ENERGY BLOWUP OF SAMPLING-BASED APPROXIMATION METHODS

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ABSTRACT

This paper considers the problem of approximating continuous functions of finite Dirichlet energy from samples of these functions. It will be shown that there exists no sampling-based method which is able to approximate every function in this space from its samples. Specifically, we are going to show that for any sampling based approximation method, the energy of the approximation tends to infinity as the number of samples is increased for almost every continuous function of finite energy. As an application, we study the problem of solving the Dirichlet problem on a bounded region. It will be shown that if only samples of the boundary function can be processed then the energy of the solution can not be controlled for any function from a non-meager dense set.

***Index Terms*—** Sampling and reconstruction, approximation, computability, Dirichlet problem, finite energy

1. INTRODUCTION

Let f be a given function in a Banach space $\mathcal{B}$ of 2π-periodic, continuous functions considered on the interval $\mathbb{T} = [-\pi, \pi]$. Since the cardinality of $\mathbb{T}$ is uncountable, it is impossible to store or to process all values $\{f(t) : t \in \mathbb{T}\}$ on a digital computer. Instead, one retains only finitely many samples $\mathcal{W}_N(f) = \{f(\tau_n) : \tau_n \in \mathcal{Z}_N\}$ of f taken on a sampling set $\mathcal{Z}_N \subset \mathbb{T}$ of cardinality $|\mathcal{Z}_N| = N < \infty$. Then any function or operator A which has f as an input calculates effectively only with the available samples $\mathcal{W}_N(f)$. One core assumption in signal processing is then that there always exists an operator or algorithm $\mathrm{A}_N : \mathcal{W}_N(f) \mapsto \mathcal{B}$ which uses only the samples $\mathcal{W}_N(f)$ as input and which is able to approximate the output $\mathrm{A}(f)$ sufficiently well in the sense that $\mathrm{A}_N(f)$ converges to $\mathrm{A}(f)$ as the number of available samples N goes to infinity, i.e. so that

$$\lim_{N\to\infty} \|\mathrm{A}(f) - \mathrm{A}_N(f)\|_{\mathcal{B}} = 0 \qquad \text{for every } f \in \mathcal{B} ,$$

where the convergence is assumed to be in the norm of the Banach space $\mathcal{B}$ and where the norm characterizes usually important physical quantities of any $f \in \mathcal{B}$, e.g. the energy of f.

In this paper, we are going to show that this fundamental assumption in signal processing is actually not true, in general. For concreteness and clarity of the presentation, we consider a very simple but often used example for the operator A, namely we study the identity operator $\mathrm{I} : \mathcal{B} \to \mathcal{B}$ on certain Banach spaces $\mathcal{B}$. In other words, we consider the problem of reconstructing (approximating) a function $f \in \mathcal{B}$ from its samples $\mathcal{W}_N(f)$ on a finite sampling set $\mathcal{Z}_N \subset \mathbb{T}$ in such a way that the reconstruction error goes to zero as the number of available samples N goes to infinity. This problem is considered on Banach spaces $\mathcal{E}_\beta$ of continuous functions with finite Dirichlet energy $\|f\|_{\mathrm{E}}$ and where the signal energy satisfies additionally a certain concentration condition characterized by the parameter $0 \leq \beta \leq 1$. Functions in these Banach spaces play an important role in physics and engineering since for these functions the Dirichlet problem has a solution of finite Dirichlet energy.

This paper is going to show that for any arbitrary sequence of sampling sets $\{\mathcal{Z}_N\}_{N\in\mathbb{N}} \subset \mathbb{T}$ with associated approximation operators A_N there always exist functions $f \in \mathcal{E}_\beta$ of finite energy $\|f\|_{\mathrm{E}} < +\infty$ such that the energy of the sampling-based approximation $\mathrm{A}_N(f)$ gets arbitrarily large for sufficiently large N, i.e. such that

$$\limsup_{N\to\infty} \|\mathrm{A}_N(f)\|_{\mathrm{E}} = +\infty .$$

This is true, even if the signal energy satisfies additionally a certain concentration condition. This observation is then applied to sampling-based methods for solving the Dirichlet problem.

Following the publication of Shannon's seminal paper [1] on sampling of bandlimited signals, many different approaches for sampling-based signal processing where developed for a variety of applications, for different signal spaces, and for several basis functions [2–10]. So sampling theory is now one of the cornerstones in signal processing [11, 12]. Based on an axiomatic approach, which generalizes the above settings by making no restriction on the used basis functions, this paper presents a particular and very simple example showing that there exist fundamental limits of sampling-based signal processing methods for the approximation of continuous functions f of finite energy. It will be shown that such approximation methods can generally not control the energy of the approximation.

2. FUNCTIONS OF FINITE DIRICHLET ENERGY

A family of Banach spaces The Banach space of all functions continuous on $\mathbb{T}$ with the norm $\|f\|_\infty = \max_{\zeta\in\mathbb{T}} |f(\zeta)|$ is denoted by $\mathcal{C}(\mathbb{T})$. Let $f \in \mathcal{C}(\mathbb{T})$ be arbitrary with its *Fourier coefficients*

$$c_n(f) = \tfrac{1}{2\pi} \int_{\mathbb{T}} f(t)\, \mathrm{e}^{int}\, \mathrm{d}t , \qquad n = 0, \pm1, \pm2, \dots . \tag{1}$$

Then we define on $\mathcal{C}(\mathbb{T})$ for any $\beta \geq 0$ the seminorm

$$\|f\|_\beta = \sqrt{\textstyle\sum_{n\in\mathbb{Z}} |n|\, (1 + \log|n|)^\beta\, |c_n(f)|^2} \tag{2}$$

and therewith the space $\mathcal{E}_\beta = \{f \in \mathcal{C}(\mathbb{T}) : \|f\|_\beta < \infty\}$. Equipped with the norm $\|f\|_{\mathcal{E}_\beta} = \max\big(\|f\|_\infty, \|f\|_\beta\big)$, this space becomes a Banach space for every $\beta \geq 0$. This way, we obtain a family $\{\mathcal{E}_\beta\}_{\beta\geq0}$ of Banach space and it is clear from the definition that

$$\mathcal{E}_{\beta_2} \subset \mathcal{E}_{\beta_1} \subset \mathcal{E}_0 \subset \mathcal{C}(\mathbb{T}) \qquad \text{for all } \beta_2 > \beta_1 > 0 , \tag{3}$$

and Parseval's identity implies that $\mathcal{E}_\beta \subset L^2(\mathbb{T})$ for all $\beta \geq 0$.

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The Dirichlet problem and Dirichlet energy The seminorm (2) with $\beta = 0$ plays a particular important role in physics since it corresponds to the Dirichlet energy of certain potential fields. To explain this, we consider the well known Dirichlet problem on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $u : \mathbb{D} \to \mathbb{C}$ be a function defined on $\mathbb{D}$, write $z = x + \mathrm{i}y$ for any point $z \in \mathbb{D}$, and let $g \in \mathcal{C}(\mathbb{T})$ be a given function. Then the *Dirichlet problem* is to find u such that

$$(\Delta u)(z) = \frac{\partial^2 u}{\partial x^2}(z) + \frac{\partial^2 u}{\partial y^2}(z) = 0 \qquad \text{for all } z \in \mathbb{D} \tag{4a}$$

$$u(\mathrm{e}^{\mathrm{i}\theta}) = g(\theta) \qquad \text{for all } \theta \in \mathbb{T}\ . \tag{4b}$$

In other words, one looks for a function u which is harmonic in $\mathbb{D}$ and whose boundary values $u|_{\partial\mathbb{D}}$ coincide with the given $g \in \mathcal{C}(\mathbb{T})$. The unique solution of (4) is known to be the Poisson integral of g (see, e.g., [13, § 11.8]), i.e. for every $z = r\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{D}$, it is given by

$$u(z) = (\mathrm{P}g)(z) = \frac{1}{2\pi}\int_{\mathbb{T}} g(\tau)\, \frac{1 - r^2}{1 - 2r\cos(\theta - \tau) + r^2}\, \mathrm{d}\tau\ . \tag{5}$$

Nevertheless, according to the so called *Dirichlet principle*, any solution of (4) can also be obtained by minimizing a particular energy functionals under the side constraint (4b), i.e. by

$$u = \arg\min_{f \in \mathcal{D}} \mathrm{D}[f] \quad \text{s. t.} \quad f(\mathrm{e}^{\mathrm{i}\theta}) = g(\theta) \quad \text{for all } \theta \in \mathbb{T}, \tag{6}$$

where the functional $\mathrm{D}[f] = \frac{1}{2\pi}\iint_{\mathbb{D}} \|\mathrm{grad}(f)\|^2_{\mathbb{R}^2}\, \mathrm{d}A$ is called the *Dirichlet energy* of f, and $\mathcal{D} = \{u : \mathbb{D} \to \mathbb{C} \text{ with } \mathrm{D}[u] < +\infty\}$ is the set of all functions on $\mathbb{D}$ with finite Dirichlet energy [14].

So (6) yields a solution u of (4) and it is clear that every solution of (6) has finite Dirichlet energy $\mathrm{D}[u]$. Since u is harmonic in $\mathbb{D}$, it is easy to verify that the Dirichlet energy of u can also be expressed by $\mathrm{D}[u] = \sum_{n\in\mathbb{Z}} |n|\, |c_n(u)|^2$ in terms of its Fourier coefficients. Because u coincides with g on the boundary of $\mathbb{D}$, it follows that also the Dirichlet energy of the boundary function g has to be finite, i.e.

$$\mathrm{D}[u] = \mathrm{D}[g] = \textstyle\sum_{n\in\mathbb{Z}} |n|\, |c_n(g)|^2 =: \|g\|^2_{\mathrm{E}} < \infty\ ,$$

wherein $\|g\|_{\mathrm{E}} := \|g\|_{\beta=0}$ stands for the seminorm of g defined in (2) with $\beta = 0$. This illustrates the importance of the space $\mathcal{E}_0$. It contains all continuous functions with finite Dirichlet energy, i.e. all functions g for which the Dirichlet problem (4) has a solution of finite energy. Moreover, it follows from (3) that every space $\mathcal{E}_\beta$ with $\beta \geq 0$ contains functions of finite Dirichlet energy. The parameter β characterizes only how good this energy is concentrated in its Fourier coefficients. The larger β the better concentrated is the energy of f in its Fourier coefficients $c_n(f)$ with small index n. We finally notice that $\mathcal{E}_0$ coincides with the *Sobolev space* $H^{1/2}(\mathbb{T}) = W^{1/2,2}(\mathbb{T})$.

3. SAMPLING–BASED APPROXIMATION

We are going to investigate sampling-based approximation methods on the signal spaces $\mathcal{E}_\beta$. Each approximation method consists of a sequence $\boldsymbol{A} = \{\mathrm{A}_N\}_{N\in\mathbb{N}}$ of operators $\mathrm{A}_N : \mathcal{E}_\beta \to \mathcal{E}_\beta$ concentrated on certain sampling sets $\mathcal{Z}_N$. We wish that $\mathrm{A}_N(f)$ converges to f as $N \to \infty$ for every $f \in \mathcal{E}_\beta$, and it is only required that every $\boldsymbol{A}$ satisfies the two simple properties given in the following definition.

Definition 1: *Let $\beta \geq 0$ be arbitrary and let $\boldsymbol{A} = \{\mathrm{A}_N\}_{N\in\mathbb{N}}$ be a sequence of lower semicontinuous operators $\mathrm{A}_N : \mathcal{E}_\beta \to \mathcal{E}_\beta$. We say that $\boldsymbol{A}$ is a* sampling-based approximation method *on $\mathcal{E}_\beta$ if it satisfies the following two properties.*

(A) To every $N \in \mathbb{N}$ there exists a finite set $\mathcal{Z}_N \subset \mathbb{T}$ such that for all $f_1, f_2 \in \mathcal{E}_\beta$

$$f_1(\tau) = f_2(\tau) \qquad \textit{for all } \tau \in \mathcal{Z}_N$$
$$\textit{implies } [\mathrm{A}_N(f_1)](t) = [\mathrm{A}_N(f_2)](t) \quad \textit{for all } t \in \mathbb{T}\ .$$

(B) There exists a dense subset $\mathcal{M} \subset \mathcal{E}_\beta$ such that

$$\lim_{N\to\infty} \|f - \mathrm{A}_N(f)\|_{\mathcal{E}_\beta} = 0 \qquad \textit{for all } f \in \mathcal{M}\ .$$

The first condition requires that each operator A_N processes only samples of f taken on the finite sampling set $\mathcal{Z}_N$. The second property requires that $\mathrm{A}_N(f)$ converges to f for all functions f in a dense subset of $\mathcal{E}_\beta$. Since we are interested in methods $\boldsymbol{A}$ which converge for all $f \in \mathcal{E}_\beta$, Property (B) constitutes no restriction on the considered approximation methods. Moreover, we would like to emphasis that Definition 1 makes no assumption on the linearity of the operators A_N. So each A_N might be non-linear, in general.

4. ENERGY BLOWUP

We are going to prove (in Sec. 5) the following statement showing that for any sampling-based approximation method $\boldsymbol{A} = \{\mathrm{A}_N\}_{N\in\mathbb{N}}$ on $\mathcal{E}_\beta$ with $\beta \in [0, 1]$, the Dirichlet energy of the approximation $\mathrm{A}_N(f)$ may get arbitrarily large as N goes to infinity.

Theorem 1: *Let $0 \leq \beta \leq 1$ be arbitrary and let $\boldsymbol{A} = \{\mathrm{A}_N\}_{N\in\mathbb{N}}$ be a sampling-based approximation method satisfying Properties (A) and (B) of Definition 1. Then the set*

$$\mathcal{R}(\boldsymbol{A}) = \left\{f \in \mathcal{E}_\beta\ :\ \limsup\nolimits_{N\to\infty} \|\mathrm{A}_N(f)\|_{\mathrm{E}} = +\infty\right\}$$

is a residual set in $\mathcal{E}_\beta$, i.e. its complement is a meager set.

Since $\mathrm{D}[f] = \|f\|^2_{\mathrm{E}} \leq \|f\|^2_\beta$ for every $f \in \mathcal{B}_\beta$ with $\beta \geq 0$, Theorem 1 implies in particular divergence in the norm of $\mathcal{B}_\beta$, i.e.

$$\limsup_{N\to\infty} \|\mathrm{A}_N(f)\|_{\mathcal{E}_\beta} = +\infty \qquad \text{for all } f \in \mathcal{R}(\boldsymbol{A})\ .$$

Assume $\boldsymbol{A} = \{\mathrm{A}_N\}_{N\in\mathbb{N}}$ is an arbitrary given approximation method for reconstructing functions from their samples and which satisfies the conditions of Definition 1. Then there always exist functions $f \in \mathcal{E}_\beta$ which have (by the definition of $\mathcal{E}_\beta$) finite Dirichlet energy $\|f\|^2_{\mathrm{E}} < +\infty$ but such that the Dirichlet energy of the approximation $\|\mathrm{A}_N(f)\|^2_{\mathrm{E}}$ can not be controlled. Thus the Dirichlet energy of the approximation exceeds any given bound $M > 0$ as the approximation degree N gets sufficiently large. This happens even though we want that the approximation error decreases as N increases, which would imply $\lim_{N\to\infty} \|\mathrm{A}_N(f)\|_{\mathrm{E}} = \|f\|_{\mathrm{E}} < +\infty$. So Theorem 1 shows in particular that for every $f \in \mathcal{R}(\boldsymbol{A})$ always $\lim_{N\to\infty} \|f - \mathrm{A}_N(f)\|_{\mathrm{E}} \neq 0$ hold. Nevertheless, it is worth to notice that without aiming to control the energy of the approximation $\mathrm{A}_N(f)$, it will always be possible to find approximation methods such that $\lim_{N\to\infty} \|f - \mathrm{A}_N(f)\|_\infty = 0$ for all $f \in \mathcal{C}(\mathbb{T})$. Examples include the interpolation by trigonometric polynomials [15, 16] or by splines [4, 17]. So controlling the energy of a sampling-based approximation process seems to be much harder than controlling the maximum norm.

Application: Solution of the Dirichlet problem As an application, we consider the problem of calculating the solution of the Dirichlet problem (4) discussed in the last paragraph of Section 2.

Assume that for $\beta \in [0, 1]$, $g \in \mathcal{E}_\beta$ is the given boundary function of the Dirichlet problem (4). To calculate its solution on a digital computer, only finitely many values $\mathcal{W}_N(g) = \{g(\tau) : \tau \in \mathcal{Z}_N\}$

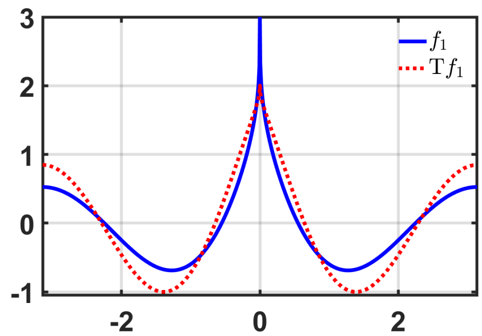

Fig. 1. The function $f_1 \in E_0$ as defined in (10) and the corresponding function $\mathrm{T}f_1 \in \mathcal{C}(\mathbb{T})$

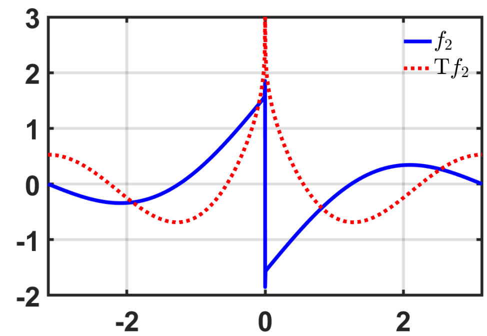

Fig. 2. The function $f_2 \in E_0 \cap L^\infty(\mathbb{T})$, defined in (11), for which $\mathrm{T}f_2 \notin L^\infty(\mathbb{T})$.

of g can be taken into account. Then a common way to determine the solution of (4) goes a follows: 1) Interpolate the given samples of g with an appropriate continuous function (e.g. with a spline or a trigonometric polynomial) to obtain an approximation g_N of g. 2) Calculate the Poisson integral (5) of g_N to obtain an approximation $u_N = \mathrm{P} g_N$ of the solution u. In other words, choose an approximation method $\boldsymbol{A} = \{\mathrm{A}_N\}_{N\in\mathbb{N}}$ as in Definition 1 and calculate

$$u_N(z) = \big(\mathrm{P}\left[\mathrm{A}_N(g)\right]\big)(z)\,, \qquad z \in \mathbb{D}\,. \tag{7}$$

However, even for this appealing method, Theorem 1 implies that it is impossible to control the Dirichlet energy $\|u_N\|_{\mathrm{E}}$ of the solution u_N. More precisely, we have the following statement.

Corollary 2: *Let $0 \leq \beta \leq 1$ be arbitrary and let $\boldsymbol{A} = \{\mathrm{A}_N\}_{N\in\mathbb{N}}$ be a sampling-based approximation method satisfying Properties (A) and (B) of Definition 1. Then the set*

$$\left\{g \in \mathcal{E}_\beta \;:\; \limsup_{N\to\infty} \left\|\mathrm{P}\left[\mathrm{A}_N(g)\right]\right\|_{\mathrm{E}} = +\infty\right\}$$

is a residual set in $\mathcal{E}_\beta$.

So even though for ever $g \in \mathcal{E}_\beta$, the Dirichlet problem (4) has a solution of finite Dirichlet energy, any algorithm of the form (7), based on samples of g, will fail to calculate a finite energy solution for almost each function in $\mathcal{E}_\beta$.

5. PROOFS AND AUXILIARY RESULTS

We are going to prove Theorem 1. To this end, some preliminary results are needed, presented and discussed in the first paragraph. The proof of Theorem 1 is then given in the second paragraph.

Auxiliary results We start by defining a particular linear operator $\mathrm{T} : \mathcal{E}_\beta \to \mathcal{C}(\mathbb{T})$ which will be of importance in the proof of Theorem 1. Let $0 \leq \beta \leq 1$ be arbitrary and let $f \in \mathcal{E}_\beta$ be arbitrary with Fourier coefficients $\{c_n(f)\}_{n\in\mathbb{Z}}$ as given in (1). Then we define $\mathrm{T} : \mathcal{E}_\beta \to \mathcal{C}(\mathbb{T})$ by

$$(\mathrm{T}f)(t) = -\sum_{n=-\infty}^{-2} \frac{c_n(f)}{\log|n|}\,\mathrm{e}^{\mathrm{i}nt} + \sum_{n=2}^{\infty} \frac{c_n(f)}{\log n}\,\mathrm{e}^{\mathrm{i}nt}, \quad t \in \mathbb{T}. \tag{8}$$

It will be important that the so defined operator is bounded.

Lemma 3: *Let $0 \leq \beta \leq 1$ be arbitrary. Then $\mathrm{T} : \mathcal{E}_\beta \to \mathcal{C}(\mathbb{T})$, as defined in* (8)*, is bounded and there exists a constant K_1 such that*

$$\|\mathrm{T}f\|_\infty \leq K_1 \|f\|_{\mathrm{E}} \leq K_1 \|f\|_\beta \qquad \textit{for all } f \in \mathcal{E}_\beta. \tag{9}$$

Proof: Let $f \in \mathcal{E}_\beta$ be arbitrary and set $g = \mathrm{T}f$. Then

$$|(\mathrm{T}f)(t)| = |g(t)| \leq \textstyle\sum_{n\in\mathbb{Z}} |c_n(g)| = \sum_{|n|\geq 2} \frac{|c_n(f)|}{\log|n|}$$

for all $t \in \mathbb{T}$ and Cauchy–Schwarz inequality yields

$$\begin{aligned} |g(t)| &\leq \Big(\textstyle\sum_{|n|\geq 2} \frac{1}{|n|(\log|n|)^2}\Big)^{1/2} \Big(\textstyle\sum_{|n|\geq 2} |n|\,|c_n(f)|^2\Big)^{1/2} \\ &\leq K_1 \|f\|_{\mathrm{E}} \leq K_1 \|f\|_\beta \leq K_1 \|f\|_{\mathcal{E}_\beta} \quad \text{ for every } t \in \mathbb{T}. \end{aligned}$$

This proves (9) with $K_1 = \big(2\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}\big)^{1/2} < \infty$ and shows that $\mathrm{T} : \mathcal{E}_\beta \to \mathcal{C}(\mathbb{T})$ is bounded. ■

We notice that T, as defined in (8), shows a very interesting behavior on the set $E_0 = \{f : \mathbb{T} \to \mathbb{C} \text{ with } \|f\|_{\mathrm{E}} < +\infty\}$ of all functions on $\mathbb{T}$ with finite Dirichlet energy. On the one hand, there exist functions $f \in E_0$ which are unbounded but for which $\mathrm{T}f$ is bounded on $\mathbb{T}$. One examples is the function

$$f_1(t) = \sum_{n=2}^{\infty} \frac{\cos(nt)}{n\log n} \quad \text{with} \quad (\mathrm{T}f_1)(t) = \mathrm{i}\sum_{n=2}^{\infty} \frac{\sin(nt)}{n(\log n)^2}. \tag{10}$$

The so defined f_1 is in $L^1(\mathbb{T})$ and continuous for $t \neq 0$ (cf. [18, Thm. 1.8, Chap. 5.1] but it is unbounded since $f_1(t) \to \infty$ as $|t| \to 0$. Nevertheless, one easily verifies that $\mathrm{T}f_1$ satisfies $\sum_{n\in\mathbb{Z}} |c_n(\mathrm{T}f_1)| = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2} < +\infty$, showing that $\mathrm{T}f_1$ belongs to the Wiener algebra and in particular showing that $\mathrm{T}f_1 \in \mathcal{C}(\mathbb{T})$. So T maps an unbounded function onto a bounded function. This property of T is visualized in Fig. 1.

On the other hand, there are bounded functions $f \in E_0$ such that $\mathrm{T}f$ is unbounded. This is illustrated by the following examples

$$f_2(t) = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \quad \text{with} \quad (\mathrm{T}f_2)(t) = \frac{1}{\mathrm{i}}\sum_{n=2}^{\infty} \frac{\cos(nt)}{n(\log n)} \tag{11}$$

and the corresponding graphs are plotted in Fig. 2.

Apart from the operator T, the following lemma, taken from [19], will be of central importance in the proof of Theorem 1.

Lemma 4 (Interpolation lemma): *Let $0 \leq \beta \leq 1$ be arbitrary and let $\mathcal{Z} \subset \mathbb{T}$ be a finite sampling set. To every $g \in \mathcal{C}(\mathbb{T})$ there exists an $f \in \mathcal{E}_\beta$ with $\|f\|_{\mathcal{E}_\beta} \leq 2\|g\|_\infty$ which satisfies*

$$f(\tau) = g(\tau) \qquad \textit{for all } \tau \in \mathcal{Z}\,.$$

Remark: We notice that the corresponding result in [19, Corollary 5] makes a much stronger statement since it gives also a bound on the maximum norm of the conjugate of the interpolating function f, which is not needed for our actual problem.

Sketch of proof of Theorem 1 For simplicity and because of space constraints, we give a full proof of Theorem 1 only under the additional assumption that all operators $\mathrm{A}_N : \mathcal{E}_\beta \to \mathcal{E}_\beta$ in the sequence $\boldsymbol{A} = \{\mathrm{A}_N\}_{N\in\mathbb{N}}$ are linear. Subsequent to this proof, we will comment on necessary extensions needed to prove the theorem also for non-linear operators A_N.

Proof: We fix an arbitrary $\beta \in [0,1]$ and prove the theorem by contradiction. Assume the theorem is false, i.e. assume the set $\left\{f \in \mathcal{E}_\beta \,:\, \limsup_{N\to\infty} \|\mathrm{A}_N f\|_{\mathrm{E}} < \infty\right\}$ is a residual set. Then the uniform boundedness principle for *linear* operators (Banach–Steinhaus Theorem) implies that there is a constant $C_{\boldsymbol{A}}$ so that

$$\|\mathrm{A}_N f\|_{\mathrm{E}} \leq C_{\boldsymbol{A}} \|f\|_{\mathcal{E}_\beta} \qquad \text{for all } f \in \mathcal{E}_\beta \,. \tag{12}$$

1) Let $\mathrm{T} : \mathcal{E}_\beta \to \mathcal{C}(\mathbb{T})$ be given as in (8). Therewith, we define a sequence $\boldsymbol{T} = \{\mathrm{T}_N\}_{N\in\mathbb{N}}$ of operators $\mathrm{T}_N : \mathcal{E}_\beta \to \mathcal{C}(\mathbb{T})$ by

$$\mathrm{T}_N f = \mathrm{T}\mathrm{A}_N f \,, \qquad f \in \mathcal{E}_\beta \,. \tag{13}$$

Applying Lemma 3, one gets from (12)

$$\begin{aligned}\|\mathrm{T}_N f\|_\infty = \|\mathrm{T}\mathrm{A}_N f\|_\infty \leq K_1 \|\mathrm{A}_N f\|_{\mathrm{E}} \leq K_1 C_{\boldsymbol{A}} \|f\|_{\mathcal{E}_\beta} \\ \leq C_{\boldsymbol{T}} \|f\|_{\mathcal{E}_\beta} \qquad \text{for all } f \in \mathcal{E}_\beta\end{aligned} \tag{14}$$

and for every $N \in \mathbb{N}$ and with the constant $C_{\boldsymbol{T}} = K_1\, C_{\boldsymbol{A}}$.

2) Now we pick an arbitrary $N \in \mathbb{N}$ with corresponding sampling set $\mathcal{Z}_N$ and we choose an arbitrary $f \in \mathcal{C}(\mathbb{T})$. According to Lemma 4, there exists an $f_N \in \mathcal{E}_\beta$ such that

$$\|f_N\|_{\mathcal{E}_\beta} \leq 2\,\|f\|_\infty \quad \text{and} \quad f_N(\tau) = f(\tau) \quad \text{for all } \tau \in \mathcal{Z}_N \,.$$

By the definition of T_N in (13), it is clear that T_N is concentrated on $\mathcal{Z}_N$. So (14) implies in particular that $\|\mathrm{T}_N f\|_\infty = \|\mathrm{T}_N f_N\|_\infty \leq C_{\boldsymbol{T}} \|f_N\|_{\mathcal{E}_\beta} \leq 2\, C_{\boldsymbol{T}} \|f\|_\infty$ for all $f \in \mathcal{C}(\mathbb{T})$, i.e.

$$\sup_{N\in\mathbb{N}} \|\mathrm{T}_N\|_{\mathcal{C}(\mathbb{T})\to\mathcal{C}(\mathbb{T})} \leq 2\, C_{\boldsymbol{T}} \,. \tag{15}$$

3) Let $\mathcal{M} \subset \mathcal{E}_\beta$ be the subset from Property (B) of $\boldsymbol{A}$. Then the boundedness of $\mathrm{T} : \mathcal{E}_\beta \to \mathcal{C}(\mathbb{T})$ (cf. Lemma 3) yields

$$\|\mathrm{T}_N f - \mathrm{T} f\|_\infty = \|\mathrm{T}\,(\mathrm{A}_N f - f)\|_\infty \leq \|\mathrm{T}\|\, \|\mathrm{A}_N f - f\|_{\mathcal{E}_\beta}$$

and so Property (B) of the sequence $\{\mathrm{A}_N\}_{N\in\mathbb{N}}$ implies

$$\lim_{N\to\infty} \|\mathrm{T}_N f - \mathrm{T} f\|_\infty = 0 \qquad \text{for all } f \in \mathcal{M} \,. \tag{16}$$

Moreover, since $\mathcal{M}$ is dense in $\mathcal{E}_\beta$, and $\mathcal{E}_\beta$ is dense in $\mathcal{C}(\mathbb{T})$, it easily follows that $\mathcal{M}$ is dense in $\mathcal{C}(\mathbb{T})$. So (15) together with (16) imply that $\mathrm{T} : \mathcal{C}(\mathbb{T}) \to \mathcal{C}(\mathbb{T})$ is a bounded linear operator with

$$\|\mathrm{T} f\|_\infty \leq 2\, C_{\boldsymbol{T}} \|f\|_\infty \qquad \text{for all } f \in \mathcal{C}(\mathbb{T}) \,. \tag{17}$$

4) Nevertheless, $\mathrm{T} : \mathcal{C}(\mathbb{T}) \to \mathcal{C}(\mathbb{T})$ is not bounded. To see this, we consider for arbitrary $L \in \mathbb{N}$ the trigonometric polynomial

$$\varphi_L(t) = \textstyle\sum_{n=1}^{L} \frac{\sin(nt)}{n} \,, \qquad t \in \mathbb{T} \,.$$

It is known [18] that there exists a constant $C_1 < +\infty$ such that $\|\varphi_L\|_\infty \leq C_1$ for all $L \in \mathbb{N}$. So (17) would give

$$\|\mathrm{T}\varphi_L\|_\infty \leq 2\, C_{\boldsymbol{T}}\, C_1 \qquad \text{for all } L \in \mathbb{N} \,. \tag{18}$$

However, since $(\mathrm{T}\varphi_L)(t) = \frac{1}{\mathrm{i}} \sum_{n=2}^{L} \frac{\cos(nt)}{n \log n}$, one obtains

$$\|\mathrm{T}\varphi_L\|_\infty \geq |(\mathrm{T}\varphi_L)(0)| = \textstyle\sum_{n=2}^{L} \frac{1}{n \log n} \geq \log\log L \,.$$

This contradicts (18) for sufficiently large L and finishes the proof.■

The previous proof assumed that the operators $\mathrm{A}_N : \mathcal{E}_\beta \to \mathcal{E}_\beta$ are all linear. This linearity allowed us to apply the uniform boundedness principle (Theorem of Banach–Steinhaus) to obtain (12). Nevertheless, for non-linear operators A_N this theorem has to be replaces by the *generalized uniform boundedness principle*. Therewith one obtains instead of (12) an inequality which does not hold for all $f \in \mathcal{E}_\beta$ but only for all f in a specific open ball in $\mathcal{E}_\beta$. Then additional steps are necessary to extend, in some sense, this inequality again onto the whole space. These steps are almost exactly the same as in the proof of [20, Theorem 3.1] and omitted here. Moreover, we refer to [20] for a precise formulation of the generalized uniform boundedness principle together with a corresponding proof, and a discussion on its relation to the uniform boundedness principle for linear operators.

6. DISCUSSION AND OUTLOOK

This paper investigated the possibility of approximating 2π-periodic continuous functions from discrete samples. There already exists an extensive literature on sampling-based signal processing methods [21–24]. These works usually aim to control the approximation error in the infinity norm and there are many concrete sampling-based approximation techniques $\{\mathrm{A}_N\}_{N\in\mathbb{N}}$ such that

$$\lim_{N\to\infty} \|f - \mathrm{A}_N(f)\|_\infty = 0 \qquad \text{for all } f \in \mathcal{C}(\mathbb{T}) \,.$$

One very well known and often used example is the B-spline interpolation [17, 25–28].

However, even thought the approximation in the maximum norm is a reasonable approach for continuous functions, it is often important to control additionally other physical quantities of the approximation $\mathrm{A}_N(f)$ such as the energy $\|\mathrm{A}_N(f)\|_{\mathrm{E}}$. This paper considered subspaces $\mathcal{E}_\beta \subset \mathcal{C}(\mathbb{T})$ of continuous functions with finite (Dirichlet) energy and showed that there exists *no* sampling-based approximation process which is able to control the energy of the approximation for all $f \in \mathcal{E}_\beta$ even thought there are approximation processes (e.g. approximations by splines or trigonometric polynomials) which do converge for all $f \in \mathcal{E}_\beta$ in the norm of $\mathcal{C}(\mathbb{T})$.

We note that a similar complicated behavior was observed for approximations of the Hilbert transform of continuous functions with finite energy [19, 20]. Together with the present paper, these works illustrate fundamental limitations of sampling-based signal processing methods. In particular, they show that continuous signals of finite energy are too complex to allow for a sampling-based approximation methods which can control the energy of the approximation. Based on a recent approach to classify the complexity of certain optimization problems [29, 30], a complete characterization of the complexity of sampling-based methods for approximation the Hilbert transform of continuous signals with finite energy was achieved in [31]. It seems an interesting open problem to obtain a similar characterization for the complexity of sampling-based approximation methods for continuous signals with finite energy.

We notice finally that Theorem 1 holds not only for continuous functions with finite energy but for every $\mathcal{E}_\beta$ with $0 < \beta \leq 1$, i.e. for all functions which satisfy additionally a certain energy concentration. It is an interesting problem to find sufficient condition on the energy concentration such that sampling-based approximation methods do exist. Our previous work [32] indicates that on spaces $\mathcal{E}_\beta$ with $\beta > 1$ sampling-based approximation might exist. However, a formal proof of this conjecture is part of our future research.

7. REFERENCES

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