames," *IEEE Transactions on Signal Processing*, vol. 61, no. 3, pp. 611–621, 2013.

[10] Dustin G Mixon, "Explicit matrices with the restricted isometry property: Breaking the square-root bottleneck," pp. 389–417, 2015.

[11] Marco Chiani, Ahmed Elzanaty, and Andrea Giorgetti, "Analysis of the restricted isometry property for gaussian random matrices," pp. 1–6, 2015.

[12] Mário AT Figueiredo, Robert D Nowak, and Stephen J Wright, "Gradient projection for sparse reconstruction: Application to compressed sensing and other inverse problems," *IEEE Journal of selected topics in signal processing*, vol. 1, no. 4, pp. 586–597, 2007.

[13] Thomas Blumensath and Mike E Davies, "Iterative hard thresholding for compressed sensing," *Applied and computational harmonic analysis*, vol. 27, no. 3, pp. 265–274, 2009.

[14] David L Donoho, Arian Maleki, and Andrea Montanari, "Message-passing algorithms for compressed sensing," *Proceedings of the National Academy of Sciences*, vol. 106, no. 45, pp. 18914–18919, 2009.

[15] T Tony Cai, Guangwu Xu, and Jun Zhang, "On recovery of sparse signals via ℓ_1 minimization," *IEEE Transactions on Information Theory*, vol. 55, no. 7, pp. 3388–3397, 2009.

[16] Tong Zhang, "Sparse recovery with orthogonal matching pursuit under rip," *IEEE Transactions on Information Theory*, vol. 57, no. 9, pp. 6215–6221, 2011.

[17] Simon Foucart, "A note on guaranteed sparse recovery via ℓ_1-minimization," *Applied and Computational Harmonic Analysis*, vol. 29, no. 1, pp. 97–103, 2010.

[18] Boris S Kashin and Vladimir N Temlyakov, "A remark on compressed sensing," *Mathematical notes*, vol. 82, no. 5-6, pp. 748–755, 2007.

[19] Gen Li, Jingkai Yan, and Yuantao Gu, *On the Lower and Upper Bounds of the RIP Constant of Gaussian Random Matrices*, preparing for submission, available at http://gu.ee.tsinghua.edu.cn/publications, 2018.

[20] Richard G Baraniuk, Mark A Davenport, Ronald Devore, and Michael B Wakin, "A simple proof of the restricted isometry property for random matrices," *Constructive Approximation*, vol. 28, no. 3, pp. 253–263, 2008.

[21] Andreas M Tillmann and Marc E Pfetsch, "The computational complexity of the restricted isometry property, the nullspace property, and related concepts in compressed sensing," *IEEE Transactions on Information Theory*, vol. 60, no. 2, pp. 1248–1259, 2014.