AN ANTIPODALLY SYMMETRIC OPTIMAL DIMENSIONALITY SAMPLING ON THE SPHERE

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ABSTRACT

We propose an antipodally symmetric sampling scheme of optimal dimensionality for the sampling of band-limited signals. The proposed scheme takes $\sim L^2$ number of samples for the sampling of spherical signal of band-limit L and the accurate computation of its spherical harmonic transform (SHT). Since the number of samples are asymptotically equal to the degrees of freedom of the signal in harmonic space, the proposed scheme attains optimal spatial dimensionality. We also formulate the SHT associated with proposed sampling scheme. We employ the antipodal symmetry of the sampling points that is exploited to separate the signal into antipodally symmetric and asymmetric signals due to which the signal splits in harmonic space into the signals of even and odd spherical harmonic degrees. The exploitation of this splitting in the formulation of the SHT makes our method computationally efficient by a factor of four in comparison with the existing methods developed for sampling schemes that attain optimal spatial dimensionality. We also analyse the numerical accuracy of the proposed SHT by conducting numerical experiments and show that the proposed sampling and its associated SHT enable accurate signal reconstruction for band-limits in the range $15 \leq L \leq 127$.

***Index Terms*—** 2-sphere, spherical harmonic transform, sampling, signal reconstruction, band-limited signals

1. INTRODUCTION

Spherical signal processing finds applications in a variety of branches of engineering and sciences including computer graphics [1], wireless communication [2], acoustics [3], medical imaging [4] and cosmology [5]. In these applications, the signals are either in spatial domain or spectral (harmonic) domain. Spectral domain is enabled by the spherical harmonic transform (SHT) which serves as a counterpart of Fourier transform for analysis of signals on the sphere. To support harmonic domain analysis, the ability to compute spherical harmonic transform of the signal from its measurements is of significant importance. Since the acquisition of measurements is time consuming, it is desirable to have a sample acquisition strategy that takes the minimum possible number of samples for the accurate computation of spherical harmonic transform, exhibits structure in the placement of samples to facilitate the acquisition and has spatially uniform distribution of samples.

Many sampling schemes on the sphere have been devised in the literature (e.g., see [4, 6–11] and the references therein). A sampling theorem is presented in [6] for the computation of the SHT of the signal band-limited at L (defined in the next section) by taking samples on the equiangular grid of spatial dimensionality (asymptotically) $\sim 4L^2$ (total number of samples). Another equiangular sampling strategy is proposed in [7] that takes $\sim 2L^2$ number of samples for the exact computation of the SHT. The computational complexity of the stable SHT associated with these sampling methods is $O(L^3)$. Recently, an optimal dimensionality sampling scheme has been proposed in [9] that takes optimal L^2 number of samples equal to the degrees of freedom of the band-limited signal in the harmonic space. However, the computational complexity of the associated SHT is of the order $O(L^4)$ due to the series of matrix inversions involved in the computation. The samples are placed in iso-latitude rings but appear unstructured and asymmetric around the equator.

In this work, we make the following research contributions:

- We propose an antipodally symmetric sampling scheme of asymptotic optimal dimensionality for the acquisition of band-limited signals. For a signal band-limited at L, the proposed scheme takes $\sim L^2$ number of samples.
- We develop the transform associated with the proposed sampling scheme for the accurate computation of the SHT. The SHT developed in this work (having complexity of the order $O(L^4)$) is computationally efficient by a factor of four thanks to the symmetry of placement of samples which is exploited to reduce the size of the matrices required to be inverted for the computation of the SHT.
- We also propose a method for iterative placement of rings of samples along co-latitude and conduct numerical experiments to analyse the accuracy of the SHT.

We organize the rest of the paper as follows. We review the mathematical background related to signals on the sphere and sampling in Section 2. Before we conclude the paper in Section 5, the proposed sampling scheme is presented in Section 3 and the associated SHT is developed in Section 4, where we also present a method to place the samples that enable accurate computation of the SHT and analyse the accuracy of the SHT.

2. MATHEMATICAL PRELIMINARIES

2.1. Signals on the Sphere and Spherical Harmonics

We consider the complex-valued square integrable functions defined on the unit sphere $\mathbb{S}^2$ of the form $f(\theta, \phi)$, where the angles $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$ symbolise co-latitude and longitude respectively. θ and ϕ parameterise a 3D point $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)' \in \mathbb{R}^3$ on $\mathbb{S}^2$. These square integrable complex-valued functions form a Hilbert space $L^2(\mathbb{S}^2)$ equipped with the inner product defined for two functions f, g as [8]

$$\langle f, g \rangle \triangleq \int_{\mathbb{S}^2} f(\theta, \phi)\overline{g(\theta, \phi)} \sin\theta \, d\theta \, d\phi, \quad (1)$$

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which induces a norm $\|f\| \triangleq \langle f, f \rangle^{1/2}$. Here the integration is carried out over the whole sphere and $\overline{(\cdot)}$ denotes the complex conjugate operation. The functions with finite energy (induced norm) are referred to as signals on the sphere.

2.2. Spherical Harmonic (Fourier) Domain Representation

For integer degree $\ell \geq 0$ and integer order $|m| \leq \ell$, the spherical harmonics, denoted by $Y_\ell^m(\theta, \phi)$, serve as a complete orthonormal set of basis functions for $L^2(\mathbb{S}^2)$ [8,12]. Due to the completeness of the spherical harmonics, any signal $f \in L^2(\mathbb{S}^2)$ on the sphere can expanded be as

$$f(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (f)_\ell^m Y_\ell^m(\theta, \phi), \tag{2}$$

where $(f)_\ell^m$ represents the spherical harmonic coefficient of degree ℓ and order m of the signal f. The coefficients $(f)_\ell^m$ form the harmonic domain representation of the signal and are given by the following spherical harmonic transform (SHT)

$$(f)_\ell^m = \langle f, Y_\ell^m \rangle = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} f(\theta, \phi) \overline{Y_\ell^m(\theta, \phi)} \sin\theta d\theta d\phi. \tag{3}$$

We refer to the signal as band-limited at degree L if the signal can be *completely* represented in terms of the spherical harmonic basis functions $Y_\ell^m(\theta, \phi)$ with $\ell < L$ and $|m| \leq \ell$. For a signal band-limited at L, the number of non-zero coefficients are L^2.

2.3. Optimal Dimensionality Sampling Scheme

We here briefly review the optimal dimensionality sampling scheme that enables accurate computation of spherical harmonic transform of the signal band-limited at L using optimal number L^2 samples of the signal [9]. This scheme, denoted by $\mathfrak{S}(L)$, for signals of band-limit L take L iso-latitude rings of samples located at the positions θ_n, $n = 0, 1, \ldots, (L-1)$ along co-latitude. In the ring placed at θ_n, the sampling grid takes $2n+1$ equiangular samples along longitude for the accurate computation of the SHT. We note that the total number of samples in $\mathfrak{S}(L)$ is L^2, that is equal to the number of degrees of freedom required to represent a signal band-limited at L.

3. ANTIPODALLY SYMMETRIC OPTIMAL DIMENSIONALITY SAMPLING

3.1. Proposed Sampling Scheme — Structure and Design

We propose to place $L+1$ iso-latitude rings (of samples) symmetric around the equator ($\theta = \pi/2$). With this consideration, we define the vector $\boldsymbol{\theta}$ containing the location of these $L+1$ iso-latitude rings as

$$\boldsymbol{\theta} \triangleq [\theta_0, \pi - \theta_0, \ldots, \theta_{L-3}, \pi - \theta_{L-3}, \theta_{L-1}, \pi - \theta_{L-1}], \tag{4}$$

for odd L and

$$\boldsymbol{\theta} \triangleq [\theta_0, \pi - \theta_0, \ldots, \theta_{L-4}, \pi - \theta_{L-4}, \theta_{L-2}, \pi - \theta_{L-2}, \pi/2], \tag{5}$$

for even L. Here $\theta_0 = 0$ for both odd and even band-limits and θ_n denotes the n-th entry in the vector $\boldsymbol{\theta}$. We shortly present the location of the remaining rings along the co-latitude. We note that the placement of rings is symmetric around equator. In the iso-latitude ring placed at θ_n, we propose to place the samples along ϕ as

$$\phi_k^n \triangleq \begin{cases} \frac{2k\pi}{2n+1}, & n = 0, 2, \ldots, L-1, \quad k \in [0, 2n], \\ \frac{\pi(2k+1)}{2n-1}, & n = 1, 3, \ldots, L, \quad k \in [0, 2(n-1)], \end{cases} \tag{6}$$

for odd L and

$$\phi_k^n \triangleq \begin{cases} \frac{2k\pi}{2n+1}, & n = 0, 2, \ldots, L-2, \quad k \in [0, 2n], \\ \frac{\pi(2k+1)}{2n-1}, & n = 1, 3, \ldots, L-1, \quad k \in [0, 2(n-1)], \end{cases} \tag{7}$$

including $2L-1$ equiangular samples along ϕ on the ring $\theta_L = \pi/2$ for even L. We use $\mathfrak{E}_L$ and $\mathfrak{O}_L$ to denote the sampling schemes defined above for even and odd L respectively.

Antipodal Symmetry of Sampling Points: We note that the samples along ϕ in the proposed sampling schemes are placed such that the samples in the ring located at θ_n are antipodal to the samples in the ring located at θ_{n-1}, that is, $(\theta_{n-1}, \phi_k^{n-1}) = (\pi - \theta_{n-1}, \pi + \phi_k^n)$ for $n = 2, 4, \ldots, L-1$. As an example, Fig. 2 shows the proposed sampling scheme for $L = 21$.

Number of Points: The total number of samples in the proposed sampling schemes is given by

$$2 \sum_{\substack{n=0 \\ n\text{even}}}^{L-1} (2n+1) = L^2 + L, \quad L \text{ odd}. \tag{8}$$

$$2L - 1 + 2 \sum_{\substack{n=0 \\ n\text{even}}}^{L-2} (2n+1) = L^2 + L - 1, \quad L \text{ even}. \tag{9}$$

We note that the proposed sampling schemes for odd and even band-limits take the optimal number of samples asymptotically. Optimal number of samples is L^2 given by the degrees of freedom of the band-limited signal in harmonic space.

4. SPHERICAL HARMONIC TRANSFORM

We develop an algorithm for the computation of the spherical harmonic transform of the signal band-limited at L from its samples taken using the proposed sampling scheme. Following the philosophy proposed in [9], we here present the spherical harmonic transform algorithm for the case when the band-limit of the signal is odd. For signals with even band-limits, an equivalent formulation can be developed.

We assume that the samples of the band-limited signal $f \in \mathcal{H}_L$ are taken on the scheme $\mathfrak{O}_L$ proposed in the previous subsection. Exploiting the antipodal structure of the sampling scheme, we first split the band-limited signal f into antipodally symmetric (f_s) and antipodally asymmetric signals f_a given by

$$f_s(\theta_n, \phi_n) = \frac{1}{2}\big(f(\theta_n, \phi_n) + f(\pi - \theta_n, \pi + \phi_n)\big), \tag{10}$$

$$f_a(\theta_n, \phi_n) = \frac{1}{2}\big(f(\theta_n, \phi_n) - f(\pi - \theta_n, \pi + \phi_n)\big) \tag{11}$$

for all $(\theta_n, \phi_n) \in \mathfrak{O}_L$. It is trivial to show that $f = f_s + f_a$. Due to the antipodal symmetry and asymmetry of the spherical harmonics of even and odd degrees respectively, we have the following expansion of f_s and f_a

$$f_s(\theta_n, \phi_n) = \sum_{\ell=0, \ell\,\text{even}}^{L-1} (f)_\ell^m Y_\ell^m(\theta_n, \phi_n), \tag{12}$$

$$f_a(\theta_n, \phi_n) = \sum_{\ell=1, \ell\,\text{odd}}^{L-2} (f)_\ell^m Y_\ell^m(\theta_n, \phi_n). \tag{13}$$

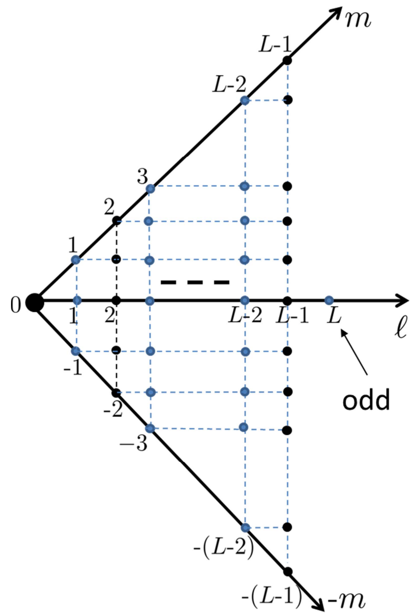


Fig. 1: Representation of non-zero coefficients of the signal band-limited at odd degree L. The coefficients of the symmetric signal f_s and asymmetric signal f_a are indicated as black and blue dots respectively.

The separation of the signal into antipodally symmetric and asymmetric signals enabled by the proposed sampling structure also splits the signal in the harmonic space into even degree harmonics and odd degree harmonics respectively. We illustrate this in Fig. 1, where we plot the spectral domain of the band-limited signal and indicate the non-zero coefficients of antipodally symmetric (even degrees, black solid dots) and antipodally asymmetric (odd degrees, blue dots).

4.1. Formulation of Spherical Harmonic Transform (SHT)

To formulate the SHT, we first define an iso-latitude transform, that is, the Fourier transform along ϕ for symmetric and asymmetric signals as

$$ {}_sG_m(\theta) \triangleq \int_0^{2\pi} f_s(\theta,\phi) e^{-im\phi} d\phi \tag{14}$$

$$ = 2\pi \sum_{\ell=2\lceil|m/2|\rceil,\ell\,\text{even}}^{L-1} (f)_\ell^m \widetilde{Y}_\ell^m(\theta), \tag{15}$$

$$ {}_aG_m(\theta) \triangleq \int_0^{2\pi} f_a(\theta,\phi) e^{-im\phi} d\phi \tag{16}$$

$$ = 2\pi \sum_{\ell=2\lfloor|m/2|\rfloor+1,\ell\,\text{odd}}^{L-2} (f)_\ell^m \widetilde{Y}_\ell^m(\theta), \tag{17}$$

where $\widetilde{Y}_\ell^m(\theta) \triangleq Y_\ell^m(\theta, 0)$. We also define vectors ${}_s\mathbf{f}_m$ and ${}_a\mathbf{f}_m$ as

$$ {}_s\mathbf{f}_m = \left[(f)^m_{2\lceil|m/2|\rceil},\, (f)^m_{2\lceil|m/2|\rceil+2},\, \ldots,\, (f)^m_{L-1}\right], \tag{18}$$

$$ {}_a\mathbf{f}_m = \left[(f)^m_{2\lfloor|m/2|\rfloor+1},\, (f)^m_{2\lfloor|m/2|\rfloor+3},\, \ldots,\, (f)^m_{L-2}\right], \tag{19}$$

containing m-th order spherical harmonic coefficients of even degrees and odd degrees respectively. Be defining

$$ {}_s\mathbf{g}_m = [{}_sG_m(\theta_{2\lceil|m/2|\rceil}),\, {}_sG_m(\theta_{2\lceil|m/2|\rceil+2}),\, \ldots,\, {}_sG_m(\theta_{L-1})], $$

$$ {}_a\mathbf{g}_m = [{}_aG_m(\theta_{2\lceil|m/2|\rceil}),\, {}_aG_m(\theta_{2\lceil|m/2|\rceil+2}),\, \ldots,\, {}_aG_m(\theta_{L-1})], $$

for each $|m| < L$, we can express ${}_s\mathbf{g}_m$ using the formulation of ${}_s\mathbf{f}_m$, which respectively contains spherical harmonic coefficients of order $|m| < L$ and even degrees $m \leq \ell < L$ and iso-latitude transforms of order $|m| < L$ evaluated along the rings placed at $\theta_{2\lceil|m/2|\rceil}, \theta_{2\lceil|m/2|\rceil+2}, \ldots, \theta_{L-1}$, we can write (14) as

$$ {}_s\mathbf{g}_m = 2\pi {}_s\mathbf{P}_L^m \mathbf{f}_m, \quad |m| \leq L, \tag{20}$$

where ${}_s\mathbf{P}_L^m$, containing $\widetilde{Y}_\ell^m(\theta_n)$ consistent with the formulation of ${}_sG_m(\theta)$ in (14), is a square matrix of number of rows (or columns) equal to $(L - 2\lceil|m/2|\rceil + 1)/2$. We can similarly express ${}_a\mathbf{g}_m$ as

$$ {}_a\mathbf{g}_m = 2\pi {}_a\mathbf{P}_L^m \mathbf{f}_m, \quad |m| \leq L. \tag{21}$$

Using (20) and (21) for each order $|m| < L$, we can recover the spherical harmonic coefficients of order m and even and odd degrees respectively provided the rings are placed such that ${}_s\mathbf{P}_L^m$ and ${}_a\mathbf{P}_L^m$ are well-conditioned (invertible) and iso-latitude transform along ϕ can be computed accurately. For the structure of the samples along ϕ of the proposed sampling scheme, an iso-latitude transform ${}_sG_m(\theta)$ and ${}_aG_m(\theta)$ can be computed accurately by taking FFT over the samples of symmetric and asymmetric signals respectively.

Remark 1. *We note that the computation of the spherical harmonic coefficients, that is, the spherical harmonic transform using the formulation proposed above requires the inversion of matrices ${}_s\mathbf{P}_L^m$ and ${}_a\mathbf{P}_L^m$ for each $|m| < L$. Therefore the computational complexity of the proposed transform is ($O(L^4)$) equal to the complexity of the transform associated with optimal dimensionality sampling without antipodal symmetry [9]. Although we require twice the number of matrices to be inverted, the size of the matrices in our formulation is half the size of matrices required to be inverted for the computation of SHT proposed in [9]. We therefore note the improvement in the SHT computation time by a factor of (approximately) 4 using the proposed scheme.*

4.2. Placement of Rings of Samples along Co-latitude

We now present a method to place the rings of samples along co-latitude, that is, we determine positions indexed in (4) such that the matrices ${}_s\mathbf{P}_L^m$ and ${}_a\mathbf{P}_L^m$ are well-conditioned. We first note that either ${}_s\mathbf{P}_L^m$ or ${}_a\mathbf{P}_L^{(m-1)}$ depend on $\theta_{2\lceil|m/2|\rceil}$, $\theta_{2\lceil|m/2|\rceil+2}, \ldots, \theta_{L-1}$ and ${}_s\mathbf{P}_L^{-m} = (-1)^m {}_s\mathbf{P}_L^m$ and ${}_a\mathbf{P}_L^{-m} = (-1)^m {}_a\mathbf{P}_L^m$.

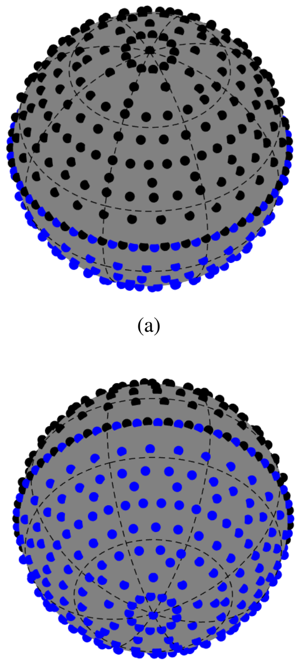
Since the locations of the rings indexed in $\boldsymbol{\theta}$ given in (4) appear in pairs due to the antipodal symmetry of the proposed sampling scheme, we are only required to find the locations of the rings in the Northern hemisphere ($\theta \in [0, \pi/2)$). We take a set of equiangular $M > L$ samples in the Northern hemisphere along co-latitude given by

$$ \Theta = \left\{ \frac{\pi(t)}{2M} \right\}, \quad t = 0, 1, \ldots, M-1. \tag{22}$$

We propose the following method to iteratively place the rings of samples along co-latitude.

- Choose $\theta_{L-1} = \pi(M-1)/2M$, that is the farthest sample from the poles in the set Θ from the poles.
- For each $m = L-3,\, L-5,\, \ldots 2$, choose θ_m from the set Θ such that sum of the condition numbers of the four matrices ${}_s\mathbf{P}^m$, ${}_s\mathbf{P}^{m-1}$, ${}_a\mathbf{P}^{m-1}$ and ${}_a\mathbf{P}^{m-2}$ is minimized.
- Choose the last ring location $\theta_0 = 0$.

As an example, we plot the sampling positions on the sphere for $L = 21$ in Fig. 2, where we use $M = 15L$ equiangular points in the set Θ. Such placement ensures that the SHT can be accurately computed by taking samples using the proposed sampling scheme and the associated SHT developed in the previous section.



(a)

(b)

Fig. 2: Samples of the proposed sampling schemes are plotted for $L = 21$ with (a) North pole view and (b) South pole view. The points in the Northern hemisphere are are shown in black and the antipodally symmetric points in the Southern hemisphere are shown in blue.

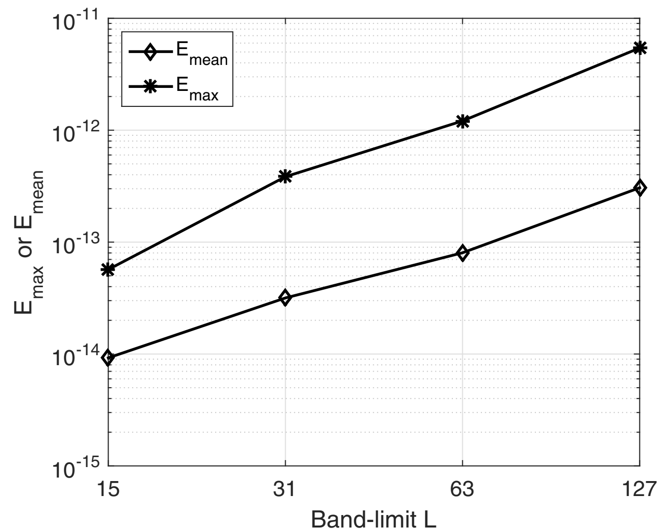


Fig. 3: Plots of the maximum error $E_{\max}$ and the mean error E_{mean}, given in (23) and (24) respectively, for band-limits $15 \leq L \leq 127$.

4.3. Numerical Accuracy Analysis

We analyse the accuracy of the SHT developed for the proposed sampling scheme in this section. To evaluate the numerical accuracy, we obtain a band-limited test signal f_{t} for band-limit $15 \leq L \leq 127$ in the harmonic domain by generating its spherical harmonic coefficients $(f_{\text{t}})_\ell^m$ for $0 < \ell < L$ with real and imaginary parts uniformly distributed in the interval $[-1, 1]$. We then use (2) to obtain the signal at the samples of proposed sampling schemes $\mathfrak{E}_L$ or $\mathfrak{O}_L$. We then apply the proposed SHT to compute the spherical harmonic coefficients, denoted by $(f_{\text{r}})_\ell^m$, of the reconstructed signal. We repeat this test 10 times for each band-limit and compute the average values for the maximum error $E_{\max}$ and the mean error E_{mean}, given by

$$E_{\max} \triangleq \max |(f_{\text{t}})_\ell^m - (f_{\text{r}})_\ell^m|, \tag{23}$$

$$E_{\text{mean}} \triangleq \frac{1}{L^2} \sum_{\ell=0}^{L-1} \sum_{m=-\ell}^{\ell} |(f_{\text{t}})_\ell^m - (f_{\text{r}})_\ell^m|, \tag{24}$$

which are plotted in Fig. 3 over the range of band-limits, where it is evident that the proposed transform enables accurate computation of SHT with errors on the order of machine (double) precision.

5. CONCLUSIONS

In this work we have proposed an antipodally symmetric sampling scheme on the unit-sphere for the sampling of band-limited signals. Using the proposed scheme, the accurate computation of the spherical harmonic transform (SHT) can be done by taking $\sim L^2$ number of samples of the signal with band-limit L. The proposed scheme attains optimal spatial dimensionality because the number of samples are asymptotically equal to the degrees of freedom of the signal in harmonic space. While formulating the SHT associated with the proposed sampling scheme, we used the antipodal symmetry of the sampling points. This separates the signal into antipodally symmetric and asymmetric signals due to which the signal also splits in harmonic space into the signals of even and odd spherical harmonic degrees. This type of splitting allows our method to be computationally efficient by a factor of four as compared to the existing sampling schemes that attain optimal spatial dimensionality. To analyse the numerical accuracy of the proposed SHT, we conducted numerical experiments and showed that the proposed sampling and its associated SHT enable accurate signal reconstruction for signals in the band-limit range $15 \leq L \leq 127$.

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