EFFICIENT THOMSON SPECTRAL ESTIMATOR WITH TIME-SHIFTED WINDOWS

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ABSTRACT

In this paper optimal spectral analysis window shapes, using weighted discrete prolate spheroidal sequences as basis functions, are proposed. These windows are not typically positive or even. The windows are time-shifted, combining the computational efficiency of the Welch method and the appealing property of predefined frequency resolution of the Thomson spectral estimator. The parameters of the optimal windows are found by minimising the resulting spectral covariances and optimising the window overlap, for the predetermined frequency resolution and number of windows. The windows are found to have low side lobes, giving small spectral leakage, and the final spectral estimate gives close to optimal variance reduction, i.e. the covariance between different sub-spectra is very small.

***Index Terms*—** Welch method, Slepian functions, DPSS, spectral leakage, variance

1. INTRODUCTION

The Welch method [1] is well known and used as spectral estimator in many different applications [2, 3, 4]. The spectral estimate has low variance as the time-shifted windows result in almost orthogonal sub-spectra that are averaged for the final estimate. Partly overlapping and smooth windows, such as the commonly used Hanning window, are more beneficial than non-overlapping rectangular windows [5, 6]. From frequency resolution and leakage viewpoint, different window shapes have been thoroughly investigated and compared [7, 8] and for a predefined frequency resolution, the discrete prolate spheroidal sequences (DPSS) are the most optimal from leakage viewpoint [9]. Multitaper estimators [10], which uses all data samples (100% overlap) for all the windowed periodograms, are also popular choices for spectral estimation. The properties of the different windows give uncorrelated periodograms and thereby reduced variance.

Optimising variance, resolution and leakage is often of great interest, where window shapes with certain properties are chosen followed by optimisation of the overlap and minimising the overall variance [11, 12]. It has been shown that the Thomson multitaper estimator, based on the DPSS, outperforms the Welch method in terms of bias and variance [13]. However, the Welch method is more efficient in real-time applications, with less computations and less memory allocation, as the windowed sequences require shorter discrete Fourier transforms and less storage [11]. The appealing Thomson multitaper property of predefined resolution was used to optimise a time-shifted window shape of the Welch estimator in [14], but the resulting windows did not fulfil the property of window orthogonality and well suppressed side lobes.

This paper proposes using the DPSS as basis functions to estimate an optimal analysis window shape for a predefined frequency resolution and a fixed number of time-shifted windows. The window shape is optimal in the sense that it minimises the variance of white Gaussian noise, under the constraint of well suppressed side lobes, i.e. low spectral leakage. A procedure to find the optimal overlap and corresponding window shapes is also proposed. The windows are not restricted to be positive or even.

2. THOMSON ESTIMATOR WITH WELCH WINDOW STRUCTURE

Given the discrete-time zero-mean stationary stochastic process, $x(n)$, with spectral density, $S_x(f)$, the spectrum can be estimated from the N samples $\mathbf{x} = [x(0) \ \dots \ x(N-1)]^T$, ($T$ denotes transpose), using the Welch method

$$\hat{S}_x(f) = \frac{1}{K} \sum_{k=1}^{K} \left| \sum_{n=0}^{N_g-1} x(n+(k-1)L)g(n)e^{-i2\pi fn} \right|^2, \tag{1}$$

where K is the number of windows, L is the time-shift in samples, $\mathbf{g} = [g(0) \ \dots \ g(N_g-1)]^T$ is the window function and the length of the window is the largest integer $N_g \leq N - L(K-1)$. Another way of estimating the spectrum is instead to consider the set of time-shifted windows

$$\mathbf{h}_k = [\underbrace{0 \ \dots \ 0}_{(k-1)L} \ \mathbf{g} \ \overbrace{0 \ \dots \ 0}^{N-((k-1)L+N_g)}]^T, \quad k = 1 \ \dots \ K. \tag{2}$$

These time-shifted windows make it possible to formulate a Thomson estimator

$$\hat{S}_x(f) = \frac{1}{K} \sum_{k=1}^{K} \left| \sum_{n=0}^{N-1} x(n)h_k(n)e^{-i2\pi fn} \right|^2. \tag{3}$$

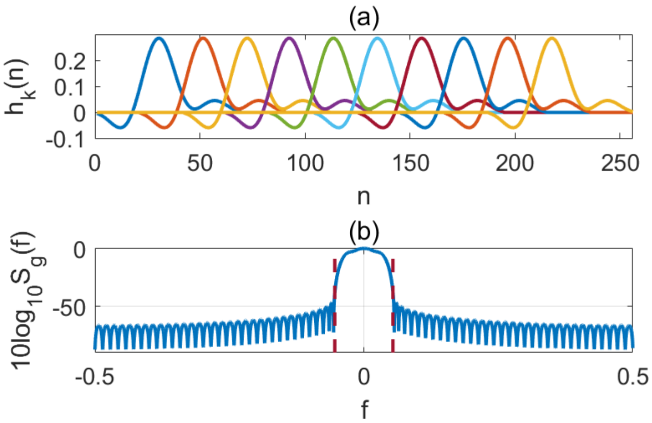

Fig. 1. Optimal time-shifted windows $\mathbf{h}_k$ for the frequency band $B = 0.108$, when $N = 256$ and $K = 10$; (a) time-shifted windows using $M = 4$ DPSS basis functions; (b) window spectrum of each single window $\mathbf{g}$.

This paper proposes that for some given frequency resolution, indicated by B, set a number of overlapping windows K, appropriate for the application, and then optimise the shape of the single window function $\mathbf{g}$ and its length N_g to reduce the (co)variance and leakage.

3. AN OPTIMAL TIME-SHIFTED WINDOW

The approach in this paper is to combine a set of M basis functions $\mathbf{q}_m = [q_m(0) \ \ldots \ q_m(N_g - 1)]^T$, $m = 1 \ldots M$, to express the single window

$$\mathbf{g} = \sum_{m=1}^{M} \alpha_m \mathbf{q}_m = \mathbf{Q}\boldsymbol{\alpha}, \tag{4}$$

where $\boldsymbol{\alpha} = [\alpha_1 \ \alpha_2 \ \ldots \ \alpha_M]^T$, is the scaling vector and $\mathbf{Q} = [\mathbf{q}_1 \ \mathbf{q}_2 \ \ldots \ \mathbf{q}_M]$ is the matrix including the basis functions as column vectors.

The DPSS [9] are used as basis functions since they are all orthogonal and localised to have maximum power inside a predefined frequency band. The spectral leakage is determined by the corresponding eigenvalue, thus making it easy to define a tolerated leakage $\epsilon_T = 1 - \lambda_i$ and at the same time determine the number of basis functions M, i.e. all eigenvectors with corresponding $\lambda_m \geq 1 - \epsilon_T, \ m = 1 \ldots M$.

The resulting shapes of the windows $\mathbf{g}$ are typically not positive or even. Figure 1(a) shows an example of this when $B = 0.108$, $N = 256$ and $K = 10$. The windows are a sum of $M = 4$ DPSS, the tolerated leakage is $\epsilon_T = 0.0005$, the overlap 70% and $\boldsymbol{\alpha}= [0.87 \ 0.28 \ -0.17 \ -0.36]^T$. Figure 1(b) shows the spectrum of the window $\mathbf{g}$. The dashed vertical lines marks the frequency band $|f| = B/2 = 0.054$, and it can be noted that for $|f| > B/2$ the spectrum is very low, the first side lobe is only -47 dB high.

An optimal spectral estimator should minimise the variance of a white Gaussian noise process, $N(0, 1)$,

$$\text{Var } \hat{S}_w(f) = \frac{1}{K^2} \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} |\mathbf{h}_{k_1}^T \mathbf{h}_{k_2}|^2, \tag{5}$$

under the constraint of zero bias,

$$E[\hat{S}_w(f)] = \frac{1}{K} \sum_{k=1}^{K} \mathbf{h}_k^T \mathbf{h}_k = 1, \tag{6}$$

[10]. As the DPSS windows are orthonormal, i.e., $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix, the zero bias constraint is simplified to

$$E[\hat{S}_w(f)] = \mathbf{g}^T\mathbf{g} = \boldsymbol{\alpha}^T\mathbf{Q}^T\mathbf{Q}\boldsymbol{\alpha} = \boldsymbol{\alpha}^T\boldsymbol{\alpha} = 1. \tag{7}$$

Thus the minimisation problem that needs to be solved is

$$\underset{|\boldsymbol{\alpha}|^2=1}{\arg\min} \text{ Var } \hat{S}_w(f), \tag{8}$$

which will give the scaling vector $\boldsymbol{\alpha}$ to construct the single window function $\mathbf{g}$. However, the shape of $\mathbf{g}$ is dependent on the window length, and thus the window overlap, a procedure to find both the optimal window shape and overlap is presented next.

3.1. Procedure

This procedure finds the optimal shape of the window $\mathbf{g}$ and the optimal overlap for the set of time-shifted windows $\mathbf{h}_k$, assuming there is a predetermined frequency band B and number of windows K.

1. Decide a range of overlaps to be evaluated.
2. Find the number of DPSS basis functions which fulfil $\lambda_m \geq 1 - 0.0005, \ m = 1 \ldots M$.
3. Solve the minimisation problem (8) for all considered overlaps and calculate the minimum variances given by the obtained $\boldsymbol{\alpha}$s.
4. Find the smallest minimum variance η. Consider all, though possible just one, of the $\boldsymbol{\alpha}$ that give variances smaller than $\eta + 0.0005^2$, choose among them the $\boldsymbol{\alpha}$ that corresponds to the smallest overlap.
5. From the chosen $\boldsymbol{\alpha}$ and corresponding overlap, construct the time-shifted windows $\mathbf{h}_k$.

The tolerated leakage

$$\epsilon_T = 1 - P_B = 1 - \int_{-B/2}^{B/2} S_h(f) df \leq 0.0005, \tag{9}$$

used in step 2, is chosen as the first sidelobe of the Hanning window spectrum is found slightly below -30 dB. In step 4 a deviation from the smallest minimum variance is allowed to balance the variance minimisation and the increased computational complexity of larger overlaps.

4. VARIANCE MINIMISATION

When solving the minimisation problem (8) it is of interest to consider the overlaps of the time-shifted windows. Lets define the lower and upper parts of the window vector $\mathbf{g}$

$$\mathbf{g}_{l_k} = [g(kL)\dots g(N_g-1)]^T, \tag{10}$$
$$\mathbf{g}_{u_k} = [g(0)\dots g(N_g-1-kL)]^T. \tag{11}$$

They can be used to rewrite the expression (5) into

$$\begin{aligned}\min \text{Var}\ \hat{S}_w(f) &= \min \frac{1}{K}|\mathbf{g}^T\mathbf{g}|^2 + \sum_{k=1}^{K_C} w_k |\mathbf{g}_{l_k}^T \mathbf{g}_{u_k}|^2 \\ &= \min \text{Var}_0\ \hat{S}_w(f) + \text{Cov}\ \hat{S}_w(f),\end{aligned} \tag{12}$$

where the number of different overlaps is $K_C < N_g/L < K$ and $w_k = 2\frac{(K-k)}{K^2}$. However, the first term Var_0 is always $1/K$ as $|\mathbf{g}^T\mathbf{g}|^2 = 1$, therefore only the covariance terms needs to be minimised. Using that

$$\mathbf{g}_{l_k} = \mathbf{Q}_{l_k}\boldsymbol{\alpha}, \tag{13}$$
$$\mathbf{g}_{u_k} = \mathbf{Q}_{u_k}\boldsymbol{\alpha}, \tag{14}$$

where $\mathbf{Q}_{l_k}$ and $\mathbf{Q}_{u_k}$ are the corresponding lower and upper part of the DPSS basis functions matrix $\mathbf{Q}$ respectively, the minimisation problem (8) can then be reformulated as

$$\underset{|\boldsymbol{\alpha}|^2=1}{\arg\min}\ \text{Var}\ \hat{S}_w(f) = \underset{|\boldsymbol{\alpha}|^2=1}{\arg\min} \sum_{k=1}^{K_C} w_k |\boldsymbol{\alpha}^T \mathbf{Q}_{l_k}^T \mathbf{Q}_{u_k}\boldsymbol{\alpha}|^2. \tag{15}$$

The DPSS are either even or odd sequences, $q_m(N_g-1-n) = (-1)^{(m-1)}q_m(n)$ [9], giving the following relationship

$$\mathbf{Q}_{l_k} = \mathbf{I}_p \mathbf{Q}_{u_k} \mathbf{I}_{cs}, \tag{16}$$

where $\mathbf{I}_{cs}$ is the $(M \times M)$ diagonal signature matrix with elements $(-1)^{(m-1)}$, and $\mathbf{I}_p$ is the $(N_g - 1 - |kL| \times N_g - 1 - |kL|)$ diagonal exchange matrix. Thus the product

$$\mathbf{A}_k = \mathbf{Q}_{l_k}^T \mathbf{Q}_{u_k} = \mathbf{I}_{cs} \mathbf{Q}_{u_k}^T \mathbf{I}_p \mathbf{Q}_{u_k} \tag{17}$$

is a $(M \times M)$ non-symmetric matrix. The minimisation problem can be solved using iterative optimisation methods for non-linear problems, but since the problem is non-convex, these methods can be costly. However, both M and K_C are assumed to always be rather small, which still make these optimisation methods a valid choice. The time-shifted windows resulting from solving (15) will be referred to as the iterative optimal time-shifted windows (I-OTSW).

4.1. Approximation

In order to achieve more computational efficiency a simplification of the minimisation problem (15) with an analytical solution is also proposed. The time-shifted windows resulting from solving this problem will be called the approximative optimal time-shifted windows (A-OTSW).

The original minimisation problem is a sum of non-negative numbers

$$\begin{aligned}\min \text{Var}\ \hat{S}_w(f) &= \underset{|\boldsymbol{\alpha}|^2=1}{\arg\min} \sum_{k=1}^{K_c} w_k |\boldsymbol{\alpha}^T \mathbf{A}_k \boldsymbol{\alpha}|^2 \\ &= \underset{|\boldsymbol{\alpha}|^2=1}{\arg\min} \sum_{k=1}^{K_c} w_k \boldsymbol{\alpha}^T \mathbf{A}_k \boldsymbol{\alpha}\boldsymbol{\alpha}^T \mathbf{A}_k^T \boldsymbol{\alpha},\end{aligned} \tag{18}$$

and a reasonable simplification is to instead consider the much simpler, although similar problem

$$\underset{|\boldsymbol{\alpha}|^2=1}{\arg\min} \sum_{k=1}^{K_c} w_k \boldsymbol{\alpha}^T \mathbf{A}_k \mathbf{A}_k^T \boldsymbol{\alpha}. \tag{19}$$

This is also a sum of non-negative numbers, since the matrix product $\mathbf{A}_k\mathbf{A}_k^T$ is Hermitian and positive semi-definite. The solution to this new problem is found by finding the right-singular vector of the $(K_cM \times M)$ block matrix

$$\begin{bmatrix} w_1 \mathbf{A}_1 \mathbf{A}_1^T \\ \vdots \\ w_{K_c} \mathbf{A}_{K_c} \mathbf{A}_{K_c}^T \end{bmatrix}, \tag{20}$$

corresponding to the smallest singular value.

At first glance it might seem tempting to first minimise the problem in (19) to get a solution $\boldsymbol{\alpha}_0$, and then use that to solve $\arg\min_{|\boldsymbol{\alpha}|^2=1} \sum_{k=1}^{K_c} w_k \boldsymbol{\alpha}^T \mathbf{A}_k \boldsymbol{\alpha}_0 \boldsymbol{\alpha}_0^T \mathbf{A}_k^T \boldsymbol{\alpha}$. However, since $\boldsymbol{\alpha}_0$ is a singular vector of the block matrix (20), the rank of the new block matrix, with rows $w_k \mathbf{A}_k \boldsymbol{\alpha}_0 \boldsymbol{\alpha}_0^T \mathbf{A}_k^T$, is reduced and the solution will always be $\boldsymbol{\alpha} = \alpha_1 = 1$.

5. EVALUATION

The proposed I-OTSW and A-OTSW estimators are evaluated with the usual Hanning window (Welch) and the first DPSS window (STSW) as Welch estimators and also to the Thomson multitaper estimator (Thomson). All windows are optimised or normalised to fulfil the zero bias condition (7). The evaluation is done for $N = 256$ and a range of different frequency bands according to Table 1 and overlap between 30% - 75%, for the I-OTSW, A-OTSW and STSW. The Nelder-Mead simplex method of Matlab (fminsearch) with a

Table 1. Range of frequency bands and corresponding number of windows for the I-OTSW and A-OTSW estimators.

B	K
$[0.040,\ 0.052]$	4
$[0.056,\ 0.072]$	6
$[0.076,\ 0.088]$	8
$[0.092,\ 0.108]$	10

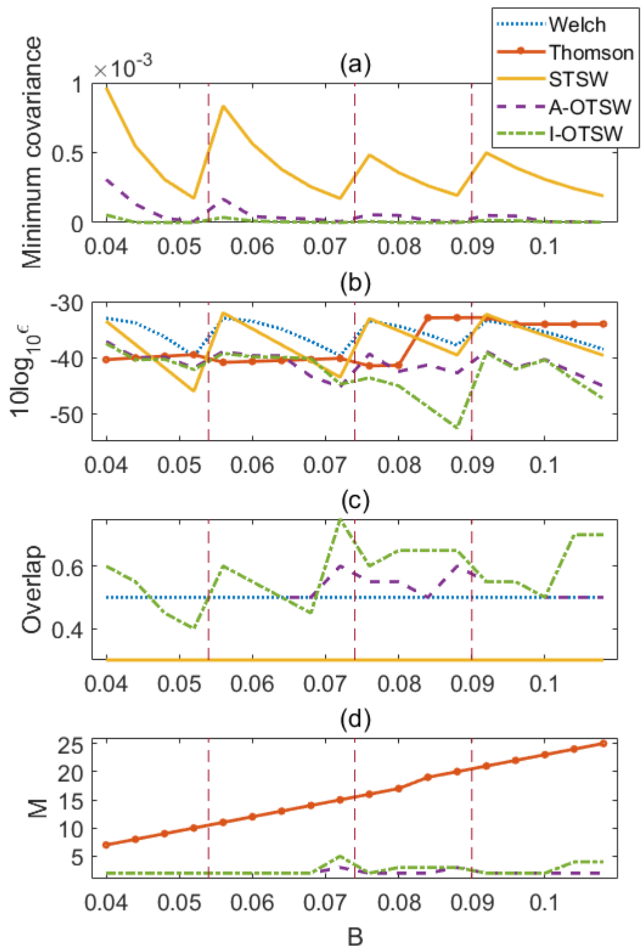

Fig. 2. Evaluation results for $N = 256$, different frequency bands B and corresponding K according to Table 1; (a) minimum covariances; (b) spectral leakage in dB; (c) optimal overlap; (d) number of DPSS basis functions.

set of random initial values is used for the I-OTSW estimator. The iterative search is repeated for reliable convergence. The Welch estimator is used with 50% overlap and the Thomson estimator with $NB - 3$ multitapers [10].

The results of the evaluation are shown in Figure 2, (a) shows the minimum covariances for the STSW, I-OTSW and A-OTSW estimators, which shows how close to the smallest possible variance $1/K$ the methods reach. The covariances for the Welch estimator are much larger than for the other methods, around 0.01 for all K. The Thomson estimator has covariance zero, however the minimum covariances for the I-OTSW and A-OTSW estimators are consistently very low and close to zero. Figure 2 (b) shows that the leakage is low for both the I-OTSW and A-OTSW estimators, often under -40 dB, and (c) shows that the overlap is mostly 55% for the A-OTSW and slightly higher for the I-OTSW. This means that most often $K_C = 2$ and $K_C \leq 4$, indicating low complexity computations. In Figure 2 (d) it can be seen that for the I-OTSW and A-OTSW estimators most often $M = 2, 3$, also indicating low complexity calculations, this is compared to the Thomson estimator that uses $7 - 25$ basis functions.

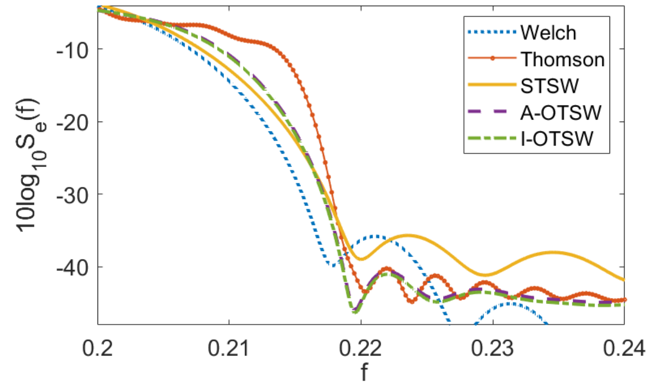

Fig. 3. Example of spectral estimates of low-pass filtered white Gaussian noise with cut-off frequency $f = 0.2$.

5.1. Low-pass filtered noise example

Low-pass filtered white Gaussian noise, $N(0, 1)$, is used to visualise the performance of the estimators on spectra with large dynamics. The spectral estimations of a filtered noise realisation, cut-off frequency $f = 0.2$ and $N = 256$, are shown in Figure 3. It can be seen that, the Welch estimator has the most narrow main lobe, with the I-OTSW, A-OTSW and STSW estimators almost as narrow. However, the I-OTSW, A-OTSW and Thomson estimators have the lowest first side lobes. In this example $B = 0.04$, $K = 4$ and the optimal overlaps presented in Figure 2 (c) are used.

6. CONCLUSION

Two methods, I-OTSW and A-OTSW, for finding the optimal shape of a time-shifted window have been presented. The window function, typically not positive or even, is a weighted sum of the DPSS, where the number of basis functions is determined by a tolerated leakage and the weights are optimised to minimise the variance of a white Gaussian noise process. Using a novel procedure to determine the optimal overlap of the time-shifted windows, the resulting optimal spectral estimators are shown to give close to optimal variance reduction and low leakage, (side lobes -40 dB). Both methods outperform the usual Welch method, using common windows, with regards to variance and leakage. Since the methods use few DPSS, low overlap and time-shifted windows, they are more computationally efficient than the Thomson estimator. The methods however still have the appealing quality of a predetermined frequency resolution, usually associated with the Thomson estimator. It is also shown that the two novel methods perform very similar to each other, making the simplified A-OTSW estimator highly beneficial.

7. REFERENCES

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