ONE-BIT UNLIMITED SAMPLING

Olga Graf†*, Ayush Bhandari*‡ *and Felix Krahmer*†

††Dept. of Mathematics, Technical University of Munich, Germany.

‡Dept. of Electrical and Electronic Engineering, Imperial College London, UK.

Email: olga.graf@ma.tum.de • a.bhandari@imperial.ac.uk • felix.krahmer@tum.de

ABSTRACT

Conventional analog–to–digital converters (ADCs) are limited in dynamic range. If a signal exceeds some prefixed threshold, the ADC saturates and the resulting signal is clipped, thus becoming prone to aliasing artifacts. Recent developments in ADC design allow to overcome this limitation: using modulo operation, the so called self-reset ADCs fold amplitudes which exceed the dynamic range. A new (unlimited) sampling theory is currently being developed in the context of this novel class of ADCs. In this paper, we make a further step in this direction by coupling modulo sampling with one-bit $\Sigma\Delta$ quantization, or, in other words, consider *one-bit unlimited sampling*. We show that our scheme overcomes the dynamic range limitations of conventional one-bit quantizer, where no recovery guarantees are possible if the signal's dynamic range substantially exceeds the range of its one-bit output. We provide a constructive recovery algorithm for bandlimited signals from one-bit modulo samples complemented with a bound on the reconstruction error.

Index Terms— Analog-to-digital converters (ADC), sigma-delta, quantization, sampling theory, modulo samples.

1. INTRODUCTION

Recently, in [1], the authors introduced the **Unlimited Sensing Framework** which overcomes the dynamic range limitation that is a fundamental bottleneck common to all formats of digital data acquisition devices. The key idea underpinning the unlimited sensing framework is that a modulo operation *folds* the high dynamic range samples, beyond the recordable range, into low dynamic range modulo samples. While the motivation behind the use of such an operation is clear—to compress the ambient dynamic range—there are two questions that must be answered so that this abstract idea can be put into practice. The first question is: can modulo non-linearity be implemented in the conventional analog-to-digital converter (ADC) format? Furthermore, can a function or a signal be recovered from modulo wrapped measurements?

In [1], the authors make the link between modulo non-linearity and its implementation with a radically different form of ADCs, the *self-reset* ADCs, which reset the voltage before saturating/clipping. The main result in [1], the *unlimited sampling theorem* for bandlimited signals, proves that a simple correction to the usual Nyquist rate linked with Shannon's sampling theorem, allows for reconstruction of an arbitrarily high dynamic range signals from modulo folded low dynamic range samples. The sampling rate is independent of the ADC threshold and purely depends on the signal's bandwidth, and *not* on the modulo threshold. This is remarkable as signals with higher dynamic range would undergo many more folds for a fixed threshold. Consequently, the number of discontinuities increases, and analogous results in sampling theory would suggest that more folds or discontinuities imply a higher sampling rate. In [1], this difficulty is overcome by designing a recovery algorithm adapted to the dynamic range, which then comes with a sampling theorem that is independent of the ADC threshold.

Subsequently, the authors generalized this approach to the problem of recovering sparse signals from low-pass filtered, modulo measurements [2] and to the signal model of a sum-of-sinusoids [3]. The common feature in [2] and [3] is that the results are based on a *local reconstruction theorem*—how to recover a finite subset of samples given modulo samples? Understandably, in case of local reconstruction results, the sampling rate depends on the sparsity as well as the dynamic range of the signal.

Several follow up papers have discussed new variations linked with the unlimited sensing framework:

- In [4], Rudresh et al. present an interesting wavelet based reconstruction scheme for unlimited sampling strategy.
- In [5], Cucuringu and Tyagi discuss denoising of modulo samples. The authors devise an elegant optimization approach based on quadratically constrained quadratic program (QCQP).
- In [6], Musa and co-workers take the unlimited sampling architecture in the direction of compressed sensing and present a recovery approach based on generalized approximate message passing.
- In [7], Ordentlich and colleagues discuss a hardware design for electronic implementation of the modulo sensing strategy with the goal of minimizing the number of bits per sample.

In this paper, our goal is to take a step towards practical implementation of the unlimited sampling theorem, but in a different light; advancing along the lines of [1], we consider the case of **quantization**. More specifically, we consider analog-to-digital conversion based on Sigma-Delta or $\Sigma\Delta$ scheme. The $\Sigma\Delta$ scheme capitalizes on the fact that oversampling with fewer bits is cheaper to implement in hardware. This is also the distinct feature of the unlimited sensing architecture; modulo mapping amounts to recording lower significant bits and reconstruction is based on a constant factor oversampling criterion. Thus integrating $\Sigma\Delta$ with unlimited sampling is a natural extension to the theory in [1] and the advantages are two fold; firstly, in doing so we overcome the main drawback of $\Sigma\Delta$ scheme which can only handle signals with prefixed dynamic range and, secondly, it gives a conceptual guideline towards making the unlimited sensing architecture practically feasible.

$\Sigma\Delta$ has been known to circuit engineers since the 1963 pioneering work [8] of Inose and Yasuda; a rigorous mathematical study was initiated by Daubechies and DeVore in [9] in the early 2000's. The work [9] as well as a number of follow up papers proved that representing a signal by only a single bit per sample via $\Sigma\Delta$ still allows for accurate reconstruction when combined with substantial oversampling; this observation allows for circuits of very low complexity. The best known error decay rate for such one-bit $\Sigma\Delta$ schemes is

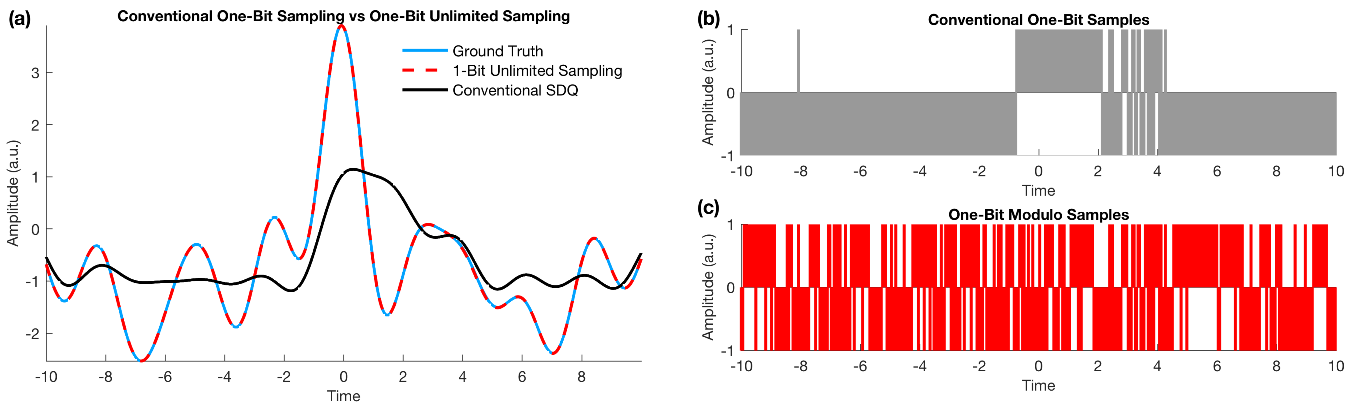

Fig. 1: Comparison between conventional one-bit sampling and one-bit unlimited sampling (both exploiting $\Sigma\Delta$ scheme). (a) Conventional method leads to reconstruction failure whenever the dynamic range of the signal exceeds the dynamic range of one-bit samples, while our method still allows for fair reconstruction. (b) Conventional one-bit samples exhibit saturation resulting from the fact that dynamic range exceeds [-1,1]. (c) Due to amplitude folding, one-bit modulo samples capture sufficiently more information about the signal than conventional one-bit samples.

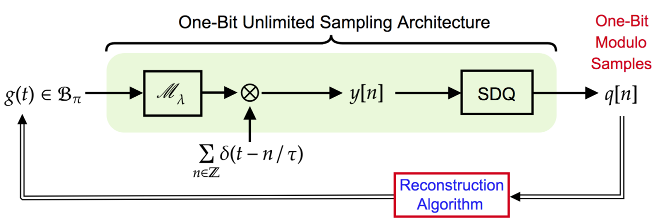

Fig. 2: System architecture for one-bit unlimited sampling.

exponential in the oversampling rate [10, 11], which is also known to be optimal [12, 13]. This accuracy of reconstruction is achieved by combining $\Sigma\Delta$ schemes of different orders. While in this paper we will focus on the classical and well-studied first order $\Sigma\Delta$ scheme, we expect that in follow-up works, our approach can also be generalized to higher orders. To the best of our knowledge, this is the first work to discuss approximation theory and recovery guarantees of quantization within the unlimited sensing framework.

Our contributions are the following:

1) We combine the advantages of modulo sampling and one-bit $\Sigma\Delta$ to obtain an ADC scheme that has low complexity due to coarseness of quantization and at the same time overcomes the dynamic range limitations of conventional one-bit $\Sigma\Delta$.

2) We provide a sufficiency condition for recovery of bandlimited signals from one-bit modulo samples, as well as an algorithm for recovery and a bound on the reconstruction error.

2. ONE-BIT QUANTIZATION FROM MODULO SAMPLES

Motivated by [1], we use the model in Fig. 2 to represent the self-reset ADC coupled with $\Sigma\Delta$ quantizer or $\Sigma\Delta$. We assume that the signal g belongs to the Paley-Wiener space $\mathrm{PW}_\Omega(\mathbb{R})$—the space of finite energy, bandlimited functions, that is, $g \in L^2(\mathbb{R})$ and $g \in \mathcal{B}_\Omega \Leftrightarrow \widehat{g}(\omega) = \mathbb{1}_{[-\Omega,\Omega]}(\omega)\widehat{g}(\omega)$ where $\widehat{g}(\omega) := \int g(t)e^{-j\omega t}dt$ denotes the Fourier transform and $\mathbb{1}_{\mathcal{D}}$ denotes the indicator function on domain $\mathcal{D}$. When a signal is not bandlimited, then it is pre-filtered with a low-pass kernel. Furthermore, we will normalize the bandwidth to π such that $g \in \mathcal{B}_\pi$.

We also assume that we have control over the superoscillation property of g. As described in [14], superoscillation is the phenomenon, that, for arbitrary times $\{t_i\}_{i=1}^N$ and amplitudes $\{a_i\}_{i=1}^N$, it is possible to find signal $f \in \mathrm{PW}_\Omega(\mathbb{R})$ with fixed Ω such that $f(t_i) = a_i$ for $i = 1, \ldots, N$, where N is arbitrarily large. In other words, local segments of such signal can oscillate at a frequency higher than 2Ω. This is not the case for many commonly used signals: images, sounds, electrical, etc [15]. However, superoscillating functions have found applications in quantum physics, metrology, antenna design, optics. If the signal exhibits superoscillating behaviour around the modulo threshold, it leads to a high number of ADC resets, making one-bit unlimited sampling setup difficult to analyze. Namely, our approach will require an increase in the sampling rate depending on the increase in oscillation. To quantify this, we define the superoscillation (SO) parameter for g as follows.

Definition 1 (SO Parameter)**.** *We say that $g \in \mathrm{PW}_\pi(\mathbb{R})$ has superoscillation (SO) parameter $c \in \mathbb{R}$ if for all $\mu, \alpha \in \mathbb{R}$ the function $g(t) - \mu$ has at most 2 zeros in the interval $[\alpha, \alpha + c]$.*

Finally, we assume that the upper bound on the infinity norm of g is known, $\beta_g \geqslant \|g\|_{L^\infty}$.

At first, the function g undergoes a non-linear amplitude folding

$$\mathscr{M}_\lambda : t \mapsto 2\lambda\left(\left[\!\left[\frac{t}{2\lambda} + \frac{1}{2}\right]\!\right] - \frac{1}{2}\right), \tag{1}$$

where $[\![t]\!] := t - \lfloor t \rfloor$ is the fractional part of t and the output of (1) is in the range $[-\lambda, \lambda]$. From now on, we will set $\lambda = 1$ in order to have a match with one-bit $\Sigma\Delta$ which requires amplitudes smaller than 1. After folding, $\mathscr{M}_1(g(t))$ is uniformly sampled:

$$y[n] := \mathscr{M}_1\left(g\left(\tfrac{n}{\tau}\right)\right) = \mathscr{M}_1\left(g[n]\right), \quad n \in \mathbb{Z}, \tag{2}$$

where $\tau \geqslant 1$ is the oversampling rate (for $g \in \mathcal{B}_\pi, \tau = 1$ implies critical sampling, that is, there is no oversampling).

In order to discretize the range, the signal is quantized via the first order, one-bit $\Sigma\Delta$ which runs the following iteration on modulo samples for $n \in \mathbb{Z}$:

$$\begin{cases} u[n] = u[n-1] + y[n] - q[n], \\ q[n] = \mathrm{sign}\left(u[n-1] + y[n]\right), \end{cases} \tag{3}$$

where $\{q[n]\}_{n\in\mathbb{Z}}$ are *one-bit modulo samples* and $\{u[n]\}_{n\in\mathbb{Z}}$ are the state variables with initialization $u_{\mathsf{init}} := 0$.

The **modulo decomposition property** (MDP) in [1] allows us to write $g(t) = \mathscr{M}_\lambda(g(t)) + \varepsilon_g(t)$, where $\varepsilon_g(t) \in 2\lambda\mathbb{Z}$ is a piecewise-constant function. Its discrete domain equivalent implies $g[n] = y[n] + \varepsilon_g[n]$. Therefore, recovering $g[n]$ (and later $g(t)$) from $y[n]$ boils down to finding $\varepsilon_g[n]$. In this paper, we consider

$$\varepsilon_q[n] = q_{\mathsf{MB}}[n] - q[n], \tag{4}$$

where $q_{\mathsf{MB}}[n] := q[n] + \varepsilon_g[n]$ is a multi-bit representation of $g(t)$. Finding $\varepsilon_q[n]$ allows to recover $q_{\mathsf{MB}}[n]$ which is then followed up by recovery of original signal $g(t)$.

3. ONE-BIT UNLIMITED SAMPLING: A SUFFICIENCY CONDITION AND A RECOVERY ALGORITHM

In contrast to [1], applying the finite difference operator $\Delta a[n] := a[n+1] - a[n]$ to the sequence $q[n]$ does not result in a sequence which is close to $\Delta\varepsilon_q[n]$ and allows for recovery of $\varepsilon_q[n]$. In this paper, instead, we develop and capitalize on an alternative to [1] which allows for recovery with accuracy $O(1/\tau)$, which is close to the best known error bound of $O(\tau^{-3/2})$ for conventional first order $\Sigma\Delta$ [16].

In [17], Candy observed that the moving average filtering of quantized measurements, with h measurements taken into account at a time, leads to the mean values of $y[n]$ up to $O(1/h)$ error bound. Here, we express the moving average filtering in terms of discrete convolution $(a * b)[n] := \sum_{k\in\mathbb{Z}} a[k]b[n-k]$ as

$$\overline{q}[n] = \left(q * h^{-1}\mathsf{B}_h^0\right)[n], \tag{5}$$

where $\mathsf{B}_h^0[n] = \mathsf{B}^0(\frac{n}{h})$ is a sampled version of B-spline of order 0 given by $\mathsf{B}^0(t) := \mathbb{1}_{[0,1)}(t)$. Let $\Delta_h a[n] := a[n+h] - a[n]$; the finite difference with step size h. Observe that the following convolution operations are equal:

$$\Delta_h\left(q * h^{-1}\mathsf{B}_h^0\right)[n] = \left(q * \Delta_h h^{-1}\mathsf{B}_h^0\right)[n] = \left(q * \Delta\mathsf{B}_h^1\right)[n] \tag{6}$$

where $\mathsf{B}_h^1[n]$ is a sampled version of B-spline of order 1 with $\operatorname{supp}\mathsf{B}^1(t) = (-1,1)$. Note that $\max\mathsf{B}^1(t) = 1$ (at $t=0$). Let B^N be a B-spline of order N with $\operatorname{supp}\mathsf{B}^N = (-\frac{N}{2}, \frac{N}{2})$. We denote its normalized and scaled version

$$\psi^N(t) := \mathsf{B}^N\left(\tfrac{N}{2}t\right)/\max\left(\mathsf{B}^N(t)\right) \tag{7}$$

and the corresponding sampled version $\psi_h^N[n] = \psi^N\left(\frac{n}{h}\right), n \in \mathbb{Z}, h \in 2\mathbb{N}$ with $\operatorname{supp}\psi_h^N = (-\frac{1}{2}, \frac{1}{2})$, $\max\psi_h^N[n] = 1$ (at $n=0$).

Motivated by (6), we consider the discrete convolution of sequence $\Delta q[n]$ with $\psi_h^N[n]$ and using (3) and MDP we get

$$\Delta q * \psi_h^N = \underbrace{\Delta g * \psi_h^N}_{1} - \underbrace{\Delta\varepsilon_g * \psi_h^N}_{2} - \underbrace{\Delta^2 u * \psi_h^N}_{3}. \tag{8}$$

Now let us analyze each of the three summands in (8) separately and start with $\Delta\varepsilon_g * \psi_h^N$. In [1], the authors give the upper bound for finite differences of $g[n]$, that is, $\|\Delta g\|_{\ell^\infty} \leqslant (\pi e/\tau)\|g\|_{L^\infty}$. Choosing $\tau > \pi e\|g\|_{L^\infty}$ guarantees that g does not cross more than one modulo threshold in one sampling interval, therefore $\Delta\varepsilon_g$ can only take values ± 2 or 0. Non-zero entries of $\Delta\varepsilon_g$ correspond to the jumps that occur due to amplitude folding. We can write

$$\left(\Delta\varepsilon_g * \psi_h^N\right)[n] = 2\left(\sum_{k\in\mathcal{I}^+}\mathscr{T}^k\psi_h^N[n] - \sum_{k\in\mathcal{I}^-}\mathscr{T}^k\psi_h^N[n]\right) \tag{9}$$

where $\mathcal{I}^+$ and $\mathcal{I}^-$ are the index sets of positive and negative entries of $\Delta\varepsilon_g$, respectively, and $\mathscr{T}^k a[n] := \sum_{n\in\mathbb{Z}} a[n]\delta[n-k]$ is the translation operator. Each summand in (9) has width $|\operatorname{supp}(\psi_h^N[n])| = h-1$, and we will refer to these summands as spikes. In case of non-overlapping spikes, the local extrema of $(\Delta\varepsilon_g * \psi_h^N)[n]$ coincide with non-zero entries of $\Delta\varepsilon_g$ and allow for an easy recovery of the latter, while in case of overlapping spikes this does not happen. Therefore, it is vital to determine under which conditions the spikes overlap.

If g is monotonically increasing or decreasing on some interval, $\|\Delta g\|_{\ell^\infty} \leqslant (\pi e/\tau)\|g\|_{L^\infty}$ implies that non-zero entries of $\Delta\varepsilon_g$ are located at least $\lceil 2\tau/\pi e\|g\|_{L^\infty}\rceil$ samples apart. Then choosing $h < 2\tau/\pi e\|g\|_{L^\infty}$ guarantees that no overlapping occurs.

Now consider the case when g crosses same threshold several times on some interval. Then g crosses μ at most twice per interval of length c (cf. Def. 1). Note that in case of two crossings per interval, the corresponding spikes must have opposite signs and will partially cancel each other in case of overlap. The distance between these consecutive crossings can be arbitrarily small and overlaps cannot be avoided in general. However, they will only occur if the distance between crossings is less than $\frac{h-1}{\tau}$. In Section 3.1, we will show that such overlaps will not cause any substantial inaccuracies in reconstruction.

In the following analysis, we assume the presence of both overlapping and non-overlapping spikes. With well-chosen τ and h we have $\|\Delta\varepsilon_g * \psi_h^N\|_{\ell^\infty} = 2$. If we guarantee that the other two summands in (8) are small enough such that

$$\|\Delta g * \psi_h^N\|_{\ell^\infty} + \|\Delta^2 u * \psi_h^N\|_{\ell^\infty} < 1, \tag{10}$$

then $\Delta\varepsilon_g * \psi_h^N$ dominates and leads to $\|\Delta q * \psi_h^N\|_{\ell^\infty} > 1$, whereas exceeding this threshold is guaranteed for all non-overlapping spikes.

For the first summand in (8), by applying Young's inequality and using $\|\Delta g\|_{\ell^\infty} \leqslant (\pi e/\tau)\|g\|_{L^\infty}$ from [1], we have,

$$\|\Delta g * \psi_h^N\|_{\ell^\infty} \leqslant (\pi e/\tau)\|g\|_{L^\infty}\left(h\|\psi_h^N\|_{L^1} + 1\right). \tag{11}$$

Finally, for the last summand we use Young's inequality and the fact that $\|u\|_{\ell^\infty} < 1$ (see [9]) and we have

$$\|\Delta^2 u * \psi_h^N\|_{\ell^\infty} \leqslant \|u\|_{\ell^\infty}\|\Delta^2\psi_h^N\|_{\ell^1} < \frac{1}{h}\|\partial^2\psi_h^N\|_{L^1}. \tag{12}$$

In the above inequalities, we omit detailed computations due to space limitations. Note that the last estimate is independent of the oversampling rate τ and depends only on the kernel's sampling rate h.

Combining (10), (11) and (12), we have the following condition:

$$\pi e\tau^{-1}\|g\|_{L^\infty}\left(h\|\psi_h^N\|_{L^1} + 1\right) + \frac{1}{h}\|\partial^2\psi_h^N\|_{L^1} < 1. \tag{13}$$

Let us at first give the lower bound for h. Splitting the sum in (13) in two equal parts results in $h \geqslant 2\|\partial^2\psi_h^N\|_{L^1}$. Recall from the previous discussion that the occurrance of overlaps depends on h and that they are more likely to occur if h is large. Therefore, our choice of h for the reconstruction algorithm is $h_r := 2\left\lceil\|\partial^2\psi_h^N\|_{L^1}\right\rceil$. Now, recalling that $\beta_g \geqslant \|g\|_{L^\infty}$ we obtain the sufficient condition for τ,

$$\tau > 4\pi e\beta_g\left(\left\lceil\|\partial^2\psi_h^N\|_{L^1}\right\rceil\|\psi_h^N\|_{L^1} + 1\right). \tag{14}$$

Now we are ready to recover the approximate residual $\tilde{\varepsilon}_q[n]$ from one-bit modulo samples. We discard the part of $(\Delta q * \psi_h^N)[n]$ which has absolute value smaller than 1 by applying

Algorithm 1 Recovery from One-Bit Modulo Samples

Data: $q[n]$, $\psi_h^N[n]$ and $\beta_g \geq \|g\|_{L^\infty}$.
Result: $\tilde{g}(t) \approx g(t)$.

1) Compute $(\Delta q * \psi_h^N)[n]$.
2) Compute $\mathscr{M}_1\left((\Delta q * \psi_h^N)[n]\right) - (\Delta q * \psi_h^N)[n]$ and retain one point from each of its non-zero neighborhoods to obtain $\Delta\tilde{\varepsilon}_q[n]$.
3) Apply S to obtain $\tilde{\varepsilon}_q[n]$.
4) Compute $\tilde{q}_{\mathsf{MB}}[n] = q[n] + \tilde{\varepsilon}_q[n]$.
5) Reconstruct $\tilde{g}(t)$ from $\tilde{q}_{\mathsf{MB}}[n]$ via low-pass filter.

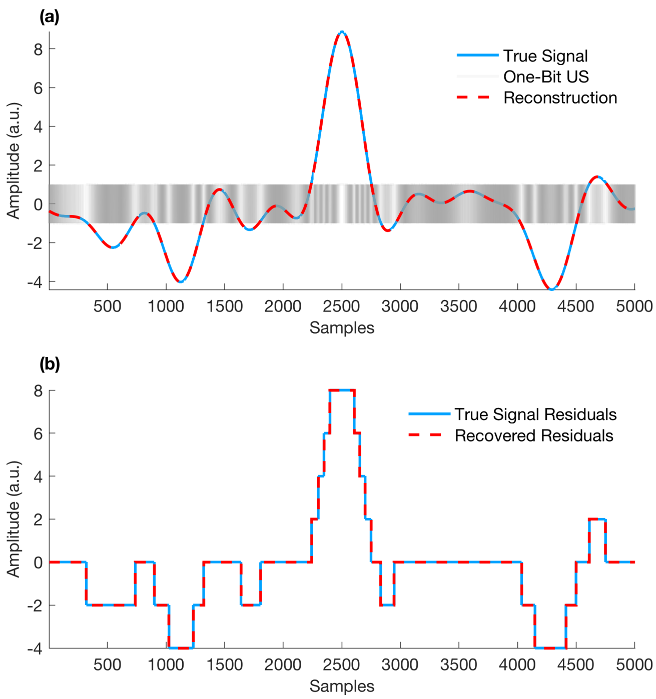

Fig. 3: One-bit modulo sampling of bandlimited signals. (a) Randomly generated bandlimited signal $g \in \mathrm{PW}_\pi(\mathbb{R})$, its one-bit modulo samples $q[n]$ acquired with $\tau = 250$ as well as the reconstructed signal $\tilde{g}$ which is obtained using second order ψ^2. The mean error $|g - \tilde{g}|$ is 2.1×10^{-3}. (b) The true residual $\varepsilon_g[n]$ and its approximate recovery $\tilde{\varepsilon}_q[n]$.

$\mathscr{M}_1\left((\Delta q * \psi_h^N)[n]\right) - (\Delta q * \psi_h^N)[n]$; then take an arbitrary point from each non-zero neighborhood of the resulting sequence to obtain $\Delta\tilde{\varepsilon}_q[n]$.[1] Finally, we apply the summation operator $\mathsf{S} : (a_i)_{i=1}^{\infty} \mapsto \sum_{i'=1}^{i}(a_{i'})_{i=1}^{\infty}$ to recover $\tilde{\varepsilon}_q[n]$ (up to additive multiples of 2). Now we can use it to obtain a multi-bit quantized representation of g, that is, $\tilde{q}_{\mathsf{MB}}[n] = q[n] + \tilde{\varepsilon}_q[n]$, and then compute the reconstruction $\tilde{g}(t)$ by low-pass filtering $\tilde{q}_{\mathsf{MB}}[n]$.

3.1. Bound on the reconstruction error

Here we formally state two results which show that we reasonably recover $\tilde{g}(t)$ from $\tilde{q}_{\mathsf{MB}}[n]$ and it will not lead to a reconstruction error bigger than $O(1/\tau)$. A standard approach to signal recovery from its $\Sigma\Delta$ samples uses the analogue of Shannon interpolation formula,

$$\tilde{g}(t) = \frac{1}{\tau}\sum\nolimits_{n\in\mathbb{Z}} \tilde{q}_{\mathsf{MB}}[n]\varphi\left(t - \frac{n}{\tau}\right), \tag{15}$$

where the reconstruction kernel $\varphi(t)$ is defined via its Fourier transform such that $\widehat{\varphi}(\omega) = \mathbb{1}_{(-\pi,\pi)}(\omega)$. However, just like the scaling function in Meyer wavelet, $\widehat{\varphi}(\omega)$ is non-zero when $\pi < |\omega| \leqslant \tau$. Additionally we require that $\varphi(t)$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$ and has maximum $\varphi(0) = 1$. The following lemma will help us to provide the estimate for the reconstruction error.

Lemma 1. *For $\varepsilon_g[n] = g[n] - y[n]$ and $\tilde{\varepsilon}_q[n]$ which is obtained via Algorithm 1, provided that h and τ are chosen according to $h_r := 2\lceil\|\partial^2\psi_h^N\|_{L^1}\rceil$ and* (14)*, respectively, and $\varphi(t)$ is a valid interpolation kernel (cf.* (15)*), we have*

$$\frac{1}{\tau}\left|\sum_{n\in\mathbb{Z}}(\varepsilon_g[n] - \tilde{\varepsilon}_q[n])\varphi\left(t - \frac{n}{\tau}\right)\right| < \frac{1}{\tau}M(c,\varphi,\psi_h^N), \tag{16}$$

where $M(c,\varphi,\psi_h^N)$ is a constant dependent on SO parameter c (cf. Def. 1) and on the choice of kernels φ and ψ_h^N.

We omit the proof due to space limitations. Essentially, this lemma tells that the error $|\varepsilon_g[n] - \tilde{\varepsilon}_q[n]|$ between true and recovered residual has the contribution of only $O(1/\tau)$ to the overall reconstruction error. Finally, we state the main result of this paper (again, the proof is omitted due to space limitation).

Theorem 1 (Error Bound for One-Bit Unlimited Sampling)**.** *Let $g \in \mathrm{PW}_\pi(\mathbb{R})$ with SO parameter c (cf. Def. 1) and $\beta_g \geqslant \|g\|_{L^\infty}$. Let $q[n]$, $n \in \mathbb{Z}$, in* (3) *be the one-bit modulo samples of $g(t)$ with oversampling rate $\tau \geqslant 1$. Let $\psi_h^N[n]$, $n \in \mathbb{Z}$, be the samples of the smoothing kernel $\psi_h^N(t)$ defined in* (7) *with sampling rate $h_r := 2\lceil\|\partial^2\psi_h^N\|_{L^1}\rceil$ and let $\varphi(t)$ be the reconstruction kernel as in* (15)*. Then, a sufficient condition for approximate recovery of $\tilde{g}(t)$ from $q[n]$ (up to additive multiples of 2) is given by* (14)*. Under these conditions, Algorithm 1 yields the reconstruction error*

$$|g(t) - \tilde{g}(t)| \leqslant \frac{1}{\tau}\left(\|\partial\varphi\|_{L^1} + M(c,\varphi,\psi_h^N)\right), \tag{17}$$

where $M(c,\varphi,\psi_h^N)$ is a constant described in Lemma 1.

We complement theoretical results with numerical demonstration in Fig. 3.

4. FUTURE WORK AND CONCLUSIONS

As a natural extension of our research, we plan to incorporate higher order $\Sigma\Delta$ in our sampling architecture and higher order finite differences in the recovery algorithm. Similar to [1], considering $\Delta^N q[n]$ with higher N will hopefully allow to discard the norm estimate of g in the sufficiency condition for the oversampling rate τ, thus, making our one-bit sampling architecture unlimited in a broader sense. Also our sufficient condition seems to have room for improvement and further research is needed to investigate sharper recovery conditions.

We conclude this paper by emphasizing the fact that we provided a new step towards making the sampling theory for self-reset ADCs practically feasible. Namely, we coupled modulo sampling with discretization in range via one-bit $\Sigma\Delta$ quantization, thus, overcoming the limitation on dynamic range inherent to conventional one-bit $\Sigma\Delta$ schemes. Our results establish a sufficient condition for the recovery of bandlimited signals from one-bit modulo samples as well as provide a recovery algorithm and a bound on the reconstruction error.

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[1] A practical and more precise approach in MATLAB; in order to find a more precise $\Delta\tilde{\varepsilon}_q[n]$, we retain only the local extrema of $(\Delta q * \psi_h^N)[n]$ and replace other values by zeroes.

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