A NOVEL APPROXIMATE LLOYD-MAX QUANTIZER AND ITS ANALYSIS

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ABSTRACT

Several distributed real-time signal sensing/monitoring systems require quantization for efficient signal representation. These distributed sensors often have computational and energy limitations. Motivated by this concern, we propose a novel quantization scheme called Approximate Lloyd-Max (ALM) that is nearly-optimal. Assuming a continuous and finite support probability distribution of the source, we show that our ALM quantizer converges to the classical Lloyd-Max quantizer with increasing bitrate. Our ALM quantizer, which is recursive, converges exponentially fast with the number of iteration. We illustrate our results using simulations for the Beta(4,2) distribution on the source.

***Index Terms*—** Quantization (signal), Piecewise linear approximation, Convergence, Probability distribution

1. INTRODUCTION

Many modern systems monitor a large amount of real-time data from distributed sensors (such as pollution, weather). With the increasing scale of such sensor deployments (due to IoTs and mobile sensing [1, 2, 3, 4]) signal compression becomes essential for storage and communication [5]. Quantization will play a significant role in efficient signal representations for these monitoring data.

Scalar quantization of a signal with known probability distribution is studied in the well-known works of Lloyd and Max [6, 7]. The classical Lloyd-Max algorithm requires integral computations in the centroid (conditional mean) update step. In this work we introduce a nearly-optimal scalar quantization algorithm, named as Approximate Lloyd-Max (ALM), that bypasses these computationally complex operations. Similar to the Lloyd-Max, the ALM algorithm has a recursive nature. We show exponentially fast convergence of ALM to a limit near the point of convergence of the Lloyd-Max quantizer. Our algorithm uses simpler update rules governed by localized mean square error optimizations.

Related work: Mean Square Error (MSE) scalar quantization at fixed bitrate with a known data distribution was studied in independent works of Lloyd and Max [6, 7]. Sharma extended the Lloyd-Max method to a general class of (convex/semiconvex) distortion measures [8]. Vector data quantization was introduced by Linde, Buzo and Gray [9], resulting in the celebrated LBG algorithm. Extensions of vector quantization to predictive and variable rate universal quantizers were performed by Gray and Ziv respectively [10, 11, 12].

The convergence aspects of the Lloyd-Max algorithm are widely analyzed in literature. Convergence at exponential decay rate to a unique global minima, under convex cost function and a log-concave probability distribution has been shown [13]. Sabin and Gray proved the absolute convergence of the Lloyd algorithm and its empirical density consistency on training data [14]. Wu has shown the convergence of the Lloyd method I for continuous, positive density function defined over a finite interval using the idea of finite state machines [15]. Quantization based on training data finds applications in (adaptive) signal processing and machine learning. Some well-known data-centric quantizers include learning vector quantizer (LVQ) and K-means clustering [16, 17, 18, 19].

Our contributions differ in the following ways.

- ALM uses piecewise linear density approximations to simplify the Lloyd-Max level updates. The quantizer is nearly-optimal at high bitrates.
- ALM has exponentially fast convergence with the number of iterations. The convergence analysis hinges on insights from Perron-Frobenius theory [20, 21]

Remark: Due to space constraints the complete proofs of results are made available in the supplementary paper [22].

2. SOURCE AND QUANTIZER MODEL

We model the scalar data to be generated from a random continuous source having a known density function $f_X(x)$. Let $f_X(x)$ be positive, differentiable and (without loss of generality) supported on a finite interval $\mathcal{D} = [0, 1]$. In addition, the following *smoothness condition* is assumed to hold,

$$|f'_X(x)| \leq m \in (0, \infty) \quad \text{for all } x \in \mathcal{D}. \tag{1}$$

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This condition ensures that the slope of the density is bounded and hence the density is smoothly varying.

Let $\mathcal{Q}(.)$ be a scalar quantizer function defined on a domain set $\mathcal{D}$. The range set of $\mathcal{Q}(.)$ is a finite discrete set consisting of the quantization levels $\vec{q} = [q_1, q_2, \cdots, q_K]$. The size of the set, K indicates the bitrate used by the quantizer. We assume the order $q_1 < q_2 < \cdots < q_K$. The *quantization error* is a distortion measure that evaluates the performance of the quantizer. In this paper we choose the distortion criteria as the mean square error (MSE), given by

$$\mathcal{R}_{\mathcal{Q}}(f_X) := \mathbb{E}\left[(\mathcal{Q}(X) - X)^2\right]. \tag{2}$$

A quantizer, $\mathcal{Q}^*$ is called *(globally) optimal* if it results in the minimum MSE among all quantizers. That is, $\mathcal{Q}^* = \arg\min_{\mathcal{Q}(.)} R_{\mathcal{Q}}(f_X)$. The objective of this work is to provide an analytically and computationally feasible quantization scheme that is nearly optimal at high bitrates. This property of the quantizer is termed as *asymptotic near-optimality*. That is $\lim_{K\to\infty} |\mathcal{Q}^*(x) - \mathcal{Q}(x)| = 0$ for all $x \in \mathcal{D}$.

The scalar quantization problem has a K dimensional search space over $[0,1]^K$. An efficient algorithmic search for the optimal quantizer is characterised by the number of iterations required to converge. An algorithm is said to have an *exponential decay rate* if the levels generated at each iteration approaches the optimal levels with exponentially decaying error. The quantizer proposed in this work satisfies both near-optimality and exponential convergence for K large.

3. APPROXIMATE LLOYD-MAX

The Lloyd-Max (LM) algorithm is known to generate the globally optimal quantizer for the class of continuous and positive source distributions on a finite support [15]. We introduce the ALM quantizer that achieves (asymptotic) near optimality on the same distribution class (with additional smoothness condition). ALM provides a computational advantage over LM, by avoiding the integration operation in the centroid update step. Similar to LM algorithm, ALM can be implemented with parallel (concurrent) level updates. This section describes the ALM algorithm, which uses a piecewise linear approximation of the source density. The features of the algorithm such as asymptotic near optimality and exponential convergence are shown in Sec. 3.2 and Sec. 4.

For elucidating our quantization algorithm, we introduce two reference levels, $q_0 := 0$ and $q_{K'} := 1$ fixed at the endpoints of $\mathcal{D}$. In this paper we assume $K \geq 2$. We develop the ALM algorithm based on the MSE cost function minimization as described below.

3.1. ALM cost minimization and level updates

The LM quantizer minimizes the MSE cost function by alternate modifications of the quantization levels $\{q_i; 1 \leq i \leq K\}$ and the boundary set $\{b_j; 1 \leq j \leq K+1\}$. The MSE minimization can be performed by locally optimizing the MSE cost in the left and right decision neighborhood of each quantization level. Cost function in (2) can be decomposed as,

$$\begin{aligned}\mathcal{R}(\mathcal{Q})(f_X) &= \int_0^1 (\mathcal{Q}(x) - x)^2 f_X(x)\mathrm{d}x \\ &= \sum_{k=1}^{K} \int_{b_k}^{b_{k+1}} (q_k - x)^2 f_X(x)\mathrm{d}x \end{aligned} \tag{3}$$

The boundary set in the expression above corresponds to $b_{j+1} = \frac{q_j + q_{j+1}}{2}$ for $j = 1, 2, \cdots, K-1$, $b_1 := q_0$ and $b_{K+1} := q_{K'}$. The MSE cost in (3) is minimized by taking partial derivatives with respect to the level q_k. Using Leibniz rule the optimal levels are obtained as, [23]

$$0 = 2\int_{b_k}^{b_{k+1}} (q_k - x) f_X(x)\mathrm{d}x, \tag{4}$$

where $1 \leq k \leq K$. The solution of q_k from the equation above does not have a closed form expression. This issue is solved in classical LM algorithm by fixing the boundary b_k and b_{k+1} according to previous iterate of q_k, then followed by the centroid computation. In ALM algorithm we use an alternative approach. That is, we apply a piecewise linear approximation of the density function $f_X(x)$ between the neighboring quantization levels q_{k-1} and q_{k+1}, allowing the boundaries to depend on the (unknown) variable q_k.

We consider the first order approximation of the density function in the nearest neighbor interval of q_k; i.e.,

$$f_{\text{app}}(x) = m_k x + c_k, \quad \text{for } x \in [q_{k-1}, q_{k+1}] \tag{5}$$

where m_k and c_k corresponds to the slope and the intercept of the approximation. These parameters are determined using the end point conditions $f_{\text{app}}(q_{k-1}) = f_X(q_{k-1})$ and $f_{\text{app}}(q_{k+1}) = f_X(q_{k+1})$. The linear approximation simplifies (4) and a computable expression for optimal q_k is obtained. The solution of q_k are the roots of a cubic polynomial, $r(u) = r_0 + r_1 u + r_2 u^2 + r_3 u^3$. The real root in the interval $[q_{k-1}, q_{k+1}]$ is chosen as the optimal (existence of such a root is always ensured; see Appendix B [22]). The coefficients $\{r_0, r_1, r_2, r_3\}$ depend on m_k, c_k, q_{k-1} and q_{k+1}. For $2 \leq k \leq K-1$, the equation becomes quadratic as $r_3 = 0$ (See Table. 1). At the boundaries, i.e. $k = 1$ and $k = K$, the partition boundaries are asymmetric and hence $r_3 \neq 0$.

The ALM scheme is summarized in the following steps:

1. $\vec{q}^{\,(i=0)}$ is initialized uniformly in $[0,1]$
2. In iteration $i \geq 0$, $\vec{q}^{\,(i)}$ is partitioned into odd and even sets, $\mathcal{Q}_{\text{odd}} = \{q_1, q_3, \cdots\}$ and $\mathcal{Q}^c_{\text{even}} = \{q_2, q_4, \cdots\}$
3. $q_k \in \mathcal{Q}_{\text{odd}}$ is updated (*in parallel*) to the real root of $r(u) = 0$ in $[q_{k-1}, q_{k+1}]$ (see Table 1), using parameters m_k, c_k chosen according to (5).

Table 1. Coefficients of the polynomial equation $r_0 + r_1 u + r_2 u^2 = r(u) = 0$, for ALM level updates; $2 \leq k \leq K-1$

Coefficients	r_0	r_1	r_2
$2 \leq k \leq K-1$	$-\frac{m_k}{24}\left(q_{k+1}^3 - q_{k-1}^3\right) - \frac{c_k}{8}\left(q_{k+1}^2 - q_{k-1}^2\right)$	$\frac{c_k}{4}\left(q_{k+1} - q_{k-1}\right)$	$\frac{m_k}{8}\left(q_{k+1} - q_{k-1}\right)$

4. $q_k \in \mathcal{Q}_{\text{even}}$ is updated (*in parallel*) similar to step 3.
5. i is incremented and convergence (or stopping rule) is checked. Algorithm is terminated if true, else iteration jumps to step 2.

The key feature of the algorithm is its simplified implementation. At each iteration, the set of odd and even indices of $\vec{q}$ are concurrently updated. We term this update rule as *alternating between evens and odds*. The stopping criteria for the ALM algorithm can be set according to the convergence requirement (more details in Sec. 4). In practice, we run the algorithm for an iteration count larger than K.

3.2. Asymptotic Near Optimality

We show that the ALM scheme approaches the MSE of the LM (optimal) for $K \gg 1$. The near-optimality result ensures that ALM generates quantization levels that are asymptotically ($K \to \infty$) closer to the LM levels. For brevity we use the following notation. Let $\vec{q}^{\,*}$ be the optimal LM quantizer with respect to the true density, $f_X(x)$ and $\vec{q}_{\text{A}}^{\,*}$ be the ALM quantizer using $f_{\text{app}}(x)$. For ease of exposition we fix $K = 2$. The Taylor expansion of the $f_X(x)$ about $x = q_2$ can be represented as $f_X(x) = f_X(q_2) + f'_X(q_2)(q_2 - x) + O((x-q_2)^2)$. At optimal q_2^*, the true and approximate density are related as

$$f_X(q_2^*) = f_{\text{app}}(q_2^*) + O(\varepsilon_K), \tag{6}$$

where $\varepsilon_K = \max_{1 \leq k \leq K-1} |q_{k+1} - q_{k-1}|^2$.

Theorem 1 (*Asymptotic optimality of ALM*)**.** *The approximate solution of the ALM quantization (see Table. 1), q_{2A}^* converges to the true solution, q_2^* as $K \to \infty$. That is, there exists a $K \geq K_0$ such that $|q_{2A}^* - q_2^*| \leq \varepsilon$ for all $\varepsilon > 0$.*

Proof. Due to space constraints we provide only an outline of the proof (full version of the proof is available in Theorem 3.1 [22]). At iteration $i = 1$ of the ALM algorithm, let the boundaries be $b_2 = \frac{q_1^{(0)} + q_2^{(0)}}{2}$ and $b_3 = \frac{q_2^{(0)} + q_3^{(0)}}{2}$. Then the one step update using the optimality condition in (4), results in $q_2^{(1)}$ and $q_{2\text{A}}^{(1)}$ for LM and ALM respectively. Using the fact that the roots of the LM and ALM lies in $[q_1^{(0)}, q_3^{(0)}]$, we observe, $|q_{2\text{A}}^{(1)} - q_2^{(1)}| \leq \varepsilon_K$. Since adjacent levels q_1 and q_3 are also bounded from their optima by ε_K, by an inductive argument it follows that,

$$|q_{2\text{A}}^* - q_2^*| = \lim_{i \to \infty} |q_{2\text{A}}^{(i)} - q_2^{(i)}| \leq \varepsilon_K.$$

As $K \to \infty$, $\varepsilon_K \to 0$, and hence the result. $\square$

4. CONVERGENCE OF ALM ALGORITHM

4.1. Level shifts as linear updates

The optimal solution for the ALM iterative update of q_k is the solution of (4) in the interval $[q_{k-1}, q_{k+1}]$. The solution at i^{th} iteration can hence be expressed as a convex combination, $q_k^{(i+1)} = \theta_k^{(i)} q_{k-1}^{(i)} + (1 - \theta_k^{(i)}) q_{k+1}^{(i)}$, where $\theta_k^{(i)} \in [0, 1]$. The above update equation will aid in the convergence analysis of ALM. In vector form the alternating even-odd update rule is,

$$\vec{q}^{\,(i+1)} = P_{\text{odd}}^{(i)} P_{\text{even}}^{(i)} \vec{q}^{\,(i)} \quad \text{where } i = 0, 1, \ldots. \tag{7}$$

In the above equation, $P_{\text{even}}^{(i)}$ and $P_{\text{odd}}^{(i)}$ represent the even and odd index update rules, entries of which are determined by convex combination of the ALM update. For $K' = 4$,

$$P_{\text{odd}}^{(i)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \theta_1^{(i)} & 0 & 1 - \theta_1^{(i)} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \theta_3^{(i)} & 0 & 1 - \theta_3^{(i)} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$
$$P_{\text{even}}^{(i)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \theta_2^{(i)} & 0 & 1 - \theta_2^{(i)} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

P_{odd} and P_{even} are row stochastic with the location of zero entries satisfying (row) symmetry. Every iteration preserves the reference levels, q_0 and $q_{K'}$. Thus the quantizer iteration governed by (7)-(8), has the required structure for Perron Frobenius theory to apply [20, 21].

4.2. Insights from Uniformly Distributed Sources

Consider a uniformly distributed source in $[0, 1]$. LM and ALM gives the same result as the piecewise linear approximations of the density are exact. For illustration let $K' = 4$ and $\vec{q}^{\,(0)}$ be initialized randomly. Then, $\vec{q}^{\,(1)} = P_2 P_1 \vec{q}^{\,(0)}$, where $P_1 = P_{\text{even}}^{(0)}$ and $P_2 = P_{\text{odd}}^{(0)}$. We observe that $\theta_1^{(i)} = 2/3$, $\theta_2^{(i)} = 1/2$ and $\theta_3^{(i)} = 1/3$ (verifiable using (4)). As P_1 and P_2 are row-stochastic, the same property holds for $P_2 P_1$. We can show the existence of a fixed point, $\vec{q}^{\,*} = P_2 P_1 \vec{q}^{\,*}$ such that $\vec{q}^{\,*}$ is the limit point of the ALM. In Theorem. 2, we show the uniqueness of $\vec{q}^{\,*}$. Also we observe an exponential decay of $O\left(1/3^n\right)$ due to the dominant eigenvalue of $P_2 P_1$ being $1/3$. Using these insights we discuss the ALM convergence.

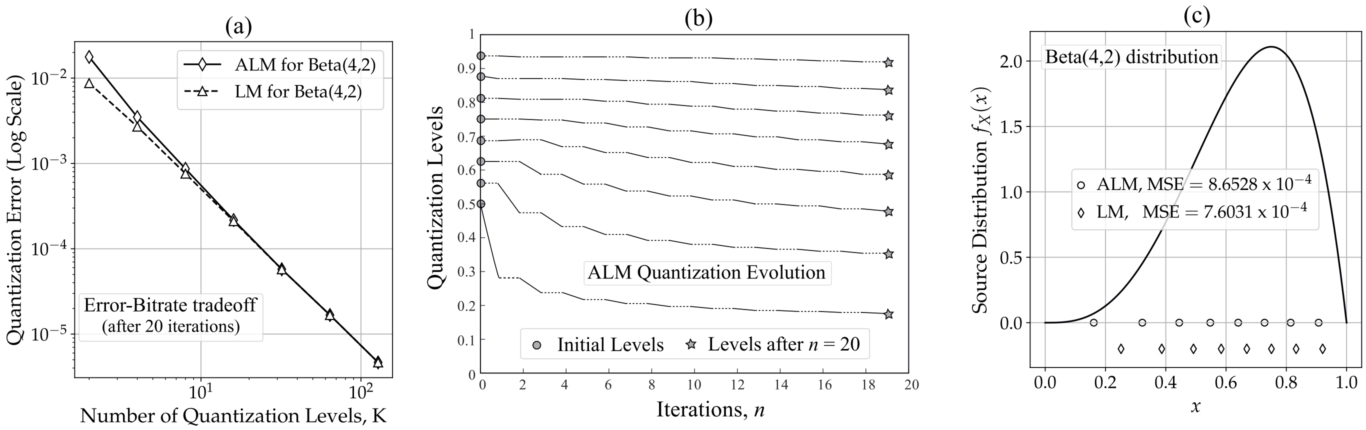

Fig. 1. (a) Error and bitrate tradeoff of ALM and LM. For $K > 8$, the MSE of the ALM is nearly same as LM scheme. (b) Quantizer evolution of ALM showing the alternating even-odd updates. $K = 8$ with uniform initialization in $[0.5, 1]$ has been used. (c) Quantization levels after $n = 100$ iterations. Levels are near optimal in regions of high probability density.

4.3. Convergence Analysis of ALM

The key idea of the convergence proof hinges on the row-stochastic nature of the product $P^{(i)} = P^{(i)}_{\text{odd}} P^{(i)}_{\text{even}}$, of the form

$$P^{(i)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \theta_1^{(i)} & 0 & \bar{\theta}_1^{(i)} & 0 & 0 \\ \theta_2^{(i)}\theta_1^{(i)} & 0 & \bar{\theta}_1^{(i)}\theta_2^{(i)} + \bar{\theta}_2^{(i)}\theta_3^{(i)} & 0 & \bar{\theta}_2^{(i)}\bar{\theta}_3^{(i)} \\ 0 & 0 & \theta_3^{(i)} & 0 & \bar{\theta}_3^{(i)} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (9)$$

where $\bar{\theta}_k^{(i)} = 1 - \theta_k^{(i)}$. From ALM update (7)-(8), after L iterations $\vec{q}^{\,(L)} = \prod_{i=1}^{L} P^{(i)} \vec{q}^{\,(0)}$. As $L \to \infty$, the odd columns of this (L term) product eventually goes to the zero vector. This is true since each $P^{(i)}$ is row-stochastic and the ALM update in (7) corresponds to a non-trivial convex combination. The smoothness assumption (1) in our source model, upper bounds the slopes of the piecewise linear approximation to m, which results in the parameter $0 < \theta_k^{(i)} < 1$. This fact will be used to state the main convergence result. The convergence analysis of ALM is similar in form to gossip algorithms and consensus models [24, 25]. However, our case differs as we get two fixed points as against one in the former.

Theorem 2. *The ALM iterations converge to a quantization vector, $\vec{q}^{\,*} = P^* \vec{q}^{\,(0)}$, where $P^* = \lim_{L\to\infty} \prod_{i=1}^{L} P^{(i)}$, and $\vec{q}^{\,*}$ is independent of the initialization $\vec{q}^{\,(0)}$.*

Proof. We note that the limiting matrix P^* converges to a matrix with all columns except first and last as zero vectors (see Proposition 2 [22]). The first and last columns, viz. $\vec{c}_1$ and $\vec{c}_{K+1}$, are non-zero vectors, as the transformation $P^{(i)}$, preserves the reference levels q_0 and q_{K+1}. P^* has the form $[\vec{c}_1\ \mathbf{0} \cdots \mathbf{0}\ \vec{c}_{K+1}]$. The all ones vector, $\mathbf{1}$ is an eigenvector corresponding to $\lambda = 1$. Then, $\vec{c}_1 + \vec{c}_{K+1} = \mathbf{1}$. That is the vector pair $\vec{c}_1$ and $\vec{c}_{K+1}$ are *order reversed*. Since rank of P^* is two, each of the vectors $\vec{c}_1$ and $\vec{c}_{K+1}$ are independent eigenvectors of $\lambda = 1$ (having geometric multiplicity of 2). Imposing the ordering constraint $0 := q_0 < \cdots < q_{K+1} := 1$, we can show that $\vec{c}_{K+1}$ corresponds to the unique global minimizer of the ALM algorithm. The iteration follow exponential rate of convergence, as every eigenvector component, decays exponentially at the rate of its eigenvalue (as $|\lambda| \leq 1$). The dominant eigenvalue, $\lambda_{(2)}$ decides the decay rate. □

5. SIMULATION RESULTS AND REMARKS

Simulation results account for the three aspects of the quantizer performance; viz. error-bitrate tradoff, convergence and accuracy. In Fig.1 (a), we compare the MSE for the ALM and LM quantizers. The quantization evolution diagram for Beta(4,2) distribution is shown in Fig.1 (b). In Fig.1 (c) the comparison of the optimal (LM) and near-optimal (ALM) levels are shown. The ALM levels compared to the LM are greater, perhaps due to of the skewness of the Beta(4,2) density considered here. Computational improvement of ALM is studied for $K = 8$. A saving of `3.4X` in terms of simulation time (in MATLAB) is observed, when the stopping criteria is chosen such that the computed MSE is within 1% of the optimal MSE (see Fig. 4(d) in [22] for details).

6. CONCLUSIONS

We introduced the ALM algorithm which is a novel approach to simplify the computations in Lloyd-Max quantization. The convergence analysis used here exploits the convex (linear) combination property of the level updates, which is a new contribution as far as we know. Extension to envelope constrained quantization [26], and its data-driven equivalents are envisaged as our future work.

7. REFERENCES

[1] A. Zanella, N. Bui, A. Castellani, L. Vangelista, and M. Zorzi, "Internet of things for smart cities," *IEEE Internet of Things Journal*, vol. 1, no. 1, pp. 22–32, Feb 2014.

[2] J. Unnikrishnan and M. Vetterli, "Sampling and reconstruction of spatial fields using mobile sensors," *IEEE Transactions on Signal Processing*, vol. 61, no. 9, pp. 2328–2340, May 2013.

[3] A. Kumar, P. Ishwar, and K. Ramchandran, "High-resolution distributed sampling of bandlimited fields with low-precision sensors," *IEEE Transactions on Information Theory*, vol. 57, no. 1, pp. 476–492, Jan 2011.

[4] C. Y. Chong and S. P. Kumar, "Sensor networks: evolution, opportunities, and challenges," *Proceedings of the IEEE*, vol. 91, no. 8, pp. 1247–1256, Aug 2003.

[5] A. Gersho and R. M. Gray, *Vector quantization and signal compression*, vol. 159, Springer Science & Business Media, 2012.

[6] S. Lloyd, "Least squares quantization in PCM," *IEEE Transactions on Information Theory*, vol. 28, no. 2, pp. 129–137, March 1982.

[7] J. Max, "Quantizing for minimum distortion," *IRE Transactions on Information Theory*, vol. 6, no. 1, pp. 7–12, March 1960.

[8] D. Sharma, "Design of absolutely optimal quantizers for a wide class of distortion measures," *IEEE Transactions on Information Theory*, vol. 24, no. 6, pp. 693–702, November 1978.

[9] Y. Linde, A. Buzo, and R. Gray, "An Algorithm for Vector Quantizer Design," *IEEE Transactions on Communications*, vol. 28, no. 1, pp. 84–95, Jan 1980.

[10] R. Gray and Y. Linde, "Vector quantizers and predictive quantizers for gauss-markov sources," *IEEE Transactions on Communications*, vol. 30, no. 2, pp. 381–389, Feb 1982.

[11] M. Effros, P. A. Chou, and R. M. Gray, "One-pass adaptive universal vector quantization," in *Proceedings of ICASSP '94. IEEE International Conference on Acoustics, Speech and Signal Processing*, April 1994, vol. v, pp. V/625–V/628 vol.5.

[12] J. Ziv, "On universal quantization," *IEEE Transactions on Information Theory*, vol. 31, no. 3, pp. 344–347, May 1985.

[13] J. Kieffer, "Exponential rate of convergence for Lloyd's method I," *IEEE Transactions on Information Theory*, vol. 28, no. 2, pp. 205–210, March 1982.

[14] M. Sabin and R. Gray, "Global convergence and empirical consistency of the generalized Lloyd algorithm," *IEEE Transactions on Information Theory*, vol. 32, no. 2, pp. 148–155, Mar 1986.

[15] X. Wu, "On convergence of Lloyd's method I," *IEEE Transactions on Information Theory*, vol. 38, no. 1, pp. 171–174, Jan 1992.

[16] T. Kohonen, "Learning vector quantization," in *Self-organizing maps*, pp. 175–189. Springer, 1995.

[17] D. Pollard, "Quantization and the method of k-means," *IEEE Transactions on Information Theory*, vol. 28, no. 2, pp. 199–205, March 1982.

[18] L. Bottou and Y. Bengio, "Convergence properties of the k-means algorithms," in *Advances in Neural Information Processing Systems 7*, G. Tesauro, D. S. Touretzky, and T. K. Leen, Eds., pp. 585–592. MIT Press, 1995.

[19] T. Kanungo, D. M. Mount, N. S. Netanyahu, C. D. Piatko, R. Silverman, and A. Y. Wu, "An efficient k-means clustering algorithm: analysis and implementation," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 24, no. 7, pp. 881–892, Jul 2002.

[20] S. U. Pillai, T. Suel, and Seunghun Cha, "The Perron-Frobenius theorem: some of its applications," *IEEE Signal Processing Magazine*, vol. 22, no. 2, pp. 62–75, March 2005.

[21] R. G. Gallager, *Stochastic processes*, Cambridge University Press, 2013.

[22] V. Anavangot and A. Kumar, "Novel near-optimal scalar quantizers with exponential decay rate and global convergence," *ArXiv Preprint*, 2018, `http://arxiv.org/abs/1810.12189`.

[23] M. H. Protter, B. Charles Jr, et al., *Intermediate calculus*, Springer Science & Business Media, 2012.

[24] D. Shah et al., "Gossip algorithms," *Foundations and Trends® in Networking*, vol. 3, no. 1, pp. 1–125, 2009.

[25] M. Huang, "Stochastic approximation for consensus: A new approach via ergodic backward products," *IEEE Transactions on Automatic Control*, vol. 57, no. 12, pp. 2994–3008, Dec 2012.

[26] G. Maheshwari and A. Kumar, "Optimal quantization of TV white space regions for a broadcast based geolocation database," in *2016 24th European Signal Processing Conference (EUSIPCO)*, Aug 2016, pp. 418–422.