ANALYSIS OF SPARSE-INTEGER MEASUREMENT MATRICES IN COMPRESSIVE SENSING

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ABSTRACT

Performance of the reconstruction algorithms in compressed sensing largely depends on the characteristics of measurement matrices. As such, the construction and analysis of the measurement matrix is of paramount interest. In this paper, for the first time, we focus on the class of sparse sensing matrices with (non-negative) integer entries. This problem, among other applications, is particularly motivated by the constraint of measuring gene regulatory expressions. We study randomly generated matrices from the integer family and analyze their properties in terms of the covariance and RIP constant. We derive bounds for the coherence and RIP constant of such measurement matrices. Further, apart from the coherence, we find that the RIP constant is closely related to the minimum non-diagonal entry ρ_n in the covariance matrix, which is rarely studied before.

1. INTRODUCTION

In *Compressive Sensing* (CS) [1,2], a signal $\boldsymbol{x}$ which is sparse in some domain is reconstructed from a (relatively) small set of linear measurements with potentially some noise $\boldsymbol{n}$, $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{n}$. This requires 1) developing a reconstruction algorithm to recover or estimate $\boldsymbol{x}$ from $\boldsymbol{y}$, and 2) constructing a practical sensing matrix $\boldsymbol{A}$. The performance of the reconstruction algorithm is highly determined by the characteristics of the sensing matrix. Hence, in this paper, we focus on the construction and analysis of sensing matrices.

Here, we consider the class of sparse sensing matrices with non-negative integer entries. This is particularly motivated by the constraint on the construction of the sensing matrix in many application such as measuring gene expressions. For example, consider the problem of designing a sensory system for miRNAs' concentrations in disease detection applications [3,4]. One viable approach is to design a *biological plasmid* with binding sites [5] onto which miRNAs will attach to. Each bio-sensor then would be a collection of handful of these biological plasmids using which we can get reading on the concentration of a miRNA. Hence, if the output of the j-th biological plasmid is proportional to the concentration of the j-th miRNA, i.e., $o_j = g_j x_j$, and the i-th sensor consists of n_{ij} such plasmids, then the aggregate reading of the sensor would be approximately[1]

$$y_i = \sum_j n_{ij}(g_j x_j) + n_i. \tag{1}$$

Moreover, the number of different sensing biological plasmids in each bio- sensor would be limited due to the practical constraints such as available space and implementation issues.

In this paper, we consider the class of sensing systems given by (1). Especially, we provide a quantitative analysis of the performance of the sensing system and provide a computationally inexpensive upper bound for the RIP constant. In [6,7], the binary sensing matrices are modeled as the LDPC parity-check matrices and deterministic construction methods are proposed. In [6] the author studied sensing matrix in the view of RIP constant [1] while [7] considered the null-space characterization [1]. In [8], the authors proposed to create the sensing matrix by permuting a binary block matrix, which is efficient and simple for hardware implementation. In [9], the binary sensing matrix is generated randomly according to the Bernoulli distribution. Then, a non-linear recovery algorithm was proposed, which can recover the signal in near-optimal times.

Our contributions can be summarized as followings:

- Because of the constraints on the sensor construction for miRNA sensing, we model the sensing matrix $\boldsymbol{A}$ as a sparse (non-negative) integer matrix. Further, due to the spatial limitation, each sensor only consists of a limited number of different biological plasmids and the majority of entries in each row of $\boldsymbol{A}$ are zero.
- We analyze the performance of the sparse integer sensing matrix $\boldsymbol{A}$. We consider random construction of such a sensing system and first study its covariance matrix [1]. Then, inspired by [6,10,11], we provide an upper bound for RIP-constant [1]. Note that the derived bound holds even when the sparse integer sensing matrix is constructed deterministically rather than randomly.

2. SYSTEM MODEL

As already mentioned in Section 1, we assumed that each sensor is constructed by aggregating a handful of biological

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[1]Here, we ignored the interaction of a sensing unit with non-target miRNAs.

plasmids (see (1)). Hence, the linear sensing model can be rewritten as

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{G}\boldsymbol{x} = \boldsymbol{A}\tilde{\boldsymbol{x}}, \tag{2}$$

where $\boldsymbol{A} \in \mathbb{Z}_+^{m\times n}$ is determined by the number of biological plasmids of the j-th kind used by the i-th sensor, i.e., $A_{ij} = n_{ij}$, $\boldsymbol{G} := \mathrm{diag}(g_1, \ldots, g_N)$ is a diagonal gain matrix and $\tilde{\boldsymbol{x}} = \boldsymbol{G}\boldsymbol{x}$ is the scaled signal.

For the simplicity of analysis, we assume A_{ij}'s are generated randomly, independent of others, according to

$$P(A_{ij} = 0) = 1 - p, \quad P(A_{ij} = s) = p/M, \;\; 1 \le s \le M,$$

where M is the maximum value in $\boldsymbol{A}$, i.e., maximum number of each biological plasmids in a sensor, and p $(0 < p < 1)$ controls the sparsity of matrix.

3. COHERENCE

In this section, we study the characteristics of the covariance matrix [1] of the resulting sensing matrix.

Definition (Covariance matrix $\boldsymbol{\Sigma}$)**.** The covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$ of the sensing matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ is defined as

$$\Sigma_{ij} = \frac{|\langle \boldsymbol{A}_i, \boldsymbol{A}_j\rangle|}{\|\boldsymbol{A}_i\|_2 \|\boldsymbol{A}_j\|_2},$$

where Σ_{ij} is the (i, j)-th entry of matrix $\boldsymbol{\Sigma}$, and $\boldsymbol{A}_i$ denotes the ith columns of $\boldsymbol{A}$.

Obviously, $\Sigma_{ii} = 1$. Let $\rho_1 = \max_{i\neq j} \Sigma_{ij}$ and $\rho_n = \min_{i\neq j} \Sigma_{ij}$. In CS theory, ρ_1 is known as the coherence [1]. It is known that in the noiseless measurements, the k-sparse signal $\boldsymbol{x}$ can be recovered accurately provided that $\rho_1 \le \frac{1}{2k-1}$ [1]. Further, as we will show later, ρ_n is closely related to the recovery performance, which is rarely studied before.

In the following, we provide probabilistic bounds on ρ_1 and ρ_n by analyzing the covariance matrix $\boldsymbol{\Sigma}$. For this purpose, the key is to study

$$\frac{\langle \boldsymbol{A}_i, \boldsymbol{A}_j\rangle}{\|\boldsymbol{A}_i\|_2\|\boldsymbol{A}_j\|_2} = \sum_{k=1}^{m} \frac{A_{ki}}{\|\boldsymbol{A}_i\|_2} \frac{A_{kj}}{\|\boldsymbol{A}_j\|_2}.$$

Note that although A_{ki} and A_{kj} are independent, the term $\|\boldsymbol{A}_i\|_2\|\boldsymbol{A}_j\|_2$ in the denominator makes the summands dependent and the analysis of the coherence challenging. To overcome this issue, we first study and bound the norm of an arbitrary column, $\|\boldsymbol{A}_i\|_2$. Then we bound the maximum and minimum inner products between any two arbitrary columns of $\boldsymbol{A}$.

3.1. Norm of arbitrary columns

In this subsection, we study the norm $\|\boldsymbol{A}_i\|_2^2$ and its concentration behavior, which is described by the following lemma

Lemma 1. *For an arbitrary δ, $0 < \delta < 1$, define*

$$P_1(\delta) := \min\left\{\exp\left(-\frac{2\delta^2\,\overline{s}^2}{mM^4}\right), e^{-mp}\left(\frac{emp}{\overline{s}(1-\delta)}\right)^{\overline{s}(1-\delta)}\right\},$$

$$P_2(\delta) := \min\left\{\exp\left(-\frac{2\delta^2\,\overline{s}^2}{mM^4}\right), e^{-mp}\left(\frac{empM^2}{\overline{s}(1+\delta)}\right)^{\frac{\overline{s}(1+\delta)}{M^2}}\right\}.$$

Then, for any arbitrary column $\boldsymbol{a}$ of the sensing matrix $\boldsymbol{A}$, we have

$$\mathbb{P}\left\{\|\boldsymbol{a}\|_2^2 \le (1-\delta)\overline{s}\right\} \le P_1(\delta),$$
$$\mathbb{P}\left\{\|\boldsymbol{a}\|_2^2 \ge (1+\delta)\overline{s}\right\} \le P_2(\delta).$$

Proof. Let $S = \sum_j a_j^2$ where a_j is the j-th element of vector $\boldsymbol{a}$. Note that $\mathbb{E}S = \overline{s}$ and using Hoeffding's inequality [12], for arbitrary $t > 0$,

$$\mathbb{P}\left\{S - \overline{s} \ge t\right\} \le \exp(-\frac{2t^2}{mM^4}). \tag{3}$$

However, this bound might not be tight enough as the sparse probability p is not incorporated and mM^4 in the denominator can become too large.

In the following, we seek alternative approaches to bound $\mathbb{P}\left\{|S - \overline{s}| \ge t\right\}$ more accurately. First, note that

$$\begin{aligned}
&\mathbb{P}\left\{S - \overline{s} \ge t\right\} = \mathbb{P}\left\{\exp\left[\lambda(S - \overline{s})\right] \ge e^{\lambda t}\right\} \\
\le\, & e^{-\lambda(t+\overline{s})}\mathbb{E}\exp\left(\lambda\sum_{j=1}^{m} a_j^2\right) = e^{-\lambda(t+\overline{s})} \prod_{j=1}^{m} \mathbb{E}e^{\lambda a_j^2} \\
\overset{(a)}{=}\, & e^{-\lambda(t+\overline{s})}\left(\mathbb{E}e^{\lambda a_j^2}\right)^m \overset{(b)}{\le} e^{-\lambda(t+\overline{s})}\left(1 - p + pe^{\lambda M^2}\right)^m,
\end{aligned}$$

where (a) is because of the independence of a_j, and (b) is because $\mathbb{E}e^{\lambda a_j^2} \le 1 - p + pe^{\lambda M^2}$, which can be easily checked. Choosing λ as $M^{-2}\log\frac{t+\overline{s}}{mpM^2}$, we have

$$\mathbb{P}\left\{S - \mathbb{E}S \ge t\right\} \le\; e^{-mp}\left(\frac{empM^2}{t+\overline{s}}\right)^{(t+\overline{s})/M^2}. \tag{4}$$

Note that the term e^{-mp} in this bound is the dominant term for practical choices of M and p, causing the probability to decrease rapidly.

Similarly, an upper bound for probability $\mathbb{P}\{S \le \mathbb{E}S - t\}$, for $0 < t < \mathbb{E}S = \overline{s}$, can be obtained as

$$\mathbb{P}\left\{S - \mathbb{E}S \le -t\right\} \le e^{-mp}\left[\frac{emp}{\overline{s} - t}\right]^{\overline{s}-t}. \tag{5}$$

Combining (3), (4), and (5), we complete the proof. ∎

Lemma 1 shows that $\|\boldsymbol{A}_i\|_2^2$ is concentrated around its expected value and the probability of deviation drops rapidly. Therefore, analyzing ρ_1 and ρ_n approximately reduces to investigating the maximum and minimum of inner products, i.e., $\max_{i\neq j}\langle \boldsymbol{A}_i, \boldsymbol{A}_j\rangle$ and $\min_{i\neq j}\langle \boldsymbol{A}_i, \boldsymbol{A}_j\rangle$.

3.2. Maximum of inner product

In this subsection, we bound the maximum inner product $\langle \boldsymbol{A}_i, \boldsymbol{A}_j\rangle$, $(i \neq j)$. To achieve this, we first bound the expected value of $\max_{i\neq j}\langle \boldsymbol{A}_i, \boldsymbol{A}_j\rangle$ and then bound the probability of deviating from the expected value.

Lemma 2. *Define $C(\lambda) = 1 - p^2 + \left(\frac{p}{M}\right)^2 \frac{e^{\lambda}(1-e^{M^2\lambda})}{1-e^{\lambda}}$. Then*

$$\mathbb{E}\max_{i\neq j}\langle \boldsymbol{A}_i, \boldsymbol{A}_j\rangle \le \inf_{\lambda>0}\frac{1}{\lambda}\left[\log\binom{n}{2} + m\log C(\lambda)\right]. \tag{6}$$

Proof. Define the index set $\mathscr{T} := \{(i,\, j) :\, 1 \le i < j \le n\}$ and random variable $Z_{k,s=(s_1,s_2)} := A_{k,s_1} A_{k,s_2}$, where $1 \le k \le m$ and $s = (s_1, s_2) \in \mathscr{T}$. Let $Z := \max_{s\in\mathscr{T}} \sum_{k=1}^{m} Z_{k,s}$. Then,

$$e^{\lambda \mathbb{E} Z} \le \mathbb{E} e^{\lambda Z} \le \sum_{s\in\mathscr{T}} \mathbb{E} e^{\lambda \sum_{k=1}^{m} Z_{k,s}} = \sum_{s\in\mathscr{T}} \left(\mathbb{E} e^{\lambda Z_{1,s}}\right)^m,$$

where the first inequality is due to the convexity of $\exp(\cdot)$. Taking $\log(\cdot)$, we obtain

$$\mathbb{E} Z \le \inf_{\lambda>0} \frac{1}{\lambda} \underbrace{\left[\log \sum_{s\in\mathscr{T}} \left(\mathbb{E} e^{\lambda Z_{k,s}}\right)^m\right]}_{\le \log\binom{n}{2} + m \log \mathbb{E} e^{\lambda Z_{k,s}}}.$$

To prove the lemma, it is enough to bound $\mathbb{E} e^{\lambda Z_{k,s}}$ for an arbitrary k and $s \in \mathscr{T}$. Note that by the definition

$$\begin{aligned} \mathbb{E} e^{\lambda Z_{k,s}} &= 1 - p^2 + \left(\frac{p}{M}\right)^2 \sum_{a_1=1}^{M} \sum_{a_2=1}^{M} e^{\lambda a_1 a_2} \\ &\overset{(c)}{\le} 1 - p^2 + \left(\frac{p}{M}\right)^2 \underbrace{e^{-M\lambda} \sum_{a_1=1}^{M} e^{\lambda a_1} \sum_{a_2=1}^{M} e^{\lambda M a_2}}_{e^{\lambda}(1-e^{M^2 \lambda})/(1-e^{\lambda})}, \end{aligned}$$

where (c) is due to the fact that

$$(k_1 - M)(k_2 - 1) \le 0 \Longrightarrow k_1 k_2 \le k_1 + M k_2 - M.$$

Note that this relaxation is tight when $M = 1$, the case of binary measurement matrix. ■

The following lemma bounds the probability of exceeding $\max_{i\ne j} \langle \boldsymbol{A}_i, \boldsymbol{A}_j\rangle$ from its expected value.

Lemma 3. *Using the same notations as the previous lemma, we have*

$$\mathbb{P}\left\{Z \ge \inf_{\lambda>0} \frac{1}{\lambda}\left[\log\binom{n}{2} + m \log C(\lambda)\right] + t\right\} \le e^{-2t^2/mM^4}.$$

Proof. Note that for a fixed $s \in \mathscr{T}$, $Z_{k,s}$'s are independent of each other for different values of k. Define $a_{k,s} := 0$ and $b_{k,s} := M^2$. Obviously, we have $a_{k,s} \le Z_{k,s} \le b_{k,s},\ \forall\, 1 \le k \le m$. From [12, Thm. 12.1], we conclude that

$$\log \mathbb{E} e^{\lambda(Z - \mathbb{E} Z)} \le \frac{V\lambda^2}{8},$$

where V is defined as $V = \sum_{k=1}^{m} \sup_{s\in\mathscr{T}} (b_{k,s} - a_{k,s})^2 = mM^4$. Therefore, for an arbitrary $\lambda > 0$, using Markov inequality results in

$$\mathbb{P}\{Z - \mathbb{E} Z \ge t\} \le e^{-\lambda t} \mathbb{E} e^{\lambda(Z - \mathbb{E} Z)} \le e^{mM^4 \lambda^2/8 - \lambda t}.$$

Setting $\lambda = 4t/mM^2$ and using Lemma 2, we have completed the proof. ■

3.3. Minimum of inner product

In this subsection, we analyze the minimum of the inner product $\min_{i\ne j} \langle \boldsymbol{A}_i, \boldsymbol{A}_j\rangle$ in a similar way as Subsection 3.2.

Lemma 4. *Expectation of* $\min_{i\ne j} \langle \boldsymbol{A}_i, \boldsymbol{A}_j\rangle$ *is lower bounded by*

$$\mathbb{E} \min_{i\ne j} \langle \boldsymbol{A}_i, \boldsymbol{A}_j\rangle \ge \frac{mp^m}{M^{m-2}}.$$

Proof. For arbitrary $\lambda \ge 0$,

$$\mathbb{E} \min_{i\ne j} \langle \boldsymbol{A}_i, \boldsymbol{A}_j\rangle = \mathbb{E} \min_{i\ne j} \sum_{k=1}^{m} A_{ki} A_{kj} \ge \sum_{k=1}^{m} \mathbb{E} \min_{i\ne j} A_{ki} A_{kj},$$

for an arbitrary k (and hence, column $\boldsymbol{A}_k$). For the simplicity of our proof, we fix k and denote the elements $\{A_{ki}\}_{i=1}^m$ in column $\boldsymbol{A}_i$ as $\{a_i\}_{i=1}^m$. For each set $\{a_i\}_{i=1}^m$, we define a coupled set $\{\tilde{a}_i\}_{i=1}^m$ as

$$\begin{cases} \tilde{a}_1 = \cdots = \tilde{a}_m = 0, & \exists\, a_i \ne M; \\ \tilde{a}_i = \cdots = \tilde{a}_m = M, & \text{otherwise}. \end{cases}$$

Easily, we can prove $\inf_{i\ne j} a_i a_j \ge \inf_{i\ne j} \tilde{a}_i \tilde{a}_j = \tilde{a}^2$, where $\tilde{a}$ is an arbitrary element from $\{\tilde{a}_i\}_{i=1}^m$. Hence, we have

$$\mathbb{E} \inf_{i\ne j} a_i a_j \ge \mathbb{E} \tilde{a}_i^2 = p^m M^2 / M^m,$$

and then complete the proof of $\mathbb{E} \inf_{i\ne j} \langle \boldsymbol{A}_i, \boldsymbol{A}_j\rangle \ge \frac{mp^m}{M^{m-2}}$. ■

Lemma 5. *For* $\inf_{i\ne j} \langle \boldsymbol{A}_i, \boldsymbol{A}_j\rangle$*, we have*

$$\mathbb{P}\left\{\inf_{i\ne j} \langle \boldsymbol{A}_i, \boldsymbol{A}_j\rangle \le \frac{mp^m}{M^{m-2}} - t\right\} \le \exp\left(-2t^2/mM^4\right).$$

Proof. Define random variable $Z = \sup_{i\ne j} -\langle \boldsymbol{A}_i, \boldsymbol{A}_j\rangle$. We can verify that $-\inf_{i\ne j} \langle \boldsymbol{A}_i, \boldsymbol{A}_j\rangle = Z$. Then we get

$$\begin{aligned} &\mathbb{P}\left\{\inf_{i\ne j} \langle \boldsymbol{A}_i, \boldsymbol{A}_j\rangle \le \frac{mp^m}{M^{m-2}} - t\right\} \\ \le\ &\mathbb{P}\left\{\inf_{i\ne j} \langle \boldsymbol{A}_i, \boldsymbol{A}_j\rangle \le \mathbb{E} \inf_{i\ne j} \langle \boldsymbol{A}_i, \boldsymbol{A}_j\rangle - t\right\} = \mathbb{P}\{Z - \mathbb{E} Z \ge t\}. \end{aligned}$$

Similar to the above, we construct $a_{k,s} = -M^2$ and $b_{k,s} = 0$, and verify that $a_{k,s} \le Z \le b_{k,s}$. Then we have

$$\mathbb{P}\{Z - \mathbb{E} Z \ge t\} \le \exp\left(-2t^2/mM^4\right), \tag{7}$$

which completes the proof via combining above together. ■

Combining the results from the previous three subsections, we obtain bounds for ρ_1 and ρ_n as:

Theorem 6. *For* ρ_1 *and* ρ_n*, we have*

$$\begin{aligned} &\mathbb{P}\left\{\rho_1 \le \frac{\inf_{\lambda>0} \frac{1}{\lambda} \left[\log\binom{n}{2} + m \log C(\lambda)\right]}{(1-\delta)\,\bar{s}}\right\} \\ &\ge [1 - P_1(\delta)] \times \left[1 - e^{-\frac{2t^2}{mM^4}}\right], \\ &\mathbb{P}\left\{\rho_n \ge \frac{mp^m}{M^{m-2}(1+\delta)\,\bar{s}}\right\} \ge [1 - P_2(\delta)]\left[1 - e^{-\frac{2t^2}{mM^4}}\right], \end{aligned}$$

where $P_1(\delta)$ *and* $P_2(\delta)$ *are defined in Lemma 1.*

In the following section, we will relate these two constants to the RIP constant [1], another widely used concept in CS.

4. ANALYSIS OF RIP CONSTANT

In this section, we investigate the RIP constant δ_k [1] associated with the random measurement matrix $\boldsymbol{A}$.

Definition (RIP constant, δ_k)**.** The RIP-constant associated with a matrix $\boldsymbol{A}$ is defined as the minimum $\delta_k \in [0, 1]$ that satisfies

$$(1-\delta_k)\|\boldsymbol{x}\|_2^2 \le \|\boldsymbol{A}\boldsymbol{x}\|_2^2 \le (1+\delta_k)\|\boldsymbol{x}\|_2^2,$$

for all k-sparse signals, i.e., $\boldsymbol{x} \in \mathbb{R}^n$ and $\|\boldsymbol{x}\|_0 \le k$.

The accuracy of recovered signal $\hat{\boldsymbol{x}}$ from noisy measurements through matrix $\boldsymbol{A}$ is closely related to δ_{2k}, as follows [1,2,13]. Let α_k and β_k be the minimum and maximum singular values of an arbitrary sub-matrix $\boldsymbol{A}_{(k)} \in \mathbb{R}^{m\times k}$ of $\boldsymbol{A}$, then the RIP constant can be calculated as [1, 13] $\delta_k = \frac{\beta_k^2-\alpha_k^2}{\beta_k^2+\alpha_k^2}$.

In the following subsection, we study bounds on the α_k and β_k, which corresponds to the minimum and maximum singular values, respectively.

4.1. Minimum singular value

In this subsection, we study the minimum singular value α_k. Consider an arbitrary sub-matrix $\boldsymbol{A}_{(k)}$ and its associated covariance matrix, $\boldsymbol{\Sigma_{(k)}}$ (as defined in Section III). Let $\boldsymbol{B}_{(k)}$ be defined as $\boldsymbol{A}_{(k)}$ with normalized columns, i.e., the i-th column is given as $\boldsymbol{B}_{(k),i} = \boldsymbol{A}_{(k),i}/\|\boldsymbol{A}_{(k),i}\|_2$. Obviously, $\boldsymbol{\Sigma}_{(k)} = \boldsymbol{B}_{(k)}^\mathsf{T}\boldsymbol{B}_{(k)}$ and

$$\|\boldsymbol{A}_{(k)}\boldsymbol{x}\|_2^2 \geq (\min\|\boldsymbol{A}_{(k),i}\|_2)^2\ \|\boldsymbol{B}_{(k)}\boldsymbol{x}\|_2^2.$$

Therefore, $\alpha_k \geq \sqrt{\sigma_{\min}(\boldsymbol{\Sigma}_{(k)})}\min_i\|\boldsymbol{A}_{(k),i}\|_2$, where $\sigma_{\min}(\cdot)$ denotes the minimum eigenvalue.

Lemma 7. *The minimum eigenvalue of $\boldsymbol{A}_{(k)}$ satisfies*

$$\sigma_{\min}(\boldsymbol{\Sigma}_{(k)}) \geq\ 1-\rho_n + \frac{k(\rho_n-\rho_1)}{2}.$$

Proof. As in [6], we denote the eigenvector corresponding to the smallest eigenvalue of $\boldsymbol{\Sigma}_{(k)}$ as $\boldsymbol{x}_e$, i.e., $\boldsymbol{x}_e = \operatorname{argmin}_{\boldsymbol{x}\in\mathbb{S}^{k-1}}\ \boldsymbol{x}^\mathsf{T}\boldsymbol{\Sigma}_{(k)}\boldsymbol{x}$, where $\mathbb{S}^{k-1}$ is the k-dimensional unit-sphere. Decompose $\boldsymbol{x}_e$ into $[\boldsymbol{x}_p\ \boldsymbol{x}_n]$, where $\boldsymbol{x}_p$ denotes the positive part of $\boldsymbol{x}_e$ and $\boldsymbol{x}_n$ is the remaining. Hence, $\boldsymbol{x}_e\boldsymbol{x}_e^\mathsf{T}$, can be decomposed as

$$\begin{bmatrix}\boldsymbol{x}_p\boldsymbol{x}_p^\mathsf{T} & \boldsymbol{x}_n\boldsymbol{x}_p^\mathsf{T}\\ \boldsymbol{x}_p\boldsymbol{x}_n^\mathsf{T} & \boldsymbol{x}_n\boldsymbol{x}_n^\mathsf{T}\end{bmatrix}.$$

Clearly, the entries in both $\boldsymbol{x}_p\boldsymbol{x}_p^\mathsf{T}$ and $\boldsymbol{x}_n\boldsymbol{x}_n^\mathsf{T}$ are all positive, while the entries of the other two sub-matrices are negative. Define $\boldsymbol{D} := \boldsymbol{\Sigma}_{(k)} - (1-\rho_n)\boldsymbol{I}$ and rearrange it as

$$\boldsymbol{D} = \begin{bmatrix}\boldsymbol{D}_{(pp)} & \boldsymbol{D}_{(pn)}\\ \boldsymbol{D}_{(np)} & \boldsymbol{D}_{(nn)}\end{bmatrix},$$

where the sub-matrices correspond to $\boldsymbol{x}_p\boldsymbol{x}_p^\mathsf{T}$, $\boldsymbol{x}_p\boldsymbol{x}_n^\mathsf{T}$, $\boldsymbol{x}_n\boldsymbol{x}_p^\mathsf{T}$, and $\boldsymbol{x}_n\boldsymbol{x}_n^\mathsf{T}$, respectively. Similarly, construct $\tilde{\boldsymbol{D}}$ as

$$\tilde{\boldsymbol{D}} = \begin{bmatrix}\rho_n\boldsymbol{1} & \rho_1\boldsymbol{1}\\ \rho_1\boldsymbol{1} & \rho_n\boldsymbol{1}\end{bmatrix} = \begin{bmatrix}\tilde{\boldsymbol{D}}_{(pp)} & \tilde{\boldsymbol{D}}_{(pn)}\\ \tilde{\boldsymbol{D}}_{(np)} & \tilde{\boldsymbol{D}}_{(nn)}\end{bmatrix},$$

where $\boldsymbol{1}$ denotes an all-one matrix of an appropriate size such that the sub-matrices in $\boldsymbol{D}$ and $\tilde{\boldsymbol{D}}$ have the same size. It can be easily verified that $\tilde{\boldsymbol{D}}_{(pp)} \leq \boldsymbol{D}_{(pp)}$ and $\tilde{\boldsymbol{D}}_{(nn)} \leq \boldsymbol{D}_{(nn)}$, which corresponds to positive entries in $\boldsymbol{x}_e\boldsymbol{x}_e^\mathsf{T}$. Similarly, $\tilde{\boldsymbol{D}}_{(np)} \geq \boldsymbol{D}_{(np)}$ and $\tilde{\boldsymbol{D}}_{(pn)} \geq \boldsymbol{D}_{(pn)}$, corresponding to the non-positive terms in $\boldsymbol{x}_e\boldsymbol{x}_e^\mathsf{T}$. Therefore, $\boldsymbol{x}_e^\mathsf{T}(\boldsymbol{D}-\tilde{\boldsymbol{D}})\boldsymbol{x}_e \geq 0$, and

$$\sigma_{\min}(\tilde{\boldsymbol{D}}) = \min_{\boldsymbol{x}\in\mathbb{S}^{k-1}}\boldsymbol{x}^\mathsf{T}\tilde{\boldsymbol{D}}\boldsymbol{x} \leq \boldsymbol{x}_e^\mathsf{T}\tilde{\boldsymbol{D}}\boldsymbol{x}_e \leq \boldsymbol{x}_e^\mathsf{T}\boldsymbol{D}\boldsymbol{x}_e = \sigma_{\min}(\boldsymbol{D}).$$

To calculate $\sigma_{\min}(\tilde{\boldsymbol{D}})$, first, note that $\operatorname{rank}(\tilde{\boldsymbol{D}}) = 2$ and $\|\tilde{\boldsymbol{D}}\|_F^2 = \sum_{i=1}^{\operatorname{rank}(\tilde{\boldsymbol{D}})}\sigma_i^2(\tilde{\boldsymbol{D}})$. Therefore,

$$\sigma_{\max}^2(\tilde{\boldsymbol{D}}) + \sigma_{\min}^2(\tilde{\boldsymbol{D}}) = \rho_n^2(k_1^2+k_2^2) + 2\rho_1^2k_1k_2, \qquad (8)$$

where k_1 and k_2 are the lengths of $\boldsymbol{x}_p$ and $\boldsymbol{x}_n$, respectively. On the other hand,

$$\operatorname{trace}(\tilde{\boldsymbol{D}}) = \sigma_{\max}(\tilde{\boldsymbol{D}}) + \sigma_{\min}(\tilde{\boldsymbol{D}}) = \rho_n(k_1+k_2). \qquad (9)$$

Solving for $\sigma_{\min}$ from (8) and (9) results in

$$\sigma_{\min}(\tilde{\boldsymbol{D}}) = \frac{\rho_nk - \sqrt{\rho_n^2k^2 + 4k_1k_2(\rho_1^2-\rho_n^2)}}{2}.$$

Note that as $k_1+k_2 = k$, we have $\sigma_{\min}(\boldsymbol{D}) \geq \sigma_{\min}(\tilde{\boldsymbol{D}}) \geq \frac{k(\rho_n-\rho_1)}{2}$, where the equality holds when $k_1 = k_2 = \frac{k}{2}$. Finally, by the definition of $\boldsymbol{D}$,

$$\sigma_{\min}(\boldsymbol{\Sigma}_{(k)}) = 1-\rho_n + \sigma_{\min}(\boldsymbol{D}) \geq 1-\rho_n + \frac{k(\rho_n-\rho_1)}{2},$$

which completes the proof. ■

4.2. Maximum singular value

Analyzing the maximum singular value is similar as the previous section. Note that

$$\|\boldsymbol{A}_{(k)}\boldsymbol{x}\|_2^2 \leq (\max\|\boldsymbol{A}_{(k),i}\|_2)^2\ \|\boldsymbol{B}_{(k)}\boldsymbol{x}\|_2^2,$$

hence, $\beta_k^2 \leq \sigma_{\max}(\boldsymbol{\Sigma}_{(k)})\max\|\boldsymbol{A}_{(k),i}\|_2^2$.

Lemma 8. *The maximum singular value can be bounded as*

$$\beta_k^2 = \sigma_{\max}(\boldsymbol{\Sigma}_{(k)}) \leq\ 1-\rho_1+\rho_1k.$$

Proof. Define $\boldsymbol{D} := \boldsymbol{\Sigma}_{(k)} - (1-\rho_1)\boldsymbol{I}$ and consider the principal eigenvector $\boldsymbol{x}_E$ corresponding to maximum eigenvalue of $\boldsymbol{\Sigma}_{(k)}$, i.e., $\boldsymbol{x}_E = \operatorname{argmax}_{\boldsymbol{x}\in\mathbb{S}^{k-1}}\boldsymbol{x}^\mathsf{T}\boldsymbol{\Sigma}_{(k)}\boldsymbol{x}$. Construct matrix $\tilde{\boldsymbol{D}}$ as

$$\tilde{\boldsymbol{D}} = \begin{bmatrix}\rho_1\boldsymbol{1} & \rho_n\boldsymbol{1}\\ \rho_n\boldsymbol{1} & \rho_1\boldsymbol{1}\end{bmatrix}.$$

Following similar argument as Lemma 7, we get

$$\sigma_{\max}(\tilde{\boldsymbol{D}}) \geq \boldsymbol{x}_E^\mathsf{T}\tilde{\boldsymbol{D}}\boldsymbol{x}_E \geq \boldsymbol{x}_E^\mathsf{T}\boldsymbol{D}\boldsymbol{x}_E = \sigma_{\max}(\boldsymbol{D}).$$

Hence, we have $\sigma_{\max}(\boldsymbol{\Sigma}_{(k)}) = \sigma_{\max}(\boldsymbol{D})+1-\rho_1 \leq \sigma_{\max}(\tilde{\boldsymbol{D}})+1-\rho_1$, where $\sigma_{\max}(\tilde{\boldsymbol{D}})$ is given by

$$\sigma_{\max}(\tilde{\boldsymbol{D}}) = \frac{\rho_1k + \sqrt{\rho_1^2k^2 - 4k_1k_2(\rho_1^2-\rho_n^2)}}{2} \leq \rho_1k.$$

Therefore, $\sigma_{\max}(\boldsymbol{\Sigma}_{(k)}) \leq 1-\rho_1+\rho_1k$. ■

Combining the results for α_k and β_k, the following theorem for the RIP constant of sparse-integer sensing matrix can be obtained:

Theorem 9. *For an arbitrary $\delta > 0$, with probability at least $[1-P_1(\delta)][1-P_2(\delta)][1-\exp(-2\delta^2\overline{s}/mM^2)]^2$, the R.I.P. constant δ_k is bounded as*

$$\delta_k \leq \frac{2\delta + \rho_1(k\delta - 3/2\delta + k - 1/2) - (k/2-1)(1-\delta)\rho_n}{2 + \rho_1(k\delta - 1/2\delta + k - 3/2) + (k/2-1)(1-\delta)\rho_n}.$$

5. CONCLUSIONS

In this paper, motivated by biological miRNA sensing systems, we have studied sparse-integer sensing matrices. We derived bounds for the covariance matrix and the RIP constant of the sensing matrix. These bounds can then be used to estimate the number of sensors, m, for reliable signal recovery.

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