ANALYTIC PROPERTIES OF DOWNSAMPLING FOR BANDLIMITED SIGNALS

Holger Boche and Ullrich J. Mönich

Technische Universität München
Lehrstuhl für Theoretische Informationstechnik

ABSTRACT

In this paper we study downsampling for bandlimited signals. Downsampling in the discrete-time domain corresponds to a removal of samples. For any downsampled signal that was created from a bandlimited signal with finite energy, we can always compute a bandlimited continuous-time signal such that the samples of this signal, taken at Nyquist rate, are equal to the downsampled discrete-time signal. However, as we show, this is no longer true for the space of bounded bandlimited signals that vanish at infinity. We explicitly construct a signal in this space, which after downsampling does not have a bounded bandlimited interpolation. This shows that downsampling in this signal space is an operation that can lead out of the set of discrete-time signals for which we have a one-to-one correspondence with continuous-time signals.

***Index Terms*—** bandlimited signal, downsampling, bandlimited interpolation, boundedness

1. INTRODUCTION

Downsampling or decimation is a fundamental operation in signal processing that is used in numerous applications, for example in filter banks [1, 2], image processing [3–5], and communication systems [6, 7]. Downsampling is the process of reducing the sampling rate of a discrete-time signal by removing samples. In this work we consider only one-dimensional downsampling. If we downsample a signal $\{x_k\}_{k\in\mathbb{Z}}$ by a factor of two, we only keep the samples $\{x_{2k}\}_{k\in\mathbb{Z}}$, and the downsampled signal is given by $\{x_k^{\text{down}}\}_{k\in\mathbb{Z}} = \{x_{2k}\}_{k\in\mathbb{Z}}$. In practice, often the discrete-time signal $\{x_k\}_{k\in\mathbb{Z}}$ is obtained by sampling a bandlimited continuous-time signal f. If f is has finite energy and no frequencies larger than 2π, then we can reconstruct the continuous-time signal f from the samples $\{f(k/2)\}_{k\in\mathbb{Z}}$ by means of the Shannon sampling series. In this paper we analyze whether this is still true for the downsampled signal $\{x_k^{\text{down}}\}_{k\in\mathbb{Z}} = \{f(k)\}_{k\in\mathbb{Z}}$, i.e., if we can find a signal f_π with bandwidth π that interpolates the downsampled signal $\{x_k^{\text{down}}\}_{k\in\mathbb{Z}}$, i.e., satisfies $f_\pi(k) = x_k^{\text{down}}, k \in \mathbb{Z}$. In the literature such a signal is known as the bandlimited interpolation [8–11].

The correspondence between discrete-time and continuous-time signals is a useful property in signal processing. For any discrete-time signal we would like to have a bounded bandlimited continuous-time signal, the samples of which taken at Nyquist rate are equal to the discrete-time signal. In other words, we want to be able to identify any discrete-time signal with a bounded bandlimited continuous-time signal.

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Whether or not this is possible, clearly depends on the properties of the discrete-time signal. For example, if the discrete-time signal was obtained by sampling a bandlimited signal with finite energy, then we can use the Shannon sampling series to uniquely recover the bandlimited continuous-time signal. More precisely, a finite energy signal f with bandwidth π can be reconstructed from the samples $\{f(k)\}_{k\in\mathbb{Z}}$ by means of the Shannon sampling series

$$f(t) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}. \tag{1}$$

Further, for any sequence $\{x_k\}_{k\in\mathbb{Z}}$ with finite energy, there exists exactly one signal f with finite energy and bandwidth π such that $f(k) = x_k$ for all $k \in \mathbb{Z}$. Again, this signal can be obtained by using the Shannon sampling series.

In general, the analysis of downsampling as a signal processing operation is not given much attention because it is assumed that this procedure does not lead to any fundamental problems. In many signal processing books [8, p. 52 and p. 162] and [9, p. 144], the bandlimited interpolation, i.e., the continuous-time signal that corresponds to the downsampled sequence, is formally obtained by using a convolution theorem and distribution theory. First, the discrete-time sequence of a continuous-time signal f is created by multiplying f with a Dirac comb

$$f_{Ш}(t) = f(t)\cdot Ш(t) = f(t) \cdot \sum_{k=-\infty}^{\infty} \delta(t-k) = \sum_{k=-\infty}^{\infty} f(k)\delta(t-k).$$

Then, the bandlimited interpolation is obtained by convolving $f_{Ш}$ with the impulse response of the ideal low-pass filter

$$f_\pi(t) = (f_{Ш} * \operatorname{sinc})(t) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}. \tag{2}$$

It is assumed that all above operations are well-defined, at least if they are treated in a distributional setting [8,12,13]. The use of distributions has a long history in engineering [14–19], however, sometimes distributions are used only formally without a proper mathematical justification. For example, for signal spaces other than the space of bandlimited signals with finite energy, it is a priori not clear whether the above manipulations and expressions are well-defined, even when they are treated in the sense of distributions [20]. In [20] it has been shown, in the context of sampling based system representations, that the sequence of partial sums diverges for certain signals and systems, even when it is treated in a distributional sense. The result was generalized in [21] by proving that for certain spaces of continuous signals with finite energy it is impossible to define a distributional convolution sum system representation. This shows that the above calculations based on distributions that led to (2), are not true in the full generality that is claimed in some books [8, 13].

2. NOTATION

By c_0 we denote the set of all sequences that vanish at infinity, and by $C_0^\infty[0,1]$ the space of all functions that have continuous derivatives of all orders and are zero outside $[0,1]$. For $\Omega \subseteq \mathbb{R}$, let $L^p(\Omega)$, $1 \le p < \infty$, be the space of all measurable, pth-power Lebesgue integrable functions on Ω, with the usual norm $\|\cdot\|_p$, and $L^\infty(\Omega)$ the space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite. The Bernstein space $\mathcal{B}_\sigma^p$, $\sigma > 0$, $1 \le p \le \infty$, consists of all functions of exponential type at most σ, whose restriction to the real line is in $L^p(\mathbb{R})$ [22, p. 49]. The norm for $\mathcal{B}_\sigma^p$ is given by the L^p-norm on the real line. A function in $\mathcal{B}_\sigma^p$ is called bandlimited to σ. $\mathcal{B}_{\sigma,0}^\infty$ denotes the space of all functions in $\mathcal{B}_\sigma^\infty$ that vanish at infinity. By $\mathcal{PW}_\sigma^p$, $1 \le p \le \infty$, we denote the Paley–Wiener space of functions f with a representation $f(z) = 1/(2\pi)\int_{-\sigma}^{\sigma} g(\omega)\,\mathrm{e}^{iz\omega}\ \mathrm{d}\omega$, $z \in \mathbb{C}$, for some $g \in L^p[-\sigma,\sigma]$. If $f \in \mathcal{PW}_\sigma^p$ then $g(\omega) = \hat{f}(\omega)$. The norm for $\mathcal{PW}_\sigma^p$ is given by $\|f\|_{\mathcal{PW}_\sigma^p} = (1/(2\pi)\int_{-\sigma}^{\sigma}|\hat{f}(\omega)|^p\ \mathrm{d}\omega)^{1/p}$. $\mathcal{PW}_\sigma^2$ is the frequently used space of bandlimited functions with bandwidth σ and finite energy.

Distributions are continuous linear functionals on a space of test functions. $\mathcal{D}$ is the space of all test functions $\phi : \mathbb{R} \to \mathbb{C}$ that have continuous derivatives of all orders and are zero outside some finite interval. $\mathcal{D}'$ denotes the dual space of $\mathcal{D}$, i.e., the space of all distributions that can be defined on $\mathcal{D}$. For locally integrable functions g we can define the linear functional

$$\phi \mapsto \int_{-\infty}^{\infty} g(t)\phi(t)\ \mathrm{d}t \tag{3}$$

on the space $\mathcal{D}$. It can be proven that this functional is continuous and thus defines a distribution [23]. Distributions of the type (3) are called regular distributions. A sequence of distributions $\{f_k\}_{k\in\mathbb{N}}$ in $\mathcal{D}'$ is said to converge in $\mathcal{D}'$ if for every $\phi \in \mathcal{D}$ the sequence of numbers $\{f_k\phi\}_{k\in\mathbb{N}}$ converges. Thus, a sequence of regular distributions, which is induced by a sequence of functions $\{g_k\}_{k\in\mathbb{N}}$ according to (3), converges in $\mathcal{D}'$ if for every $\phi \in \mathcal{D}$ the sequence of numbers $\{\int_{-\infty}^{\infty} g_k(t)\phi(t)\ \mathrm{d}t\}_{k\in\mathbb{N}}$ converges.

3. PRELIMINARY CONSIDERATIONS

Let $f \in \mathcal{PW}_{2\pi}^2$ be a bandlimited signal with bandwidth 2π and finite energy. Then f is completely determined by its samples $\{f(\frac{k}{2})\}_{k\in\mathbb{Z}}$. Removing every second sample, i.e., keeping only the samples $\{x_k^{\text{down}}\}_{k\in\mathbb{Z}} = \{f(k)\}_{k\in\mathbb{Z}}$ corresponds to a downsampling factor of two. The continuous-time signal f_π that corresponds to the downsampled discrete-time signal $\{x_k^{\text{down}}\}_{k\in\mathbb{Z}}$ is given by

$$f_\pi(t) = \sum_{k=-\infty}^{\infty} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}, \quad t \in \mathbb{R}. \tag{4}$$

The series in (4) converges in the L^2-norm, as well as uniformly on the real axis, and we have $f_\pi \in \mathcal{PW}_\pi^2 \subset \mathcal{PW}_{2\pi}^2$. Hence, for the signal space $\mathcal{PW}_{2\pi}^2$ downsampling and bandlimited interpolation of the downsampled signal are well-defined.

However, downsampling is often used also for other, larger signal spaces, like $\mathcal{B}_{2\pi,0}^\infty$ or $\mathcal{B}_{2\pi}^\infty$, both of which are important for example in communications. In the present paper we study downsampling for these larger spaces. A signal $f \in \mathcal{B}_{2\pi,0}^\infty$ is uniquely determined by its samples $f(\frac{k}{2})$, $k \in \mathbb{Z}$, and for all $T > 0$ we have

$$\lim_{N\to\infty}\max_{[-T,T]}\left|f(t) - \sum_{k=-N}^{N} f\left(\frac{k}{2}\right)\frac{\sin(2\pi(t-\frac{k}{2}))}{2\pi(t-\frac{k}{2})}\right| = 0,$$

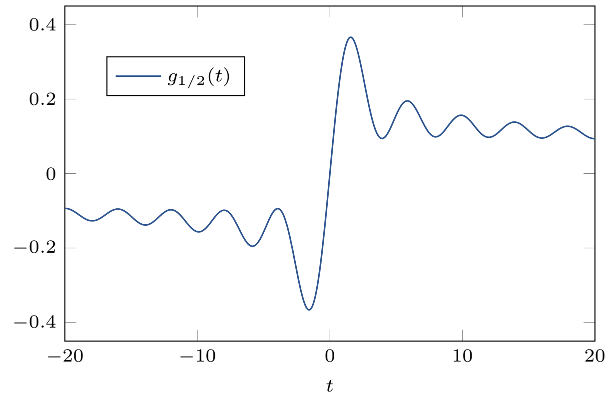

Fig. 1. Plot of the signal $g_\delta(t)$ for $\delta = 1/2$.

i.e., the Shannon sampling series converges locally uniformly to the signal f [24]. Further we have $\{f(\frac{k}{2})\}_{k\in\mathbb{Z}} \in c_0$. Clearly, the downsampled discrete-time signal also satisfies $\{f(k)\}_{k\in\mathbb{Z}} \in c_0$. However, the question is whether a continuous-time signal f_π that interpolates $\{f(k)\}_{k\in\mathbb{Z}}$ can be constructed, and if yes if it is still a well-behaved signal.

It is well-known that there exist sequences that do not possess a bounded bandlimited interpolation. For example, for the sequence

$$x_k = \begin{cases} 0, & k \le 0, \\ \frac{(-1)^k}{\log(1+k)}, & k \ge 1, \end{cases}$$

there exists no signal $f_\pi \in \mathcal{B}_\pi^\infty$ with $f_\pi(k) = x_k$ for all $k \in \mathbb{Z}$ [25]. This already shows that the distributional approach in [8, 9], which we discussed in Section 1, is flawed.

Note that the situation that is analyzed in the present paper is more complicated. Here, the sequence is not freely chosen but obtained by downsampling a bounded bandlimited signal. In fact, the signal that we will construct later is a bandpass signal with arbitrarily small effective bandwidth.

4. DOWNSAMPLING FOR BANDLIMITED SIGNALS

In the following two theorems the signal

$$\gamma_\delta(t) = \mathrm{e}^{i\pi t}\, g_\delta(t), \quad t \in \mathbb{R}, \tag{5}$$

with

$$g_\delta(t) = \frac{1}{\pi}\int_0^{\delta\pi} \frac{\sin(\omega t)}{\omega\log(\frac{\pi}{\omega})}\ \mathrm{d}\omega, \quad t \in \mathbb{R},$$

will play a central role. $\delta \in (0,1)$ is a parameter that specifies the bandwidth of the signal. The signal $g_{1/2}$ is visualized in Fig. 1. We postpone all proofs until Section 5.

Theorem 1. *Let $\delta \in (0,1)$, and let $\gamma_\delta \in \mathcal{B}_{(1+\delta)\pi,0}^\infty$ be the signal defined in* (5)*. There exists no $f_\pi \in \mathcal{B}_\pi^\infty$ with $f_\pi(k) = \gamma_\delta(k)$ for all $k \in \mathbb{Z}$. That is, there exists no bounded bandlimited interpolation for the downsampled sequence $\{\gamma_\delta(k)\}_{k\in\mathbb{Z}}$.*

As the next theorem shows, for the downsampled sequence $\{\gamma_\delta(k)\}_{k\in\mathbb{Z}}$, the Shannon sampling series diverges for all $t \in \mathbb{R}\setminus\mathbb{Z}$. Moreover, the divergence even holds in a distributional setting.

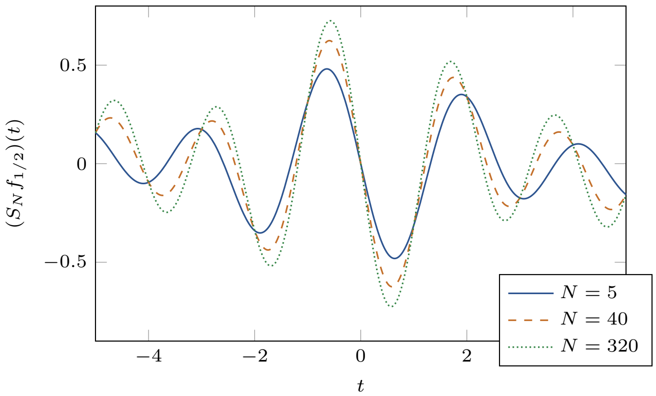

Fig. 2. Plot of the sums $(S_N\gamma_\delta)(t)$ for $\delta = 1/2$ and $N = 5, 40, 320$.

Theorem 2. *Let $\delta \in (0,1)$, and let $\gamma_\delta \in \mathcal{B}^\infty_{(1+\delta)\pi,0}$ be the signal defined in (5). Then, for all $t \in \mathbb{R}\setminus\mathbb{Z}$, we have*

$$\lim_{N\to\infty}\left|\sum_{k=-N}^{N}\gamma_\delta(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right| = \infty.$$

Further, there exists a $\phi_1 \in C_0^\infty[0,1]$ such that

$$\lim_{N\to\infty}\left|\int_{-\infty}^{\infty}\sum_{k=-N}^{N}\gamma_\delta(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\phi_1(t)\,\mathrm{d}t\right| = \infty,$$

i.e., the series diverges in $\mathcal{D}'$.

This shows that the distributional approach, discussed in the introduction, does not work, because the expression (2) diverges even in $\mathcal{D}'$. In order to illustrate the divergence observed in Theorem 2, the partial sums of the Shannon sampling series

$$(S_N\gamma_\delta)(t) = \sum_{k=-N}^{N}\gamma_\delta(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}$$

are plotted in Fig. 2 for $\delta = 1/2$ and $N = 5, 40, 320$.

Remark 1. The signal γ_δ has a remarkably simple structure. It is not constructed as an infinite series, but defined as a simple integral expression.

Remark 2. γ_δ is a bandpass signal that is created by modulating the lowpass signal g_δ. Since the spectrum of the lowpass signal g_δ is concentrated on $[-\delta\pi, \delta\pi]$, g_δ is completely determined by its samples $\{g_\delta(k/\delta)\}_{k\in\mathbb{Z}}$. Further, the effective bandwidth of the bandpass signal γ_δ is $2\delta\pi$.

5. PROOFS

We start with stating several properties of $g_\delta(t)$, which is illustrated in Fig. 1. In particular, it is important that g_δ, $\delta \in (0,1)$, is a bounded bandlimited signal that vanishes at infinity.

Lemma 1. *Let $\delta \in (0,1)$. Then we have $g_\delta \in \mathcal{B}^\infty_{\delta\pi,0}$. Further, g_δ satisfies $g_\delta(0) = 0$ and $g_\delta(t) = -g_\delta(-t)$ for all $t \in \mathbb{R}$.*

Due to space constraints we omit the proof of Lemma 1. For the proofs of our main theorems we need several auxiliary results. We start with a fact about the local behavior of the Shannon sampling series for signals in $\mathcal{B}^\infty_\pi$ [24, Theorem 1].

Fact 1. *For all $T > 0$ there exists a constant $C_1(T)$ such that for all $f \in \mathcal{B}^\infty_\pi$ and all $N \in \mathbb{N}$ we have*

$$\max_{t\in[-T,T]}\left|\sum_{k=-N}^{N} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right| \le C_1(T)\|f\|_{\mathcal{B}^\infty_\pi}.$$

Second, we need two facts about $\sum_{k=1}^\infty \sin(k\omega)/k$, which is the Fourier series of the function

$$u(\omega) = \begin{cases}\frac{1}{2}(\pi-\omega), & 0 < \omega < 2\pi,\\ 0, & \omega = 0 \text{ or } \omega = 2\pi.\end{cases}$$

On all closed intervals, excluding the jump discontinuities, we have uniform convergence [26, p. 4, Theorem 2.6].

Fact 2. *For all $\gamma > 0$ we have*

$$\lim_{N\to\infty}\max_{\omega\in[\gamma,2\pi-\gamma]}\left|u(\omega)-\sum_{k=1}^{N}\frac{\sin(k\omega)}{k}\right| = 0.$$

Further, the partial sums are strictly positive on the interval $(0,\pi)$ [26, p. 62, Theorem 9.4].

Fact 3. *For all $N \ge 1$ and all $\omega \in (0,\pi)$ we have*

$$\sum_{k=1}^{N}\frac{\sin(k\omega)}{k} > 0.$$

Now we are in the position to prove Theorem 1.

Sketch of the proof of Theorem 1. Let $\delta \in (0,1)$ be arbitrary but fixed and γ_δ the signal defined in (5). Then we have $\gamma_\delta \in \mathcal{B}^\infty_{(1+\delta)\pi,0}$. We further have $\gamma_\delta(k) = \mathrm{e}^{ik\pi}\, g_\delta(k) = (-1)^k g_\delta(k)$, $k \in \mathbb{Z}$. Thus, for $t \in \mathbb{R}\setminus\mathbb{Z}$, we obtain

$$\begin{aligned}(S_N\gamma_\delta)(t) &= \sum_{k=-N}^{N}(-1)^k g_\delta(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\\ &= \frac{\sin(\pi t)}{\pi}\sum_{k=1}^{N} g_\delta(k)\left(\frac{1}{t-k}-\frac{1}{t+k}\right),\end{aligned}$$

where we used that $\sin(\pi(t-k)) = (-1)^k\sin(\pi t)$. It follows that

$$\begin{aligned}&(S_N\gamma_\delta)(t) + \frac{\sin(\pi t)}{\pi}\sum_{k=1}^{N}\frac{2g_\delta(k)}{k}\\ &\quad= \frac{\sin(\pi t)}{\pi}\sum_{k=1}^{N} g_\delta(k)\left(\frac{t}{(t-k)k}+\frac{t}{(t+k)k}\right).\end{aligned}$$

For $t \in [1/4, 3/4]$, using basic calculations, it can be shown that

$$\left|(S_N\gamma_\delta)(t) + \frac{\sin(\pi t)}{\pi}\sum_{k=1}^{N}\frac{2g_\delta(k)}{k}\right| < \frac{12+\pi^2}{4\pi}\|g_\delta\|_{\mathcal{B}^\infty_{\delta\pi,0}}. \quad (6)$$

It follows that

$$(S_N\gamma_\delta)(t) \le -\frac{2\sin(\frac{\pi}{4})}{\pi}\sum_{k=1}^{N}\frac{g_\delta(k)}{k} + \frac{12+\pi^2}{4\pi}\|g_\delta\|_{\mathcal{B}^\infty_{\delta\pi,0}} \quad (7)$$

for all $t \in [1/4, 3/4]$, where in the last equality we used the fact that $\sin(\pi t) \ge \sin(\pi/4)$ for all $t \in [1/4, 3/4]$.

Let $0 < \gamma < \delta\pi$ be arbitrary. Then, according to Fact 3, we have

$$\sum_{k=1}^{N} \frac{g_\delta(k)}{k} = \frac{1}{\pi} \int_0^{\delta\pi} \frac{1}{\omega \log(\frac{\pi}{\omega})} \sum_{k=1}^{N} \frac{\sin(\omega k)}{k} \, \mathrm{d}\omega$$
$$\geq \frac{1}{\pi} \int_\gamma^{\delta\pi} \frac{1}{\omega \log(\frac{\pi}{\omega})} \sum_{k=1}^{N} \frac{\sin(\omega k)}{k} \, \mathrm{d}\omega.$$

Since, according to Fact 2, the series $\sum_{k=1}^{\infty} \sin(\omega k)/k \, \mathrm{d}\omega$ converges uniformly on $[\gamma, \delta\pi]$ to $(\pi - \omega)/2$, we obtain

$$\lim_{N\to\infty} \sum_{k=1}^{N} \frac{g_\delta(k)}{k} \geq \frac{1}{\pi} \int_\gamma^{\delta\pi} \frac{1}{\omega \log(\frac{\pi}{\omega})} \frac{1}{2}(\pi - \omega) \, \mathrm{d}\omega$$
$$> \frac{1}{2} \int_\gamma^{\delta\pi} \frac{1}{\omega \log(\frac{\pi}{\omega})} \, \mathrm{d}\omega - \frac{\delta}{2\log(\frac{1}{\delta})}$$
$$= \frac{1}{2} \log\left(\frac{\log(\frac{\pi}{\gamma})}{\log(\frac{1}{\delta})} \right) - \frac{\delta}{2\log(\frac{1}{\delta})}$$

for all γ with $0 < \gamma < \delta\pi$. Taking the limit $\gamma \to 0$ shows that

$$\lim_{N\to\infty} \sum_{k=1}^{N} \frac{g_\delta(k)}{k} = \infty. \tag{8}$$

Combining (7) and (8), we see that

$$\lim_{N\to\infty} (S_N \gamma_\delta)(t) = -\infty \tag{9}$$

for all $t \in [1/4, 3/4]$.

Assume that there exists a signal $f_\pi \in \mathcal{B}_\pi^\infty$ with $f_\pi(k) = \gamma_\delta(k)$, $k \in \mathbb{Z}$. Then, according to Fact 1, we have

$$\max_{t\in[-T,T]} \left| \sum_{k=-N}^{N} \gamma_\delta(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right|$$
$$= \max_{t\in[-T,T]} \left| \sum_{k=-N}^{N} f_\pi(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| \leq C_1(T) \|f\|_{\mathcal{B}_\pi^\infty}$$

for all $N \in \mathbb{N}$ and $T > 0$. This is a contradiction (9). Thus, there exists no signal $f_\pi \in \mathcal{B}_\pi^\infty$ with $f_\pi(k) = \gamma_\delta(k)$, $k \in \mathbb{Z}$. □

Sketch of the proof of Theorem 2. From the proof of Theorem 1 we already know that for the signal $\gamma_\delta \in \mathcal{B}_{(1+\delta)\pi,0}^\infty$, $\delta \in (0, 1)$, we have

$$\lim_{N\to\infty} (S_N \gamma_\delta)(1/2) = -\infty. \tag{10}$$

Let $t_1 \in \mathbb{R} \setminus \mathbb{Z}$. We have

$$\left| \frac{(S_N \gamma_\delta)(\frac{1}{2})}{\sin(\frac{\pi}{2})} - \frac{(S_N \gamma_\delta)(t_1)}{\sin(\pi t_1)} \right| \leq \frac{1}{\pi} \sum_{k=-N}^{N} |g_\delta(k)| \frac{|t_1 - \frac{1}{2}|}{|\frac{1}{2} - k||t_1 - k|}$$
$$\leq \frac{\|g_\delta\|_{\mathcal{B}_{\delta\pi,0}^\infty}}{\pi} \sum_{k=-N}^{N} \frac{|t_1 - \frac{1}{2}|}{|\frac{1}{2} - k||t_1 - k|}$$
$$\leq \|g_\delta\|_{\mathcal{B}_{\delta\pi,0}^\infty} C_2(t_1),$$

where $C_2(t_1)$ is a positive constant that depends on t_1 but not on N. It follows that

$$\left| \frac{(S_N \gamma_\delta)(t_1)}{\sin(\pi t_1)} \right| \geq \left| \frac{(S_N \gamma_\delta)(\frac{1}{2})}{\sin(\frac{\pi}{2})} \right| - \|g_\delta\|_{\mathcal{B}_{\delta\pi,0}^\infty} C_2(t_1),$$

which, using (10), implies that $\lim_{N\to\infty} |(S_N \gamma_\delta)(t_1)| = \infty$. This proves the fist assertion.

Let ϕ_1 be a function in $C_0^\infty[0, 1]$ with $\phi_1(t) \geq 0$ for all $t \in \mathbb{R}$ and

$$\phi_1(t) = \begin{cases} 1, & \frac{2}{5} \leq t \leq \frac{3}{5}, \\ 0, & t \in \mathbb{R} \setminus (\frac{1}{4}, \frac{3}{4}). \end{cases}$$

From (7) we know that

$$(S_N \gamma_\delta)(t) \leq -\frac{2\sin(\frac{\pi}{4})}{\pi} \sum_{k=1}^{N} \frac{g_\delta(k)}{k} + \frac{12 + \pi^2}{4\pi} \|g_\delta\|_{\mathcal{B}_{\delta\pi,0}^\infty}.$$

for all $t \in [1/4, 3/4]$. It follows that

$$\int_{-\infty}^{\infty} (S_N \gamma_\delta)(t)\phi_1(t) \, \mathrm{d}t = \int_{1/4}^{3/4} (S_N \gamma_\delta)(t)\phi_1(t) \, \mathrm{d}t$$
$$\leq -\int_{1/4}^{3/4} \frac{2\sin(\frac{\pi}{4})}{\pi} \sum_{k=1}^{N} \frac{g_\delta(k)}{k} \phi_1(t) \, \mathrm{d}t$$
$$+ \int_{1/4}^{3/4} \frac{12 + \pi^2}{4\pi} \|g_\delta\|_{\mathcal{B}_{\delta\pi,0}^\infty} \phi_1(t) \, \mathrm{d}t$$
$$\leq -\frac{2}{5} \frac{\sin(\frac{\pi}{4})}{\pi} \sum_{k=1}^{N} \frac{g_\delta(k)}{k} + \frac{12 + \pi^2}{4\pi} \|g_\delta\|_{\mathcal{B}_{\delta\pi,0}^\infty} \|\phi_1\|_{L^1(\mathbb{R})},$$

and, using (8), that

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} (S_N \gamma_\delta)(t)\phi_1(t) \, \mathrm{d}t = -\infty.$$

This completes the proof of Theorem 2. □

6. RELATION TO PRIOR WORK

In classical signal processing books, the theoretical treatment of the two operations, downsampling and bandlimited interpolation, is not given special attention, despite their high importance in applications. There are no studies of the analytical properties of downsampling for bandlimited signals that vanish at infinity. The usual narrative is that the bandlimited interpolation always exists [8, p. 52 and p. 162] and [9, p. 144]. That this cannot be true for arbitrary signals, has been demonstrated in [25], where a sequence in c_0 was constructed that possesses no bounded bandlimited interpolation. In the present paper we go much further and study the existence of the bandlimited interpolation for sequences that are created by downsampling a discrete-time signal that has been generated by sampling a bandlimited signals. By proving that this bandlimited interpolation does not exist in general, we show that downsampling needs to be treated carefully when considering more general signal spaces than $\mathcal{PW}_\sigma^2$. We explicitly construct a signal that exhibits this behavior. To the best of our knowledge, there have been no rigorous studies of this problem so far, and our result is the first in this direction.

One reason for this lack of attention could be the fact that for $\mathcal{PW}_\sigma^2$ and sequences in ℓ^2, the analysis is trivial and no problems occur. However, applications in signal processing, such as stochastic processes and time-discrete systems, require the study of larger classes of bandlimited and non-bandlimited signals. Already for the space $\mathcal{PW}_\pi^1$ there exist problems with the reconstruction of signals from this space with the Shannon sampling series [27, 28]. It is even possible that the series diverges strongly [29, 30].

The results in this paper continue our research on the foundations of signal processing, and in particular show that the rash application of distributions, is problematic.

7. REFERENCES

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