# ON THE ACCURACY LIMIT OF TIME-DELAY ESTIMATION WITH A BAND-LIMITED SIGNAL

*P. Das*[†⋆], *J. Vilà-Valls*[†], *E. Chaumette*[†], *F. Vincent*[†], *L. Davain*[⋆], *S. Bonnabel*[‡]

[†] University of Toulouse/Isae-Supaero, Toulouse, France ([jordi.vila-valls, eric.chaumette, francois.vincent]@isae.fr)

[⋆] Safran Electronics & Defense, Eragny, France ([priyanka.das, loic.davain]@safrangroup.com)

[‡] MINES ParisTech, PSL Research University, Centre for robotics, Paris, France (silvere.bonnabel@mines-paristech.fr)

## ABSTRACT

The derivation of tight estimation lower bounds is a key player to design and assess the performance of new estimators. Considering a generic band-limited signal formulation and constant transmitter to receiver propagation delay, we propose a novel compact closed-form expression of the Cramér-Rao bound for time-delay estimation. This new formulation, especially easy to use, allows to derive the best (lowest) Cramér-Rao bound for a band-limited signal of given length and energy, which provides an estimation performance loss metric. These results are illustrated with two representative band-limited signals.

***Index Terms***— Time-delay estimation, Band-limited signals, Mean squared error, Cramér-Rao bound.

## 1. INTRODUCTION

Time-delay estimation has been a research topic of significant practical importance in many fields (radar, sonar, ultrasonics, communications, navigation, ...) [1–5]. It is often a first stage that feeds into subsequent processing blocks for identifying, localizing, and tracking radiating sources [6]. Ranging from frequentist approaches to the most advanced Bayesian techniques, several estimators exist to perform such task [7–13]. When designing and assessing new time-delay estimation techniques, it is fundamental to know the optimal estimation performance. The deterministic Cramér-Rao Bounds (CRB) [14, 15] has been shown to give accurate estimation lower bounds on the mean squared error (MSE) in estimating deterministic parameters under certain conditions (i.e., in the high signal-to-noise (SNR) regime) [16]. Even if several time-delay estimation CRB expressions have been already derived for different applications [17–21], an easy to use compact CRB for band-limited signals is not available, whereas a compact CRB for infinite bandwidth signals has already been given in [22].

As a contribution to this research field, assuming a constant transmitter to receiver propagation delay, we recast the well known CRB for time-delay estimation [17,23] into a novel (to the best of our knowledge) compact closed-form expression fitted for generic band-limited transmitted signal. The first merit of this new formulation is to be especially easy to use since it involves only matrix computations. Its second merit is to allow to derive the best (lowest) CRB for a band-limited signal of given length and energy, which is new as well. It is noteworthy that this accuracy limit of time-delay estimation provides an estimation performance loss metric which could be used for an optimal signal design. The proposed new results are illustrated with two representative band-limited signals.

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## 2. SIGNAL MODEL

As an example of time-delay estimation problem, let us consider the line-of-sight transmission of a band-limited signal $c(t)$ with bandwidth $B$, from a transmitter T at position $\mathbf{p}_T(t)$ to a receiver R at position $\mathbf{p}_R(t)$, expressed both in time and frequency as

$$c(t) = \sum_{n=-N_1'}^{N_2'} c\left(\frac{n}{B}\right) \operatorname{sinc}\left(\pi B \left(t - \frac{n}{B}\right)\right), \tag{1a}$$

$$c(f) = \frac{1}{B} \sum_{n=-N_1'}^{N_2'} c\left(\frac{n}{B}\right) e^{-j2\pi n \frac{f}{B}}, \ \frac{-B}{2} \leq f \leq \frac{B}{2}. \tag{1b}$$

Considering that the signal is transmitted over a carrier with frequency $f_c$ (wavelength $\lambda_c = c/f_c$), the complex analytic signal at the output of the receiver's antenna can be written as, $x_A(t) = \alpha_R c_R(t) + n_A(t)$, with $n_A(t)$ a zero-mean white complex circular Gaussian noise, $\alpha_R = \beta \frac{\mathbf{f}_R(f_c, -\mathbf{u}(t)) \cdot \mathbf{f}_T(f_c, \mathbf{u}(t))}{\|\mathbf{p}_{TR}(t)\|}$, $\beta = \frac{\lambda_c}{4\pi} \sqrt{P_T} \sqrt{g_T(f_c, \mathbf{u}(t))} \sqrt{g_R(f_c, -\mathbf{u}(t))}$, $P_T$ the transmitted signal power, $\{g_T, g_R, \mathbf{f}_T, \mathbf{f}_R\}$ the transmitter/receiver antenna gains and polarization vectors, $\mathbf{u}(t) = \mathbf{p}_{TR}(t) / \|\mathbf{p}_{TR}(t)\|$, $\mathbf{p}_{TR}(t)$ the radial distance between T and R [24, 25]. If the transmitter to receiver distance is constant (constant propagation delay), i.e. $\|\mathbf{p}_{TR}(t)\| \triangleq \|\mathbf{p}_R(t) - \mathbf{p}_T(t - \tau(t))\| = c\tau$, then:

$$c_R(t) = e^{-j2\pi f_c \tau} e^{j2\pi f_c t} c(t - \tau), \tag{2}$$

and the baseband output of the receiver's Hilbert filter is

$$x(t) = x_A(t) e^{-j2\pi f_c t} = \alpha c(t - \tau) + n(t), \tag{3}$$

with $f \in \left[-\frac{F_s}{2}, \frac{F_s}{2}\right]$, $F_s \geq B$ the Hilbert filter bandwidth, $n(t)$ a complex white circular Gaussian noise within this bandwidth with unknown variance $\sigma_n^2$, and $\alpha = \alpha_R e^{-j2\pi f_c \tau}$. The discrete vector signal model is build from $N = N_1 + N_2 + 1$ ($N_1/F_s \gg N_1'/B$, $N_2/F_s \gg N_2' B$) samples at $T_s = 1/F_s$,

$$\mathbf{x} = \alpha \mathbf{c}(\tau) + \mathbf{n}, \tag{4}$$

with signal samples $\mathbf{x} = (x(-N_1 T_s), \ldots, x(N_2 T_s))^\top$, noise samples $\mathbf{n} = (n(-N_1 T_s), \ldots, n(N_2 T_s))^\top$, and code samples $\mathbf{c}(\tau) = (c(-N_1 T_s - \tau), \ldots, c(N_2 T_s - \tau))^\top$. Since $\{g_T, g_R, \mathbf{f}_T, \mathbf{f}_R\}$ are generally not perfectly known, classically $\alpha_R$, and hence $\alpha$, is assumed to be an unknown complex parameter as well [1–3, 25, 26].

Thus, the unknown deterministic parameters [27] can be gathered in vector $\underline{\boldsymbol{\epsilon}} = \left(\sigma_n^2, \tau, \alpha, \alpha^*\right)^\top$, where $\alpha^*$ is the complex conjugate of $\alpha$. Note that the same signal model (4) can be obtained if instead of line-of-sight transmission, one considers transmission via diffraction (scatterer), reflexion (reflector) or combination of the three (multi-paths) for static scenarios [2, 25].

## 3. BACKGROUND

Let $S = span(\mathbf{A})$, with $\mathbf{A}$ a matrix, be the linear span of the set of its column vectors, $\mathbf{\Pi_A} = \mathbf{A}\left(\mathbf{A}^H\mathbf{A}\right)^{-1}\mathbf{A}^H$ the orthogonal projection over $S$, and $\mathbf{\Pi}_\mathbf{A}^\perp = \mathbf{I} - \mathbf{\Pi_A}$. The CRB is given by the inverse of the Fisher information (FI), $\mathrm{CRB}_{\tau|\underline{\boldsymbol{\epsilon}}}\left(\underline{\boldsymbol{\epsilon}}^0\right) = 1/\mathrm{F}_{\tau|\underline{\boldsymbol{\epsilon}}}\left(\underline{\boldsymbol{\epsilon}}^0\right)$, with $\underline{\boldsymbol{\epsilon}}^0$ a selected value of $\underline{\boldsymbol{\epsilon}}$. If we define the function

$$\Phi(\tau) = \frac{\partial \mathbf{c}(\tau)}{\partial \tau}^H \mathbf{\Pi}_{\mathbf{c}(\tau)}^\perp \frac{\partial \mathbf{c}(\tau)}{\partial \tau} = \left\| \frac{\partial \mathbf{c}(\tau)}{\partial \tau} \right\|^2 - \frac{\left| \mathbf{c}(\tau)^H \frac{\partial \mathbf{c}(\tau)}{\partial \tau} \right|^2}{\|\mathbf{c}(\tau)\|^2} \tag{5}$$

the FI associated to the problem of interest is given by [23, 26]

$$F_{\tau|\underline{\boldsymbol{\epsilon}}}(\underline{\boldsymbol{\epsilon}}) = \frac{2|\alpha|^2 \Re\{\Phi(\tau)\}}{\sigma_\mathbf{n}^2} = \frac{|\alpha|^2 \|\mathbf{c}(\tau)\|^2}{\sigma_\mathbf{n}^2} \left( \frac{2\Re\{\Phi(\tau)\}}{\|\mathbf{c}(\tau)\|^2} \right). \tag{6}$$

Since $c(t)$ is a band-limited signal, by virtue of the Nyquist-Shannon theorem we have that [17]

$$\lim_{\min(N_1,N_2)\to\infty} \Re\{\Phi(\tau)\} = F_s \left( w_2 - \frac{|w_3|^2}{w_1} \right), \tag{7a}$$

$$w_1 = \int_{-\infty}^{+\infty} c(t)\, c(t)^* \, dt, \tag{7b}$$

$$w_2 = \int_{-\infty}^{+\infty} c^{(1)}(t)\, c^{(1)}(t)^* \, dt, \tag{7c}$$

$$w_3 = \int_{-\infty}^{+\infty} c^{(1)}(t)\, c(t)^* \, dt, \tag{7d}$$

where $w_1 = \mathbb{E}$ is the energy of the signal, and $c^{(1)}(t) = \frac{dc(t)}{dt}$. Thus (6) can be recast as:

$$F_{\tau|\underline{\boldsymbol{\epsilon}}}(\underline{\boldsymbol{\epsilon}}) = \frac{2|\alpha|^2}{\left(\frac{\sigma_\mathbf{n}^2}{F_s}\right)} w_1 \left( \frac{w_2}{w_1} - \left| \frac{w_3}{w_1} \right|^2 \right) = 2\frac{|\alpha|^2 \mathbb{E}}{\left(\frac{\sigma_\mathbf{n}^2}{F_s}\right)} \left(4\pi^2 \mathbb{B}^2\right) \tag{8a}$$

where $\mathbb{B}$ is the equivalent bandwidth of $c(t)$, defined as the square-root of the variance of its spectrum where the probability density function $p(f)$ is defined as follows [28, §2.8.5]:

$$\mathbb{B} = \sqrt{Var(f)},\; p(f) = \frac{|c(f)|^2}{\int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} |c(f)|^2 \, df} = \frac{|c(f)|^2}{\mathbb{E}}. \tag{8b}$$

It is then easy to check that $\mathbb{B} \leq \frac{F_s}{2}$, leading to:

$$F_{\tau|\underline{\boldsymbol{\epsilon}}}(\underline{\boldsymbol{\epsilon}}) \leq 2\frac{|\alpha|^2 \mathbb{E}}{\left(\frac{\sigma_\mathbf{n}^2}{F_s}\right)} F_s^2 \pi^2 \tag{8c}$$

### 3.1. Maximum Likelihood Estimation and Ambiguity Function

Considering the signal model (4), the maximum likelihood (ML) time-delay estimate is defined as [26]

$$\widehat{\tau} = \arg\min_\tau \left\{ \mathbf{x}^H \mathbf{\Pi}_{\mathbf{c}(\tau)}^\perp \mathbf{x} \right\} = \arg\max_\tau \left\{ \frac{\left| \mathbf{c}(\tau)^H \mathbf{x} \right|^2}{\mathbf{c}(\tau)^H \mathbf{c}(\tau)} \right\} \tag{9}$$

$$\underset{\min(N_1,N_2)\to\infty}{=} \arg\max_\tau \left\{ \frac{\left| \int_{-\infty}^{+\infty} c(t-\tau)^* x(t)\, dt \right|^2}{\int_{-\infty}^{+\infty} |c(t)|^2 \, dt} \right\}, \tag{10}$$

and the maximum SNR at the output of the ML matched filter is:

$$SNR_{out} = \frac{\left| \int_{-\infty}^{+\infty} c(t-\tau)^* \alpha c(t-\tau)\, dt \right|^2}{E\left[ \left| \int_{-\infty}^{+\infty} c(t-\tau)^* n(t)\, dt \right|^2 \right]} = \frac{|\alpha|^2 \left( \int_{-\infty}^{+\infty} |c(t)|^2 \, dt \right)^2}{\left(\frac{\sigma_\mathbf{n}^2}{F_s}\right) \int_{-\frac{B}{2}}^{\frac{B}{2}} |c(f)|^2 \, df} = \frac{|\alpha|^2 \mathbb{E}}{\left(\frac{\sigma_\mathbf{n}^2}{F_s}\right)}. \tag{11}$$

Last, the corresponding ambiguity function is given by [25]

$$\Xi\left(\tau; \tau^0\right) = \left| \frac{\mathbf{c}(\tau)^H \mathbf{c}\left(\tau^0\right)}{\|\mathbf{c}(\tau)\| \left\|\mathbf{c}\left(\tau^0\right)\right\|} \right|^2 \tag{12}$$

which can be approximated by its 2nd order Taylor expansion

$$\Xi\left(\tau^0 + d\tau; \tau^0\right) \simeq 1 - \frac{1}{2} \left( \frac{2\Re\left\{\Phi\left(\tau^0\right)\right\}}{\left\|\mathbf{c}\left(\tau^0\right)\right\|^2} \right) d\tau^2, \tag{13}$$

where the second term is directly related to (6).

## 4. ACCURACY LIMIT OF TIME-DELAY ESTIMATION WITH A BAND-LIMITED SIGNAL

From a practical point of view, it may be interesting to obtain a CRB expression in terms of the $SNR_{out}$, which is typically used to characterize the receiver operation point. Thus (8a) can be recast as:

$$F_{\tau|\underline{\boldsymbol{\epsilon}}}(\underline{\boldsymbol{\epsilon}}) = 2SNR_{out}\left(4\pi^2 \mathbb{B}^2\right) \leq 2SNR_{out} F_s^2 \pi^2. \tag{14}$$

So far the only known FI upper bound defining the accuracy limit of time-delay estimation with a band-limited signal was given by (8c), as explictly recalled in (14). However (8c) does not take into account the length of the support of the time-series, e.g. $\{c(nT_s)\}_{n\in\mathbb{Z}}$, associated to a band-limited signal, e.g. $c(t)$ (3). Therefore, from a system design point of view, it may be desirable to obtain a tighter FI upper bound depending on the length of the band-limited signal[1].

[1]In following, the length of the support of the time-series associated to a band-limited signal is simply referred to as the length of a band-limited signal.

Indeed, in many transmission scenarios, especially in communications, only part of the samples transmitted is known, the so-called "training sequence" [3, 4], and are used for time-delay estimation (a.k.a synchronization in that context). In order to optimize the data transfer rate for a given frequency band $(F_s)$, the length of the "training sequence" must be as small as possible while satisfying a given time-delay estimation accuracy. In most monostatic radar [25, 29], the radar blind range corresponds to the situation in which a radar transmitter is on and hence the receiver must be off, so that the radar transmitted signal does not saturate, i.e., does not blind, its own receiver. Similarly, in order to reduce the blind range, the length of the radar pulse must be as small as possible while satisfying a given time-delay estimation accuracy.
To obtain a tighter FI upper bound depending on the length of the band-limited signal, we introduce a novel (to the best of our knowledge) closed-form of the time-delay FI (14) depending only on the samples of a band-limited signal.

### 4.1. Closed-form Time-delay CRB

The goal is to obtain closed-form expressions of (7b-7d) depending only on the samples of a band-limited signal, which derivations are detailed in the sequel. Let $\mathbf{c} \triangleq \mathbf{c}(0)$; then:

$$w_1 = \int_{-\infty}^{+\infty} c(t) c(t)^* dt = \mathbf{c}^H \mathbf{c} T_s = \frac{1}{F_s} \mathbf{c}^H \mathbf{c} \tag{15}$$

$$\begin{aligned} w_2 &= \int_{-\infty}^{+\infty} \left| c^{(1)}(t) \right|^2 dt = \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left| (j2\pi f) c(f) \right|^2 df \\ &= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} (2\pi f)^2 \left| \frac{1}{F_s} \sum_{-N_1}^{N_2} c(nT_s) e^{-j2\pi f n T_s} \right|^2 df \\ &= F_s \int_{-\frac{1}{2}}^{\frac{1}{2}} (2\pi f)^2 \left| \mathbf{c}^T \boldsymbol{\nu}^*(f) \right|^2 df = F_s \mathbf{c}^H \mathbf{V} \mathbf{c}, \end{aligned} \tag{16}$$

where $\boldsymbol{\nu}(f) = \left( e^{j2\pi f(-N_1)}, \ldots, e^{j2\pi f(0)}, \ldots, e^{j2\pi f(N_2)} \right)^\top$ and $\mathbf{V} = 4\pi^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f^2 \boldsymbol{\nu}(f) \boldsymbol{\nu}(f)^H df$. The diagonal elements of $\mathbf{V}$, i.e. $V_{n,n'}$ $(n' = n)$, are given by:

$$4\pi^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f^2 df = 4\pi^2 \left[ \frac{f^3}{3} \right]_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{\pi^2}{3},$$

and the off-diagonal elements, i.e. $V_{n,n'}$ $(n' \neq n)$, are given by:

$$4\pi^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f^2 e^{j2\pi(n-n')f} df = (-1)^{|n-n'|} \frac{2}{(n-n')^2}.$$

Moreover, since $w_2 > 0$ if $\mathbf{c} \neq \mathbf{0}$, $\mathbf{V}$ is a symmetric positive definite real-valued matrix. Following a similar approach as above, we have

$$\begin{aligned} w_3 &= \int_{-\infty}^{+\infty} c^{(1)}(t) c(t)^* dt = \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} (j2\pi f) \left| c(f) \right|^2 df \\ &= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} (j2\pi f) \left| \frac{1}{F_s} \sum_{-N_1}^{N_2} c(nT_s) e^{-j2\pi f n T_s} \right|^2 df \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} (j2\pi f) \left| \mathbf{c}^T \boldsymbol{\nu}^*(f) \right|^2 df = \mathbf{c}^H \boldsymbol{\Lambda} \mathbf{c}, \end{aligned} \tag{17}$$

where $\boldsymbol{\Lambda} = j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f \boldsymbol{\nu}(f) \boldsymbol{\nu}(f)^H df$. The diagonal elements of $\boldsymbol{\Lambda}$, i.e. $\Lambda_{n,n'}$ $(n' = n)$, are given by:

$$j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f df = j2\pi \left[ \frac{f^2}{2} \right]_{-\frac{1}{2}}^{\frac{1}{2}} = 0,$$

and the off-diagonal elements, i.e. $\Lambda_{n,n'}$ $(n' \neq n)$, are given by:

$$j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f e^{j2\pi(n-n')f} df = \frac{(-1)^{|n-n'|}}{(n-n')}.$$

Finally, another closed-form of (8a)(14) is:

$$\boxed{F_{\tau|\underline{\boldsymbol{\epsilon}}}(\underline{\boldsymbol{\epsilon}}) = 2SNR_{out} F_s^2 \left( \frac{\mathbf{c}^H \mathbf{V} \mathbf{c}}{\mathbf{c}^H \mathbf{c}} - \left| \frac{\mathbf{c}^H \boldsymbol{\Lambda} \mathbf{c}}{\mathbf{c}^H \mathbf{c}} \right|^2 \right).} \tag{18a}$$

Note that for real band-limited signals $c(t)$, (18a) reduces to

$$F_{\tau|\underline{\boldsymbol{\epsilon}}}(\underline{\boldsymbol{\epsilon}}) = 2SNR_{out} F_s^2 \left( \frac{\mathbf{c}^H \mathbf{V} \mathbf{c}}{\mathbf{c}^H \mathbf{c}} \right), \tag{18b}$$

since $w_3 = 0$. It is noteworthy that (18a-18b) is especially easy to use since it involves only matrix computations.

### 4.2. Optimal Band-limited Signal for Time-Delay Estimation

The novel analytic expression of the FI for time-delay estimation with a band-limited signal (18a) can be used to find the optimal signal with given length $N$ $(N = N_2 + N_1 + 1)$ and energy $\mathbb{E}$ (or $SNR_{out}$) which minimizes the CRB. Mathematically this reads,

$$\mathbf{c}^b = \arg\max_{\mathbf{c}} \left\{ \frac{\mathbf{c}^H \mathbf{V} \mathbf{c}}{\mathbf{c}^H \mathbf{c}} - \left| \frac{\mathbf{c}^H \boldsymbol{\Lambda} \mathbf{c}}{\mathbf{c}^H \mathbf{c}} \right|^2 \text{ s.t. } \mathbf{c}^H \mathbf{c} = F_s \mathbb{E} \right\}. \tag{19}$$

Let $\mathbf{U}\mathbf{D}\mathbf{U}^\top = \mathbf{V} \in \mathbb{R}^{N \times N}$ be the eigendecomposition of $\mathbf{V}$, where $D_{1,1} \geq D_{2,2} \geq \cdots \geq D_{N,N} > 0$ are the eigenvalues of $\mathbf{V}$ and the columun vectors of $\mathbf{U} = [\mathbf{u}_1\, \mathbf{u}_2 \ldots \mathbf{u}_N] \in \mathbb{R}^{N \times N}$ are the associated eginvectors. Then

$$\forall \mathbf{c} \in \mathbb{C}^N : \frac{\mathbf{c}^H \mathbf{V} \mathbf{c}}{\mathbf{c}^H \mathbf{c}} - \left| \frac{\mathbf{c}^H \boldsymbol{\Lambda} \mathbf{c}}{\mathbf{c}^H \mathbf{c}} \right|^2 \leq \frac{\mathbf{c}^H \mathbf{V} \mathbf{c}}{\mathbf{c}^H \mathbf{c}} \leq D_{1,1}. \tag{20}$$

As a consequence, the band-limited signal $\mathbf{c}^b \in \mathbb{C}^N$ which minimizes the CRB (i.e., maximizes the FI) must verify the following conditions: 1) $\frac{(\mathbf{c}^b)^H \mathbf{V} \mathbf{c}^b}{(\mathbf{c}^b)^H \mathbf{c}^b} = D_{1,1}$, 2) $(\mathbf{c}^b)^H \boldsymbol{\Lambda} \mathbf{c}^b = 0$, and 3) the energy constraint $(\mathbf{c}^b)^H \mathbf{c}^b = \mathbb{E}/T_s$.
The signal which verifies these three conditions is $\mathbf{c}^b \triangleq e^{j\phi} \sqrt{F_s \mathbb{E}} \mathbf{u}_1$:
○ $\mathbf{u}_1 \in \mathbb{R}^N \Rightarrow \mathbf{u}_1^H \boldsymbol{\Lambda} \mathbf{u}_1 = 0 \Rightarrow \left(\mathbf{c}^b\right)^H \boldsymbol{\Lambda} \mathbf{c}^b = 0$.
○ $\frac{\left(\mathbf{c}^b\right)^H \mathbf{V} \mathbf{c}^b}{\left(\mathbf{c}^b\right)^H \mathbf{c}^b} = \frac{\mathbf{u}_1^H \mathbf{V} \mathbf{u}_1}{\mathbf{u}_1^H \mathbf{u}_1} = D_{1,1}$ s.t. $\left(\mathbf{c}^b\right)^H \mathbf{c}^b = F_s \mathbb{E}$.
Thus, the maximum FI for time-delay estimation with given length $N$ and energy $\mathbb{E}$ (or $SNR_{out}$) is given by:

$$F_{\tau|\underline{\boldsymbol{\epsilon}}}(\underline{\boldsymbol{\epsilon}}) \leq \frac{2|\alpha|^2 \mathbb{E}}{\left( \frac{\sigma_{\mathbf{n}}^2}{F_s} \right)} F_s^2 D_{1,1} = 2SNR_{out} F_s^2 D_{1,1}. \tag{21}$$

The variation of $D_{1,1}$ as a function of the length of a band-limited signal is displayed in Fig. 1. One can notice that $\lim_{N \to \infty} D_{1,1} = \pi^2$,

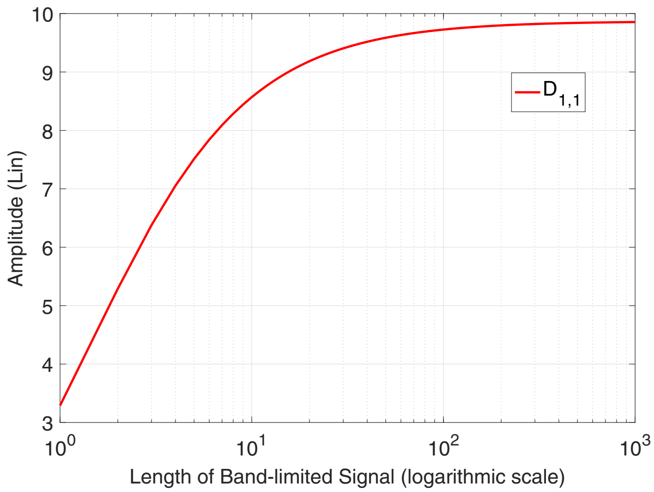


**Fig. 1**: $D_{1,1}$ versus the length of a band-limited signal

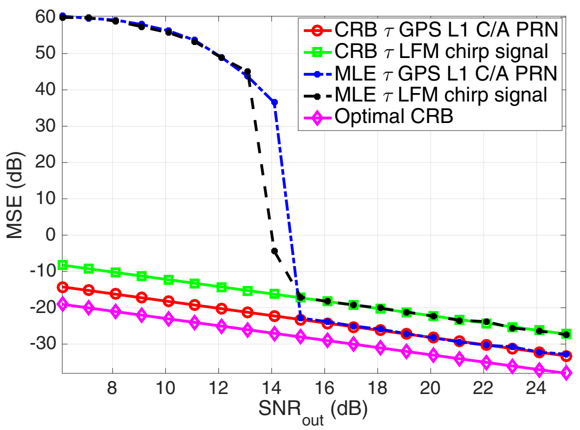


**Fig. 2**: CRB and MLE for two band-limited signals, 1) a GPS L1 C/A PRN code and 2) a LFM chirp signal.

which means that for a given energy $\mathbb{E}$ the highest FI (8c), and thus the lowest CRB, is obtained when the band-limited signal length becomes large. Moreover, Fig. 1 allows to quantify how large $N$ must be; for instance if $N \geq 14$ then $D_{1,1} \geq 0.9\pi^2$, which means that for $N \geq 14$ the accuracy limit of time-delay estimation with a band-limited signal of given energy $\mathbb{E}$ does not depend on its length. Last, for signal design purpose, (21) can be used to define a *performance loss* or *accuracy* index $\rho \in [0, 1]$ for a given band-limited signal:

$$\boxed{\rho = \frac{\text{CRB}_{opt}}{\text{CRB}_c} = \frac{\frac{\mathbf{c}^H \mathbf{V} \mathbf{c}}{\mathbf{c}^H \mathbf{c}} - \left| \frac{\mathbf{c}^H \mathbf{\Lambda} \mathbf{c}}{\mathbf{c}^H \mathbf{c}} \right|^2}{D_{1,1}}} \tag{22}$$

## 5. VALIDATION

We first assess the validity of the new closed-form CRB in (18a) with two representative band-limited signals, namely, i) a GPS L1 C/A PRN code with length 1023, and ii) a Linear Frequency Modulated (LFM) chirp signal, with a bandwidth equal to half the sampling frequency and length 1023. The CRB and the corresponding ML estimate (MLE) in (9), obtained from 1000 Monte Carlo runs and with $\alpha = ((1+j)/\sqrt{2})\sqrt{\text{SNR}_{in}}$, are shown in Fig. 2. For comparison we also plot the optimal CRB (8c) obtained with $\mathbf{c}^b = \sqrt{\mathbb{E}F_s}\mathbf{u}_1$. Since (4) belongs to the set of conditional signal models [30], the MLE converges to the CRB at high SNR [16]. Therefore Fig. 2 confirms the exactness of the proposed CRB (18a). If we compare these bounds to the optimal, we obtain that $\rho_{PRN} = 0.331$ and $\rho_{LFM} = 0.0833$, which represent a performance loss of 4.8 dB and 10.8 dB, respectively. To complete the demonstration of the exactness of (18a), we plot the ambiguity function for both signals in Fig. 3, which is compared with its 2nd order Taylor expansion (13). To compute the latter, $\Phi(\tau)$ has been updated according to (15-17). The perfect match of the 2nd order Taylor expansion is also a proof of the exactness of (18a).

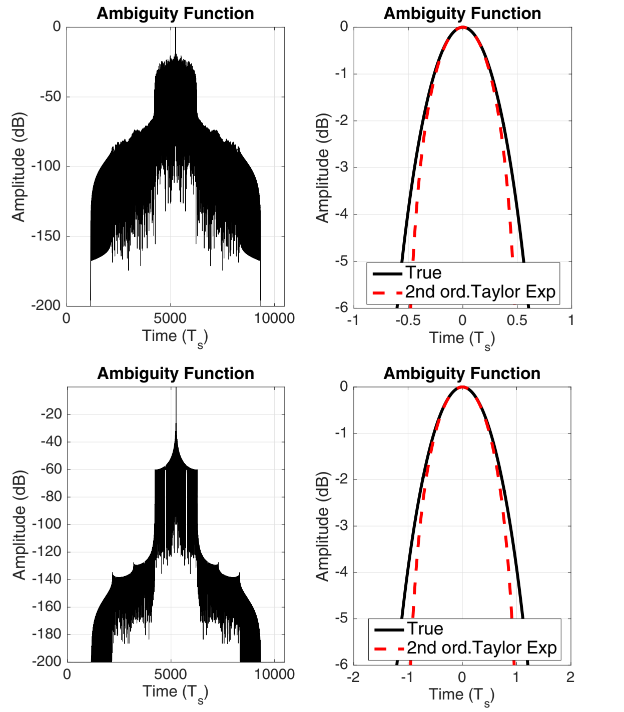


**Fig. 3**: Ambiguity function and its 2nd order Taylor expansion for a GPS code (top) and a LFM chirp signal (bottom).

## 6. CONCLUSION

In this contribution, we first derived a novel CRB closed-form expression for the time-delay estimation, considering a generic band-limited transmitted signal and constant transmitter to receiver propagation delay. This new formulation is especially easy to use and allows to derive the best (lowest) CRB for a signal of given length and energy, which provides an estimation performance loss metric. This estimation performance loss metric could be used for an optimal signal design, when optimality is expressed in terms of estimation accuracy and other features such as delay matched filter side lobes level [29] and/or peak-to-average power ratio [31].

## 7. REFERENCES


[1] H. L. Van Trees, Detection, *Estimation, and modulation theory, Part III: Radar Sonar Signal Processing and Gaussian Signals in Noise*, John Wiley & Sons, 2001.

[2] J. Chen, Y. Huang Y., and J. Benesty, "Time delay estimation," in Audio Signal Processing for Next-Generation Multimedia Communication Systems, Y. Huang and J. Benesty, Eds., chapter 8, pp. 197–227. Springer, Boston, MA, USA, 2004.

[3] B. C. Levy, *Principles of Signal Detection and Parameter Estimation*, Springer, 2008.

[4] U. Mengali and A. N. D'Andrea, *Synchronization Techniques for Digital Receivers*, Plenum Press, New York, USA, 1997.

[5] C. Fernandez-Prades, L. Lo Presti, and E. Falletti, "Satellite radiolocalization from GPS to GNSS and beyond: novel technologies and applications for civil mass market," Proc. of the IEEE, vol. 99, no. 11, pp. 1882–1904, Nov. 2011

[6] J. Yan et al., "Review of range-based positioning algorithms," IEEE AES Magazine, vol. 28, no. 8, pp. 2-27, Aug. 2013

[7] C. H. Knapp and G. C. Carter, "The generalized correlation method for estimation of time delay," IEEE Trans. on ASSP, vol. 24, no. 4, pp. 320-327, Aug. 1976.

[8] A. H. Quazi, "An overview on the time delay estimate in active and passive system for target localization," IEEE Trans. on ASSP, vol. 29, no. 3, pp. 527-533, June 1981.

[9] D. Boudreau and P. Kabal, "Joint time-delay estimation and adaptive recursive least squares filtering," IEEE Trans. on SP, vol. 41, no. 2, pp. 592–601, Feb. 1993.

[10] H. C. So, "On time delay estimation using an FIR filter," Signal Processing, vol. 81, pp. 1777-1782, Aug. 2001.

[11] M. Chakraborty, H. C. So, and J. Zheng, "New adaptive algorithm for delay estimation of sinusoidal signals," IEEE Signal Processing Letters, vol. 14, no. 12, pp. 972–975, Dec. 2007.

[12] H. L. Van Trees and K. L. Bell, Eds., *Bayesian Bounds for Parameter Estimation and Nonlinear Filtering/Tracking*, Wiley/ IEEE Press, NY, USA, 2007.

[13] R. Tao, X. M. Li, and Y. Wang, "Time delay estimation of chirp signals in the fractional fourier domain," IEEE Trans. on SP, vol. 57, no. 3, pp. 2852-2855, Mar. 2009

[14] H. L. Van Trees, *Detection, estimation, and modulation theory: Part I*, Wiley, New York, USA, 1968.

[15] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*, Prentice-Hall, Englewood Cliffs, New Jersey, USA, 1993.

[16] A. Renaux, P. Forster, E. Chaumette, and P. Larzabal, "On the high-SNR conditional maximum-likelihood estimator full statistical characterization," IEEE Trans. on SP, vol. 54, no. 12, pp. 4840-4843, Dec. 2006.

[17] A. Dogandzic and A. Nehorai, "Cramér-Rao bounds for estimating range, velocity, and direction with an active array," IEEE Trans. on SP, vol. 49, no. 6, pp. 1122–1137, June 2001.

[18] N. Noels, H. Wymeersch, H. Steendam, and M. Moeneclaey, "True Cramér-Rao bound for timing recovery from a bandlimited linearly modulated waveform with unknown carrier phase and frequency," IEEE Trans. on Communications, vol. 52, no. 3, pp. 473-483, March 2004.

[19] B. M. Sadler and R. J. Kozick, "A Survey of Time Delay Estimation Performance Bounds", IEEE Workshop on Sensor Array and Multichannel Processing, 2006.

[20] P. Closas, C. Fernandez-Prades, and J. A. Fernandez-Rubio, "Cramér-rao bound analysis of positioning approaches in GNSS receivers," IEEE Trans. on Signal Processing, vol. 57, no. 10, pp. 3775-3786, Nov. 2009

[21] C. Enneking, M. Stein, M. Castañeda, F. Antreich, and J. A. Nossek, "Multi-satellite time-delay estimation for reliable high-resolution GNSS receivers," in Proc. of the IEEE/ION PLANS, Myrtle Beach, SC, USA, April 2012.

[22] A. Bartov and H. Messer, "Lower bound on the achievable DSP performance for localizing step-like continuous signals in noise," IEEE Trans. on SP, vol. 46, no. 8, pp. 2195-2201, Aug. 1998.

[23] S. F. Yau and Y. Bresler, "A compact Cramér-Rao bound expression for parametric estimation of superimposed signals," IEEE Trans. on SP, vol. 40, no. 5, pp. 1226–1230, May 1992.

[24] S. J. Orfanidis, *Electromagnetic Waves and Antennas*, Rutgers University.

[25] M. I. Skolnik, *Radar Handbook, 3rd edition*, McGraw-Hill, New York, USA,1990.

[26] B. Ottersten, M. Viberg, P. Stoica, and A. Nehorai, "Exact and large sample maximum likelihood techniques for parameter estimation and detection in array processing," in Radar Array Processing, S. Haykin, J. Litva, and T. J. Shepherd, Eds., chapter 4, pp. 99–151. Springer-Verlag, Heidelberg, 1993.

[27] T. Menni, E. Chaumette, P. Larzabal, and J.P. Barbot, "New results on deterministic Cramér-Rao bounds for real and complex parameters," IEEE Trans. on SP, vol. 60, no. 3, pp. 1032-1049, March 2012.

[28] P. Stoica and R.L. Moses, *Spectral Analysis of Signals*, Pearson, 2005

[29] N. Levanon and E. Mozeson, *Radar Signals*, J. Wiley & Sons (Interscience Div.) New York, 2004

[30] P. Stoica and A. Nehorai, "Performances study of conditional and unconditional direction of arrival estimation," IEEE Trans. on ASSP, vol. 38, no. 10, pp. 1783-1795, Oct. 1990.

[31] M.A. Sebt, A. Cheikhi, M.M. Nayebi, "Orthogonal frequency-division multiplexing radar signal design with optimised ambiguity function and low peak-to-average power ratio", IET Radar, Sonar and Navigation, vol. 3. No. 2, pp 122-132, 2009.