PERFORMANCE ANALYSIS OF DISCRETE-VALUED VECTOR RECONSTRUCTION BASED ON BOX-CONSTRAINED SUM OF L1 REGULARIZERS

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ABSTRACT

In this paper, we analyze the asymptotic performance of a convex optimization-based discrete-valued vector reconstruction from linear measurements. We firstly propose a box-constrained version of the conventional sum of absolute values (SOAV) optimization, which uses a weighted sum of ℓ_1 regularizers as a regularizer for the discrete-valued vector. We then derive the asymptotic symbol error rate (SER) performance of the box-constrained SOAV (Box-SOAV) optimization theoretically by using convex Gaussian min-max theorem. Simulation results show that the empirical SER performances of Box-SOAV and the conventional SOAV are very close to the theoretical result for Box-SOAV when the problem size is sufficiently large.

Index Terms— Discrete-valued vector reconstruction, convex optimization, convex Gaussian min-max theorem

1. INTRODUCTION

Reconstruction of a discrete-valued vector from its linear measurements often arises in various communications systems [1–5]. In some applications such as overloaded multiple-input multiple-output (MIMO) systems [6–9], the number of measurements is less than that of the unknown variables. In such underdetermined problems, the performance of simple linear methods, e.g., linear minimum mean-square-error (LMMSE) method, have poor performance. Although the maximum likelihood (ML) method with the exhaustive search can achieve good performance, the computational complexity increases exponentially along with the problem size.

To obtain good performance with reasonable computational complexity, some convex optimization-based methods have been proposed for large-scale discrete-valued vector reconstruction. Box relaxation method [10, 11] considers the ML method under the hypercube including all possible discrete-valued vectors. Regularization-based method and transform-based method [12] apply the idea of compressed sensing [13, 14] to discrete-valued vector reconstruction. Sum of absolute values (SOAV) optimization [15] takes a similar approach and uses a weighted sum of ℓ_1 regularizers as a regularizer for the discrete-valued vector. Unlike the other convex optimization-based methods, the SOAV optimization can take the probability distribution of unknown variables into consideration. The SOAV optimization has been applied to some practical problems [7, 16–20], whereas only a few theoretical aspects are known for the performance of the SOAV optimization.

In this paper, we analyze the asymptotic performance of discrete-valued vector reconstruction based on the SOAV optimization. To make the analysis simpler, we firstly modify the conventional SOAV optimization to obtain box-constrained SOAV (Box-SOAV) by using the boundness of the unknown vector. We then investigate the performance of Box-SOAV by using convex Gaussian min-max theorem (CGMT) [11, 21, 22], which has been used for the analysis of several convex optimization problems. We provide the asymptotic symbol error rate (SER) of Box-SOAV in the large system limit, where the number of unknown variables and the number of measurements increase infinitely with a fixed ratio. The asymptotic SER is characterized by the probability distribution of the unknown vector, the measurement ratio, the parameters of Box-SOAV, and the optimizer of a scalar optimization problem associated with the original Box-SOAV optimization. The result enables us to predict the performance of Box-SOAV in large-scale discrete-valued vector reconstruction. Simulation results show that the empirical SER performance of Box-SOAV and the conventional SOAV optimization agree well with the theoretical result for Box-SOAV in large-scale problems.

In the rest of the paper, we use the following notations. we denote the transpose by $(\cdot)^{\mathsf{T}}$, the identity matrix by $\boldsymbol{I}$, the vector whose elements are all 1 by $\mathbf{1}$, and the vector whose elements are all 0 by $\mathbf{0}$. For a vector $\boldsymbol{z} = [z_1 \cdots z_N]^{\mathsf{T}} \in \mathbb{R}^N$, the ℓ_1 norm and the ℓ_2 norm are given by $\|\boldsymbol{z}\|_1 = \sum_{n=1}^{N} |z_n|$ and $\|\boldsymbol{z}\|_2 = \sqrt{\sum_{n=1}^{N} z_n^2}$, respectively. We denote the number of nonzero elements of $\boldsymbol{z}$ by $\|\boldsymbol{z}\|_0$ and the nth element of $\boldsymbol{z}$ by $[\boldsymbol{z}]_n$. For a convex function $\zeta : \mathbb{R}^N \to \mathbb{R}$, we

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define the Moreau envelope and the proximity operator as $\mathrm{env}_{\zeta}(\boldsymbol{z}) = \min_{\boldsymbol{u}\in\mathbb{R}^N}\left\{\zeta(\boldsymbol{u}) + \frac{1}{2}\|\boldsymbol{u}-\boldsymbol{z}\|_2^2\right\}$ and $\mathrm{prox}_{\zeta}(\boldsymbol{z}) = \arg\min_{\boldsymbol{u}\in\mathbb{R}^N}\left\{\zeta(\boldsymbol{u}) + \frac{1}{2}\|\boldsymbol{u}-\boldsymbol{z}\|_2^2\right\}$, respectively. When a sequence of random variables $\{Z_n\}$ $(n = 1, 2, \dots)$ converges in probability to Z, we denote $Z_n \xrightarrow{\mathrm{P}} Z$.

2. PRELIMINARIES

2.1. SOAV Optimization

We consider the reconstruction of an N dimensional discrete-valued vector $\boldsymbol{x} = [x_1 \ \cdots \ x_N]^{\mathsf{T}} \in \mathcal{R}^N \subset \mathbb{R}^N$ from its linear measurements $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{v} \in \mathbb{R}^M$. Here, $\mathcal{R} = \{r_1, \dots, r_L\}$ $(r_1 < \cdots < r_L)$ is a finite set of possible values for the elements of the unknown vector. The distribution of $\boldsymbol{x}$ is given by $\Pr(x_n = r_\ell) = p_\ell$, where $\sum_{\ell=1}^L p_\ell = 1$. $\boldsymbol{A} \in \mathbb{R}^{M\times N}$ is a known measurement matrix composed of independent and identically distributed (i.i.d.) Gaussian random variables with zero mean and variance $1/N$. $\boldsymbol{v} \in \mathbb{R}^M$ is an additive Gaussian noise vector with mean $\boldsymbol{0}$ and covariance matrix $\sigma_{\mathrm{v}}^2\boldsymbol{I}$.

The SOAV optimization [15] for the reconstruction of $\boldsymbol{x}$ is given by

$$\min_{\boldsymbol{s}\in\mathbb{R}^N}\left\{\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A}\boldsymbol{s}\|_2^2 + \sum_{\ell=1}^{L} q_\ell \|\boldsymbol{s}-r_\ell\boldsymbol{1}\|_1\right\}, \tag{1}$$

where q_ℓ (≥ 0) is a parameter. The SOAV optimization uses the regularizer $\sum_{\ell=1}^L q_\ell \|\boldsymbol{s}-r_\ell\boldsymbol{1}\|_1$ for the unknown discrete-valued vector. The idea of the regularizer comes from compressed sensing [13, 14] and the fact that the vector $\boldsymbol{x} - r_\ell\boldsymbol{1}$ has some zero elements.

2.2. CGMT

CGMT [11,21,22] is a theorem that associates the primary optimization (PO) problem and the auxiliary optimization (AO) problem given by

$$\text{(PO): } \Phi(\boldsymbol{G}) = \min_{\boldsymbol{w}\in\mathcal{S}_{\mathrm{w}}}\max_{\boldsymbol{u}\in\mathcal{S}_{\mathrm{u}}}\left\{\boldsymbol{u}^{\mathsf{T}}\boldsymbol{G}\boldsymbol{w} + \psi(\boldsymbol{w},\boldsymbol{u})\right\},$$

$$\text{(AO): } \phi(\boldsymbol{g},\boldsymbol{h}) = \min_{\boldsymbol{w}\in\mathcal{S}_{\mathrm{w}}}\max_{\boldsymbol{u}\in\mathcal{S}_{\mathrm{u}}}\left\{\|\boldsymbol{w}\|_2\,\boldsymbol{g}^{\mathsf{T}}\boldsymbol{u} - \|\boldsymbol{u}\|_2\,\boldsymbol{h}^{\mathsf{T}}\boldsymbol{w} + \psi(\boldsymbol{w},\boldsymbol{u})\right\},$$

respectively, where $\boldsymbol{G} \in \mathbb{R}^{M\times N}$, $\boldsymbol{g} \in \mathbb{R}^M$, $\boldsymbol{h} \in \mathbb{R}^N$, $\mathcal{S}_{\mathrm{w}} \subset \mathbb{R}^N$, $\mathcal{S}_{\mathrm{u}} \subset \mathbb{R}^M$, and $\psi : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$. We assume that $\mathcal{S}_{\mathrm{w}}$ and $\mathcal{S}_{\mathrm{u}}$ are closed compact sets, and $\psi(\cdot,\cdot)$ is a continuous convex-concave function on $\mathcal{S}_{\mathrm{w}} \times \mathcal{S}_{\mathrm{u}}$. The elements of $\boldsymbol{G}$, $\boldsymbol{g}$, and $\boldsymbol{h}$ are i.i.d. standard Gaussian random variables. The following theorem relates the optimizer $\hat{\boldsymbol{w}}_\Phi(\boldsymbol{G})$ of (PO) with the optimal value of (AO) in the limit of $M, N \to \infty$ with a fixed ratio $\Delta = M/N$, which we simply denote $N \to \infty$ in this paper.

Theorem 1 (CGMT [11, 22])**.** Let $\mathcal{S}$ be an open set in $\mathcal{S}_{\mathrm{w}}$ and $\mathcal{S}^{\mathrm{c}} = \mathcal{S}_{\mathrm{w}} \setminus \mathcal{S}$. Also, let $\phi_{\mathcal{S}^{\mathrm{c}}}(\boldsymbol{g},\boldsymbol{h})$ be the optimal value of (AO) with the constraint $\boldsymbol{w} \in \mathcal{S}^{\mathrm{c}}$. If there are constants $\eta > 0$ and $\bar{\phi}$ satisfying (i) $\phi(\boldsymbol{g},\boldsymbol{h}) \leq \bar{\phi} + \eta$ and (ii) $\phi_{\mathcal{S}^{\mathrm{c}}}(\boldsymbol{g},\boldsymbol{h}) \geq \bar{\phi} + 2\eta$ with probability approaching 1, then we have $\lim_{N\to\infty}\Pr\left(\hat{\boldsymbol{w}}_\Phi(\boldsymbol{G}) \in \mathcal{S}\right) = 1$.

3. MAIN RESULT

In this paper, to make the analysis simpler, we newly consider the Box-SOAV optimization given by

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{s}\in\mathbb{R}^N}\left\{\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A}\boldsymbol{s}\|_2^2 + \sum_{\ell=1}^{L} q_\ell\|\boldsymbol{s}-r_\ell\boldsymbol{1}\|_1 + \mathcal{I}(\boldsymbol{s})\right\}, \tag{2}$$

where the function $\mathcal{I}(\cdot)$ denotes the indicator function given by $\mathcal{I}(\boldsymbol{s}) = 0$ if $\boldsymbol{s} \in [r_1, r_L]^N$, otherwise $\mathcal{I}(\boldsymbol{s}) = \infty$. This modification is reasonable because $\boldsymbol{x} \in [r_1, r_L]^N$ and does not change the value of the objective function for $\boldsymbol{s} \in [r_1, r_L]^N$. Let $f(\boldsymbol{s}) = \sum_{\ell=1}^L q_\ell\|\boldsymbol{s}-r_\ell\boldsymbol{1}\|_1 + \mathcal{I}(\boldsymbol{s})$. By modifying the result in [18], the nth element of the proximity operator $\mathrm{prox}_{\gamma f}(\boldsymbol{z})$ $(\gamma \geq 0)$ can be obtained as

$$\left[\mathrm{prox}_{\gamma f}(\boldsymbol{z})\right]_n = \begin{cases} z_n - \gamma Q_k & (r_{k-1} + \gamma Q_k \leq z_n < r_k + \gamma Q_k) \\ r_k & (r_k + \gamma Q_k \leq z_n < r_k + \gamma Q_{k+1}) \end{cases}, \tag{3}$$

where z_n is the nth element of $\boldsymbol{z}$ and $Q_k = \left(\sum_{\ell=1}^{k-1} q_\ell\right) - \left(\sum_{\ell'=k}^{L} q_{\ell'}\right)$ $(k = 2, \dots, L)$, $Q_1 = -\infty$, $Q_{L+1} = \infty$.

The SER of Box-SOAV is given by $\frac{1}{N}\|\mathcal{Q}(\hat{\boldsymbol{x}}) - \boldsymbol{x}\|_0$, where the quantizer $\mathcal{Q}(\cdot)$ maps each element of the vector to the nearest value in $\mathcal{R}$, i.e., $\mathcal{Q}(\hat{\boldsymbol{x}}) = \arg\min_{\boldsymbol{z}\in\mathcal{R}^N}\|\boldsymbol{z}-\hat{\boldsymbol{x}}\|_1$. For the asymptotic SER, we have the following theorem.

Theorem 2. The measurement matrix $\boldsymbol{A} \in \mathbb{R}^{M\times N}$ is assumed to be composed of i.i.d. Gaussian random variables with zero mean and variance $1/N$. The distribution of the noise vector $\boldsymbol{v} \in \mathbb{R}^M$ is also assumed to be Gaussian with mean $\boldsymbol{0}$ and covariance matrix $\sigma_{\mathrm{v}}^2\boldsymbol{I}$. If the optimization problem $\max_{\beta>0}\min_{\alpha>0} F(\alpha,\beta)$ has a unique optimizer (α^*, β^*), we have

$$\frac{1}{N}\|\mathcal{Q}(\hat{\boldsymbol{x}}) - \boldsymbol{x}\|_0 \xrightarrow{\mathrm{P}} 1 - \sum_{\ell=1}^{L} p_\ell \Pr\left[\mathcal{Q}\left(\mathrm{prox}_{\frac{\alpha^*}{\beta^*\sqrt{\Delta}}f}\left(r_\ell + \frac{\alpha^*}{\sqrt{\Delta}}H\right)\right) = r_\ell\right] \tag{4}$$

as $N \to \infty$, where $F(\alpha,\beta) = \left\{\frac{\alpha\beta\sqrt{\Delta}}{2} + \frac{\sigma_{\mathrm{v}}^2\beta\sqrt{\Delta}}{2\alpha} - \frac{1}{2}\beta^2 - \frac{\alpha\beta}{2\sqrt{\Delta}} + \frac{\beta\sqrt{\Delta}}{\alpha}\mathrm{E}\left[\mathrm{env}_{\frac{\alpha}{\beta\sqrt{\Delta}}f}\left(X + \frac{\alpha}{\sqrt{\Delta}}H\right)\right]\right\}$. Here, X is the random variable with the same distribution as the unknown variables, i.e., $\Pr(X = r_\ell) = p_\ell$. H is the standard Gaussian random variable independent of X.

The function $F(\alpha,\beta)$ and the asymptotic SER in (4) can be calculated by using the probability density function $p(z) = \frac{1}{\sqrt{2\pi}}\exp(-z^2/2)$ and cumulative distribution function $P(z) = \int_{-\infty}^{z} p(z')dz'$ of the standard Gaussian distribution. For example, the asymptotic SER is given by

$$1 - \left\{p_L + \sum_{\ell=1}^{L-1} p_\ell P\left(\frac{(-r_\ell + r_{\ell+1})\sqrt{\Delta}}{2\alpha^*} + \frac{Q_{\ell+1}}{\beta^*}\right)\right\} + \left\{\sum_{\ell=2}^{L} p_\ell P\left(\frac{(r_{\ell-1} - r_\ell)\sqrt{\Delta}}{2\alpha^*} + \frac{Q_\ell}{\beta^*}\right)\right\}. \tag{5}$$

4. PROOF OUTLINE

In this section, we show a sketch of the proof of Theorem 2. Although the procedure of the proof roughly follows the approaches using CGMT in [11,21,22], we need to modify several parts for our problem. Some details of the proof are omitted due to space limitations.

4.1. (PO)

To obtain the (PO) problem for the proof, we firstly define the error vector as $\boldsymbol{w} := \boldsymbol{s} - \boldsymbol{x}$ and rewrite the Box-SOAV optimization (2) as

$$\min_{\boldsymbol{w}\in\mathcal{S}_{\mathrm{w}}} \frac{1}{N}\left\{\frac{1}{2}\left\|\boldsymbol{A}\boldsymbol{w}-\boldsymbol{v}\right\|_2^2+\sum_{\ell=1}^{L} q_\ell \left\|\boldsymbol{x}+\boldsymbol{w}-r_\ell \boldsymbol{1}\right\|_1\right\}, \quad (6)$$

where $\mathcal{S}_{\mathrm{w}} = \left\{\boldsymbol{z} \in \mathbb{R}^N : r_1 - x_n \le z_n \le r_L - x_n \ (n = 1,\dots,N)\right\}$ and the objective function is normalized by N. The optimization problem (6) is equivalent to

$$\min_{\boldsymbol{w}\in\mathcal{S}_{\mathrm{w}}} \max_{\boldsymbol{u}\in\mathbb{R}^M} \frac{1}{N}\left\{\sqrt{N}\boldsymbol{u}^\mathsf{T}\left(\boldsymbol{A}\boldsymbol{w}-\boldsymbol{v}\right)-\frac{N}{2}\left\|\boldsymbol{u}\right\|_2^2 + \sum_{\ell=1}^{L} q_\ell \left\|\boldsymbol{x}+\boldsymbol{w}-r_\ell \boldsymbol{1}\right\|_1\right\}. \quad (7)$$

Let $\boldsymbol{w}^*$ and $\boldsymbol{u}^*$ be the optimal values of $\boldsymbol{w}$ and $\boldsymbol{u}$, respectively. Since $\boldsymbol{u}^*$ satisfies $\boldsymbol{u}^* = \frac{1}{\sqrt{N}}(\boldsymbol{A}\boldsymbol{w}^* - \boldsymbol{v})$ and $\boldsymbol{w}^*$ is bounded, there exists a constant C_{u} such that $\left\|\boldsymbol{u}^*\right\|_2 \le C_{\mathrm{u}}$ with probability approaching 1. We can thus rewrite (7) as

$$\min_{\boldsymbol{w}\in\mathcal{S}_{\mathrm{w}}} \max_{\boldsymbol{u}\in\mathcal{S}_{\mathrm{u}}} \left\{\frac{1}{N}\boldsymbol{u}^\mathsf{T}\left(\sqrt{N}\boldsymbol{A}\right)\boldsymbol{w}-\frac{1}{\sqrt{N}}\boldsymbol{v}^\mathsf{T}\boldsymbol{u}-\frac{1}{2}\left\|\boldsymbol{u}\right\|_2^2 + \frac{1}{N}\sum_{\ell=1}^{L} q_\ell \left\|\boldsymbol{x}+\boldsymbol{w}-r_\ell \boldsymbol{1}\right\|_1\right\}, \quad (8)$$

where $\mathcal{S}_{\mathrm{u}} = \left\{\boldsymbol{z}\in\mathbb{R}^M : \left\|\boldsymbol{z}\right\|_2 \le C_{\mathrm{u}}\right\}$.

4.2. (AO)

The (AO) problem corresponding to (8) is given by

$$\min_{\boldsymbol{w}\in\mathcal{S}_{\mathrm{w}}} \max_{\boldsymbol{u}\in\mathcal{S}_{\mathrm{u}}} \left\{\frac{1}{N}\left(\left\|\boldsymbol{w}\right\|_2\boldsymbol{g}^\mathsf{T}\boldsymbol{u}-\left\|\boldsymbol{u}\right\|_2\boldsymbol{h}^\mathsf{T}\boldsymbol{w}\right)-\frac{1}{\sqrt{N}}\boldsymbol{v}^\mathsf{T}\boldsymbol{u} - \frac{1}{2}\left\|\boldsymbol{u}\right\|_2^2+\frac{1}{N}\sum_{\ell=1}^{L} q_\ell \left\|\boldsymbol{x}+\boldsymbol{w}-r_\ell \boldsymbol{1}\right\|_1\right\}. \quad (9)$$

The objective function in (9) can be written as $\frac{1}{\sqrt{N}}\left(\frac{\|\boldsymbol{w}\|_2}{\sqrt{N}}\boldsymbol{g}^\mathsf{T} - \boldsymbol{v}^\mathsf{T}\right)\boldsymbol{u} - \frac{1}{N}\left\|\boldsymbol{u}\right\|_2\boldsymbol{h}^\mathsf{T}\boldsymbol{w} - \frac{1}{2}\left\|\boldsymbol{u}\right\|_2^2 + \frac{1}{N}\sum_{\ell=1}^{L} q_\ell\left\|\boldsymbol{x}+\boldsymbol{w}-r_\ell\boldsymbol{1}\right\|_1$. Since both $\boldsymbol{g}$ and $\boldsymbol{v}$ are Gaussian, $\frac{\|\boldsymbol{w}\|_2}{\sqrt{N}}\boldsymbol{g} - \boldsymbol{v}$ is also Gaussian distributed with mean $\boldsymbol{0}$ and covariance matrix $\left(\frac{\|\boldsymbol{w}\|_2^2}{N}+\sigma_{\mathrm{v}}^2\right)\boldsymbol{I}$. We can thus rewrite $\left(\frac{\|\boldsymbol{w}\|_2}{\sqrt{N}}\boldsymbol{g}^\mathsf{T} - \boldsymbol{v}^\mathsf{T}\right)\boldsymbol{u}$ as $\sqrt{\frac{\|\boldsymbol{w}\|_2^2}{N}+\sigma_{\mathrm{v}}^2}\,\boldsymbol{g}^\mathsf{T}\boldsymbol{u}$, where we use the slight abuse of notation $\boldsymbol{g}$ as the random vector with i.i.d. standard Gaussian elements. By setting $\left\|\boldsymbol{u}\right\|_2 = \beta$, the (AO) problem can be further rewritten as

$$\min_{\boldsymbol{w}\in\mathcal{S}_{\mathrm{w}}} \max_{0\le\beta\le C_{\mathrm{u}}} \left\{\sqrt{\frac{\|\boldsymbol{w}\|_2^2}{N}+\sigma_{\mathrm{v}}^2}\,\frac{\beta\left\|\boldsymbol{g}\right\|_2}{\sqrt{N}} - \frac{1}{N}\beta\boldsymbol{h}^\mathsf{T}\boldsymbol{w} - \frac{1}{2}\beta^2 + \frac{1}{N}\sum_{\ell=1}^{L} q_\ell \left\|\boldsymbol{x}+\boldsymbol{w}-r_\ell \boldsymbol{1}\right\|_1\right\}. \quad (10)$$

We use the identity $\chi = \min_{\alpha>0}\left(\frac{\alpha}{2}+\frac{\chi^2}{2\alpha}\right)$ for $\chi\ (>0)$ and obtain

$$\max_{\beta>0}\min_{\alpha>0}\left\{\frac{\alpha\beta}{2}\frac{\|\boldsymbol{g}\|_2}{\sqrt{N}} + \frac{\sigma_{\mathrm{v}}^2\beta}{2\alpha}\frac{\|\boldsymbol{g}\|_2}{\sqrt{N}} - \frac{1}{2}\beta^2 - \frac{1}{N}\sum_{n=1}^{N}\frac{\alpha\beta h_n^2}{2}\frac{\sqrt{N}}{\|\boldsymbol{g}\|_2} + \frac{\beta}{\alpha}\frac{\|\boldsymbol{g}\|_2}{\sqrt{N}}\min_{\boldsymbol{w}\in\mathcal{S}_{\mathrm{w}}}\frac{1}{N}\sum_{n=1}^{N} J_n(w_n)\right\}, \quad (11)$$

where $J_n(w_n) = \frac{1}{2}\left(w_n - \alpha h_n \frac{\sqrt{N}}{\|\boldsymbol{g}\|_2}\right)^2 + \frac{\alpha}{\beta}\frac{\sqrt{N}}{\|\boldsymbol{g}\|_2}\sum_{\ell=1}^{L} q_\ell |x_n + w_n - r_\ell|$. The change in the range of β does not change the optimal value. Since the optimization over $\boldsymbol{w}$ in (11) is given by $\min_{\boldsymbol{w}\in\mathcal{S}_{\mathrm{w}}}\frac{1}{N}\sum_{n=1}^{N} J_n(w_n) = \frac{1}{N}\sum_{n=1}^{N}\mathrm{env}_{\frac{\alpha}{\beta}\frac{\sqrt{N}}{\|\boldsymbol{g}\|_2}f}\left(x_n + \frac{\sqrt{N}}{\|\boldsymbol{g}\|_2}\alpha h_n\right)$, (11) can be rewritten as $\phi_N^* = \max_{\beta>0}\min_{\alpha>0} F_N(\alpha,\beta)$, where

$$F_N(\alpha,\beta) = \frac{\alpha\beta}{2}\frac{\|\boldsymbol{g}\|_2}{\sqrt{N}} + \frac{\sigma_{\mathrm{v}}^2\beta}{2\alpha}\frac{\|\boldsymbol{g}\|_2}{\sqrt{N}} - \frac{1}{2}\beta^2 - \frac{1}{N}\sum_{n=1}^{N}\frac{\alpha\beta h_n^2}{2}\frac{\sqrt{N}}{\|\boldsymbol{g}\|_2} + \frac{\beta}{\alpha}\frac{\|\boldsymbol{g}\|_2}{\sqrt{N}}\frac{1}{N}\sum_{n=1}^{N}\mathrm{env}_{\frac{\alpha}{\beta}\frac{\sqrt{N}}{\|\boldsymbol{g}\|_2}f}\left(x_n + \frac{\sqrt{N}}{\|\boldsymbol{g}\|_2}\alpha h_n\right). \quad (12)$$

The optimal value of $\boldsymbol{w}$ is given by

$$\hat{\boldsymbol{w}}_N = \mathrm{prox}_{\frac{\alpha_N^*}{\beta_N^*}\frac{\sqrt{N}}{\|\boldsymbol{g}\|_2}f}\left(\boldsymbol{x} + \frac{\sqrt{N}}{\|\boldsymbol{g}\|_2}\alpha_N^*\boldsymbol{h}\right) - \boldsymbol{x}, \quad (13)$$

where α_N^* and β_N^* are the optimal values of α and β corresponding to ϕ_N^*, respectively.

4.3. Applying CGMT

We then consider the condition (i) of Theorem 1. As $N\to\infty$, $F_N(\alpha,\beta)$ converges pointwise to $F(\alpha,\beta)$ defined in Theorem 2. Let $\phi^* = \max_{\beta>0}\min_{\alpha>0}F(\alpha,\beta)$ and denote the optimal values of α and β by α^* and β^*, respectively. By a similar discussion to Lemma IV. 1 of [11], we have $\phi_N^* \xrightarrow{\mathrm{P}} \phi^*$ and $(\alpha_N^*, \beta_N^*) \xrightarrow{\mathrm{P}} (\alpha^*, \beta^*)$. Hence, the optimal value of (AO) satisfies the condition (i) in CGMT for $\bar{\phi} = \phi^*$ and any $\eta > 0$.

Next, we define the set $\mathcal{S}$ used in CGMT. We have the following lemma for the optimizer $\hat{\boldsymbol{w}}_N$ of (AO) in (13).

Lemma 1. Let $\psi(\cdot,\cdot) : [r_1 - r_L, r_L - r_1]\times\mathcal{R}\to\mathbb{R}$. If $\psi(\cdot, r_\ell)$ is Lipschitz continuous for any r_ℓ, then $\frac{1}{N}\sum_{n=1}^{N}\psi(\hat{w}_{N,n}, x_n) \xrightarrow{\mathrm{P}} \mathrm{E}\left[\psi(W,X)\right]$, where $\hat{w}_{N,n}$ denotes the nth element of $\hat{\boldsymbol{w}}_N$ in (13) and $W = \mathrm{prox}_{\frac{\alpha^*}{\beta^*\sqrt{\Delta}}f}\left(X + \frac{\alpha^*}{\sqrt{\Delta}}H\right) - X$.

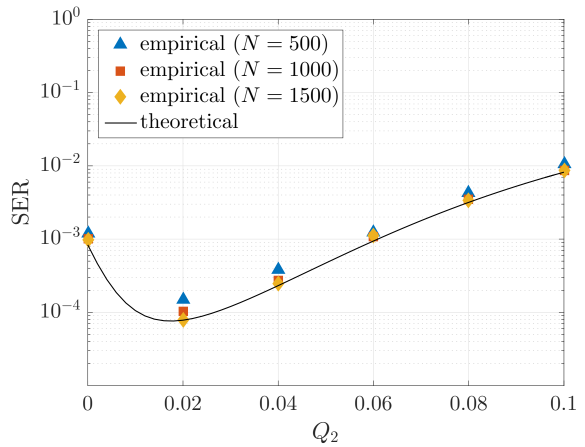

Fig. 1. SER of Box-SOAV versus Q_2 ($\Delta = 0.75$, $(r_1, r_2) = (0, 1)$, $(p_1, p_2) = (0.8, 0.2)$, SNR = 15 dB)

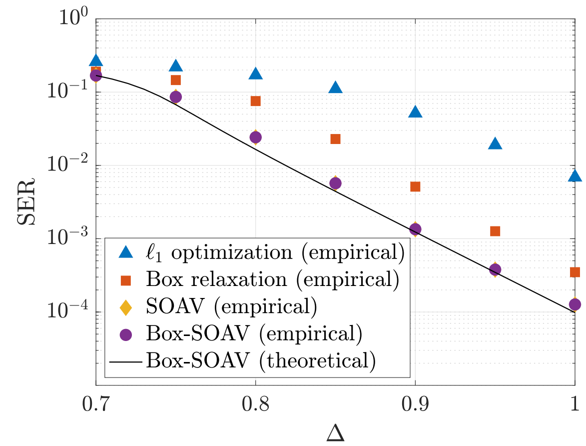

Fig. 2. SER versus Δ ($N = 1500$, $(r_1, r_2, r_3) = (-1, 0, 1)$, $(p_1, p_2, p_3) = (0.25, 0.5, 0.25)$, SNR = 20 dB)

From Lemma 1, we can define

$$\mathcal{S} = \left\{ \boldsymbol{z} \in \mathbb{R}^N : \left| \frac{1}{N} \sum_{n=1}^{N} \psi(z_n, x_n) - \mathrm{E}\left[\psi(W, X)\right] \right| < \varepsilon \right\} \quad (14)$$

and obtain $\hat{\boldsymbol{w}}_N \in \mathcal{S}$ with probability approaching 1 for any ε (> 0).

Finally, we consider the condition (ii) of CGMT. From the strong convexity in $\boldsymbol{w}$ of the objective function in (10), we can show $\phi_{\mathcal{S}^{\mathrm{c}}} \geq \phi_N^* + \tilde{\eta}$ with probability approaching 1 for a constant $\tilde{\eta}$ (> 0), where $\phi_{\mathcal{S}^{\mathrm{c}}}$ denotes the optimal value of (AO) with the restriction $\boldsymbol{w} \in \mathcal{S}^{\mathrm{c}}$. Hence, by setting $\bar{\phi} = \phi^*, \eta = \tilde{\eta}/3$ in Theorem 1, we can use CGMT for $\mathcal{S}$, i.e., Lemma 1 holds not only for the optimizer $\hat{\boldsymbol{w}}_N$ of (AO) but also for that of (PO).

We can derive the result of Theorem 2 by using CGMT for $\mathcal{S}$ in (14) with $\psi(w, x) = 1 - \chi(w + x, x)$, where the function $\chi(\cdot, \cdot)$ is given by $\chi(\hat{x}, x) = 1$ if $\mathcal{Q}(\hat{x}) = x$, otherwise $\chi(\hat{x}, x) = 0$. Although $\chi(\cdot, r_\ell)$ is not Lipschitz continuous, we can approximate it with a Lipschitz function because H is a continuous random variable and the probability measure for the discontinuity point of $\chi(\cdot, r_\ell)$ is zero (For a similar discussion, see Section IV-B and Lemma A-4 in [11]).

5. SIMULATION RESULTS

In this section, we compare the theoretical results by Theorem 2 and the empirical performance obtained by computer simulations. In the simulations, the measurement matrix $\boldsymbol{A} \in \mathbb{R}^{M \times N}$ and the noise vector $\boldsymbol{v} \in \mathbb{R}^M$ satisfy the assumptions in Theorem 2.

Figure 1 shows the SER performance for the binary vector with $(r_1, r_2) = (0, 1)$. The measurement ratio is $\Delta = 0.75$ and the distribution of unknown variable is given by $(p_1, p_2) = (0.8, 0.2)$. The signal-to-noise ratio (SNR) defined as $\sum_{\ell=1}^{L} p_\ell r_\ell^2 / \sigma_v^2$ is 15 dB. In this scenario, the Box-SOAV optimization depends only on $Q_2 = q_1 - q_2$ because $q_1 |s| + q_2 |s - 1| = (q_1 - q_2)s + (\text{const.})$ for any $s \in [0, 1]$. In the figure, 'empirical' represents the empirical performance versus Q_2 obtained by averaging the SER over 500 independent realizations of the measurement matrix. We use Douglas-Rachford algorithm [23,24] to solve the Box-SOAV optimization problem. We can see that the theoretical prediction denoted by 'theoretical' agrees well with the empirical performance for large N.

Figure 2 shows the SER performance versus Δ for the unknown discrete-valued vector with $(r_1, r_2, r_3) = (-1, 0, 1)$. We assume $N = 1500$, $(p_1, p_2, p_3) = (0.25, 0.5, 0.25)$, and SNR of 20 dB. In the figure, 'SOAV' and 'Box-SOAV' represent the conventional SOAV optimization and the Box-SOAV optimization, respectively. The parameters are given by $(q_1, q_2, q_3) = (1, 0.005, 1)$, which achieves good performance in the simulation. 'ℓ_1 optimization' and 'Box relaxation' denote the performance of the ℓ_1 optimization and the box relaxation method given by

$$\min_{\boldsymbol{s} \in \mathbb{R}^N} \left\{ \frac{1}{2} \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{s}\|_2^2 + \lambda \|\boldsymbol{s}\|_1 \right\}, \quad \min_{\boldsymbol{s} \in [-1,1]^N} \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{s}\|_2^2, \quad (15)$$

respectively. We use $\lambda = 0.005$ for the ℓ_1 optimization in the simulation. From the figure, we can see that the empirical performances of Box-SOAV and SOAV are close to the theoretical prediction of Box-SOAV. Moreover, they are better than the ℓ_1 optimization and the box relaxation method.

6. CONCLUSION

In this paper, we have derived the theoretical asymptotic performance of the discrete-valued vector reconstruction using the Box-SOAV optimization. By using the CGMT framework, we have shown that the asymptotic SER can be obtained with Theorem 2. Simulation results show that the theoretical prediction of Theorem 2 agrees well with the empirical performance. Future work includes the extension of the analysis to the reconstruction of the complex discrete-valued vector, which often appears in communications systems.

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