DISJUNCT MATRICES FOR COMPRESSED SENSING

Pradip Sasmal, Sai Subramanyam Thoota, Chandra R. Murthy

Dept. of ECE, Indian Institute of Science, Bangalore, India
email:{pradipsasmal, thoota, cmurthy}@iisc.ac.in

ABSTRACT

Disjunct matrices play a central role in non-adaptive group testing, as they provide necessary and sufficient conditions for identifying defective items from a large population using a small number of tests. In this paper, we show that binary disjunct matrices can also be very useful for recovering sparse signals from underdetermined linear measurements. They admit non-iterative, ultra-low complexity recovery of sparse signals. Binary measurement matrices have the added benefit of being friendly for hardware implementation. Further, we generalize the notion of disjunctness to matrices with arbitrary (non-binary) entries and show that such matrices also admit similar fast sparse vector recovery algorithms. We empirically demonstrate that disjunct matrices can recover denser signals than recent non-iterative sparse recovery algorithms.

Index Terms— Disjunct matrices, compressed sensing, non-iterative sparse signal recovery.

1. INTRODUCTION

Compressed Sensing (CS) [1–3] has emerged as a powerful and effective tool towards producing sparse signal representations. In CS, the goal is to recover a sparse vector $x \in \mathbb{R}^M$ from $y = \Phi x \in \mathbb{R}^m$, where the known sensing matrix $\Phi \in \mathbb{R}^{m\times M}$, with $m \ll M$. The problem of recovering sparest vector is a combinatorial problem and is known to be NP-hard in general [4].

The *spark* of a matrix, defined as the smallest number of linearly dependent columns in Φ, provides a necessary and sufficient condition for uniquely recovering an arbitrary sparse signal. If the spark$(\Phi) = k$, then sparse vectors with up to $k/2$ nonzero entries can be uniquely recovered from $y = \Phi x$. However, the recovery itself is still NP hard and one usually has to resort to greedy methods [5] or convex relaxations to recover sparse signals.

Mutual coherence and restricted isometry property (RIP) [6] based techniques are used to establish recovery guarantees for greedy algorithms such as Orthogonal Matching Pursuit (OMP) or convex relaxation based algorithms such as Basis Pursuit (BP) [7]. However, both BP and OMP are still of polynomial complexity in problem dimension, and could become impractical and expensive in high dimensional settings.

Verifying conditions based on spark and RIP is not easy, although it is known that random constructions satisfy them with very high probability. Hence, in a practical application, it remains unknown whether a given instantiation of the measurement matrix satisfies these properties. Also, due to high computational complexity of sparse recovery algorithms, it is beneficial to identify a property of a matrix that is easy to verify and also supports low computational complexity sparse recovery algorithms, while perhaps requiring a larger number of measurements for success. Motivated by this, in the present work, we make the following contributions:

1. We provide a bridge between non-adaptive group testing and compressed sensing. Specifically, disjunct matrices have been deeply investigated in non-adaptive group testing, as they are useful in detecting defective items in a large population [8, 9]. We show that the disjunctness property of binary matrices is also very useful in recovering sparse signals.
2. We exploit the disjunctness property to present an ultra-low complexity algorithm for identifying the support of the sparse signal as well as recover the nonzero coefficients. As the sparse recovery algorithm is non-iterative in nature, it is very fast in practice.
3. We extend the disjunctness property of a binary matrix to sparse matrices. We show that a similar non-iterative and fast sparse recovery algorithm is applicable.

Finally, through numerical simulations, we compare the recovery of sparse vectors with disjunct matrices with a state-of-the-art algorithm for non-iterative sparse vector recovery. We demonstrate that disjunct matrices can recover sparse vectors with a significantly larger number of nonzero entries.

Notation: The set $\{1, 2, \ldots, n\}$ is denoted by $[n]$. The i-th entry of x is denoted by x_i. $\Phi(:, i)$ and $\Phi(j, :)$ denote the i-th column and j-th row of Φ, respectively, and $\Phi(j, i)$ denotes the (j, i)th entry of Φ. The support of x is $\{i : x_i \neq 0\}$, denoted by supp(x). Let $S \subset [n]$, then $x_S \triangleq (x_i)_{i\in S}$ and

The work of P. Sasmal was supported by the National Postdoctoral Fellowship, Science and Engineering Research Board (SERB), Govt. of India (PDF/2017/002966). The work of C. R. Murthy was supported by the MeitY Young Faculty Research Fellowship, Govt. of India.

$\Phi_S \triangleq (\Phi(:,i))_{i\in S}$, the submatrix of Φ containing columns indexed by S.

2. DISJUNCT MATRICES

Definition 1. *An $m \times M$ binary matrix Φ is called t-disjunct if the support of any column is not contained in the union of the supports of any other t columns.*

In other words, if we take a submatrix Φ_S with $|S| = t+1$, then for $i \in [t+1]$, there exists j_i such that $\Phi_S(j_i, i) = 1$ and $\Phi_S(j_i, l) = 0$ for all $l \in [t+1] \setminus i$. This observation will be crucial for non-iterative recovery of almost all sparse signals.

Definition 2. *[10] A matrix Φ is t^e-disjunct if, given any $t+1$ columns of Φ with one designated column, there are $e+1$ rows with a 1 in the designated column and a 0 in each of the other t columns.*

In other words, if we take a submatrix Φ_S with $|S| = t+1$, then for $i \in [t+1]$, there exists $j_i^1, \ldots, j_i^{e+1}$ such that $\Phi_S(j_i^d, i) = 1$ and $\Phi_S(j_i^d, l) = 0$ for all $l \in [t+1] \setminus i$ and $d = 1, \ldots, e+1$. We will exploit this property for recovering all signals with a given maximum sparsity level.

Theorem 1. *[8] Let Φ be a $m \times M$ matrix with each column containing q ones and the overlap (i.e., the size of the intersection of the supports) between any two distinct columns is at most r. Then Φ is $\lfloor \frac{q-1}{r} \rfloor$-disjunct.*

2.1. Relation with other CS parameters

Theorem 2. *The spark of a t-disjunct matrix is greater than or equal to $t+1$.*

Proof. *Follows from the definition of a t-disjunct matrix.*

Theorem 3. *A matrix Φ containing the same number of ones in each column is $(\lfloor \mu_\Phi^{-1} \rfloor - 1)$-disjunct, where μ_Φ is the mutual coherence of Φ, defined as the maximum absolute inner product between any two distinct normalized columns of Φ.*

Proof. *Follows from Theorem 1 by observing that if each column of Φ contains q ones and the overlap between any two columns is at most r, then its mutual coherence $\mu_\Phi \leq \frac{r}{q}$.*

3. RECOVERY USING BINARY MATRICES

3.1. Recovery of all sparse signals

In this section, we present an ultra-low complexity algorithm for recovering all sparse signals when the matrix Φ is t^e-disjunct. Throughout this section, we assume that Φ is such that $\Phi(:,i)$ contains q_i ones for $i \in [M]$, with $q_{\min} \triangleq \min\{q_1, \ldots, q_M\}$, and that overlap between any two distinct columns is at most $r_{\max}$. Then, the disjunctiveness of Φ can be inferred from the following theorem:

Theorem 4. *Φ is t^e-disjunct for any $t < \lfloor \frac{q_{\min}}{r_{\max}} \rfloor$ and $e+1 \geq q_{\min} - tr_{\max}$.*

Proof. *Let us take Φ_S with $|S| = t+1$. Now for any $i \in [t+1]$, the total overlap between $\Phi_S(:,i)$ with all other columns of Φ_S is at most $tr_{\max}$. If $q_{\min} > tr_{\max}$, there are at least $q_{\min} - tr_{\max}$ rows where $\Phi_S(:,i)$ contains ones and all other columns contain zeros. Now, with the choice of t and e as in the theorem, it follows that Φ is t^e disjunct.*

We consider the linear system $y = \Phi x$, where Φ has the properties described above, and x is k sparse with $k < \frac{q_{\min}}{2r_{\max}}$.

3.1.1. Support Recovery

The following property allows one to almost trivially identify the support of x from y.

Claim: $S = \{j : |supp(\Phi(:,j)) \cap supp(y)| > \frac{q_{\min}}{2}\}$ is the support of x.

Proof: For $i \notin supp(x)$, $|supp(\Phi(:,i)) \cap supp(y)| \leq |supp(\Phi(:,i)) \cap \big(\cup_{l \in supp(x)} supp(\Phi(:,l))\big)| \leq kr_{max}$. On the other hand, if $s \in supp(x)$, then $|supp(\Phi(:,s)) \cap \big(\cup_{l \in supp(x), l \neq s} supp(\Phi(:,l))\big)| \leq (k-1)r_{\max}$. As a result, $|supp(\Phi(:,s)) \cap supp(y)| \geq q_{min} - (k-1)r_{\max}$. Setting $q_{min} - (k-1)r_{\max} > kr_{\max}$, one can have a clear demarcation between $|supp(\Phi(:,i)) \cap supp(y)|$ (which is $\leq \frac{q_{\min}}{2}$) and $|supp(\Phi(:,s)) \cap supp(y)|$ (which is $> \frac{q_{\min}}{2}$) with $i \notin supp(x)$ and $s \in supp(x)$. Therefore, by taking $q_{\min} > 2kr_{\max}$, the claim can be established.

3.1.2. Non-zero coefficient recovery

Suppose Φ is t^e-disjunct for some $t < \lfloor \frac{q_{\min}}{r_{\max}} \rfloor$ and $e \geq q_{\min} - tr_{\max} - 1$. Then, it is also $\lfloor \frac{q_{\min}}{r_{\max}} \rfloor^{\frac{q_{\min}}{2}}$-disjunct. As a result, whenever $s \in S$, for $\Phi_S(:,s)$ there exist $j_s^1, \ldots, j_s^{e+1}$ rows such that $\Phi_S(j_s^d, s) = 1$ and $\Phi_S(j_s^d, l) = 0$ for $l \in S \setminus s$ and $d = 1, \ldots, e+1$. Thus, we can directly recover

$$x_s = \begin{cases} y_{j_s^d},\ d = 1, \ldots, e+1 & \text{if } i \in S \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Note that the first case in the above is unambiguous because all $y_{j_s^d}$ are equal for $d = 1, \ldots, e+1$.

Theorem 5. *Let Φ be a binary matrix with every column containing at least $q_{\min}$ ones and with the overlap between any two distinct columns at most $r_{\max}$. Then any $\lfloor \frac{q_{\min}}{2r_{\max}} \rfloor$ sparse vector can be uniquely recovered.*

Remark 1. *In [11], the authors provide a non-iterative algorithm for sparse signal recovery using binary sensing matrices with every column having q ones. The result in [11] can be obtained as a special case of Theorem 5 for binary matrices with each column containing the same number of ones. However, for sparse signal recovery, the approach proposed* *in [11] is to first identify the support as above, and then for each element i in the support, find the support S_i of the i-th column of Φ, and then search for an element y_{j^i} in y_{S_i} which repeats more than $q/2$ times, and set $x_i = y_{j^i}$. In contrast, we use the disjunctness property to recover the nonzero entries of x. This not only allows us to generalize to matrices with unequal number of ones in each column, as will be shown later, it also enables the recovery of sparse vectors with much higher sparsity level than in [11].*

3.2. Recovery of almost all sparse signals

Under mild assumptions on the sparse signal, we can provide sparse signal recovery with higher sparsity level than given in section 3.1. These assumptions are made so that we can use non adaptive group testing techniques for support recovery while using disjunct matrices as the measurement matrix. Let us consider the linear system of equations $y = \Phi x = \Phi_S x_S$, where $S = supp(x)$ and Φ is a binary matrix. We get,

$$y_j = \sum_{l \in supp(\Phi_S(j,:))} x_l, \ \forall \ j \in [m].$$

We assume $y_j \neq 0$ whenever $supp(\Phi_S(j,:))$ is nonempty. This holds (a) with probability 1 if x is drawn from a generic random model; and (b) x is a non negative sparse signal. We note that these assumptions are not unduly restrictive. They are also standard in the statistical RIP (StRIP) literature [8].

3.2.1. Support recovery

Under the above assumptions on x, support recovery of x becomes similar to detection of defective items in group testing. The recovery algorithm outputs the following set

$$S = [M] \setminus \bigcup_{j: y_j = 0} supp(\Phi(j,:))$$

Irrespective of the matrix, this algorithm always provides a set that contains the support of x. Furthermore, if the matrix is disjunct, then the output exactly equals the support set.

Now, let Φ be t-disjunct and $|S| = k \leq t+1$. Then the support of x can be obtained as

$$S = \{i : supp(\Phi(:,i)) \subseteq supp(y)\}.$$

Note that, our sufficient condition here shows that we can recover the support of x for sparsity level $k \leq t+1$, while in the previous section, the sufficient condition was $k < t/2$.

3.2.2. Non-zero coefficient recovery

Once support of x is detected, we follow the following mechanism to find the coefficients $x_i, i \in S$.

As Φ is t-disjunct, for $i \in [k]$, there exists j_i such that $\Phi_S(j_i, i) = 1$ and $\Phi_S(j_i, l) = 0$ for all $l \in [k] \setminus i$. Now set

$$x_i = \begin{cases} y_{j_i}, & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Therefore, x can be recovered using the disjunct property. This way of finding the coefficients is non-iterative.

4. RECOVERY USING SPARSE MATRICES

4.1. Recovery of all sparse signals

In this section, we provide an algorithm for sparse signal recovery when the measurement matrix is sparse with arbitraty entries. First, we show that the definition of disjunctness of a binary matrix can be naturally extended to sparse matrices. We use the disjunct property of sparse matrices for support recovery and then find the non-zero coefficients. The definition of disjunctness of sparse non-binary matrices is as follows:

Definition 3. *An $m \times M$ sparse matrix Φ is called t-disjunct if the support of any column is not contained in the union of the supports of any other t columns.*

Definition 4. *A sparse matrix Φ is t^e-disjunct if given any $t+1$ columns of Φ with one designated, there are $e+1$ rows with non-zero entries in the designated column and a 0 in each of the other t columns.*

Proposition 1. *A sparse matrix Φ is t-disjunct if and only if it is t^0-disjunct.*

Now, we provide an algorithm for recovering all sparse signals. Let Φ be a sparse matrix where $\Phi(:,i)$ contains q_i number of non-zeros for $i \in [M]$ with $q_{\min} = \min\{q_1, \ldots, q_M\}$ and the cardinality of the intersection between support of any two distinct columns is at most $r_{\max}$. Then the disjunctivess of Φ can be obtained via the following theorem, whose proof is similar to that of Theorem 4:

Theorem 6. *Φ is t^e-disjunct if $t < \lfloor \frac{q_{\min}}{r_{\max}} \rfloor$ and $e+1 \geq q_{\min} - t r_{\max}$.*

We consider the linear system $y = \Phi x$, where Φ possess the properties of Theorem 6, and x is k sparse with $k < \frac{q_{\min}}{2 r_{\max}}$.

4.1.1. Support Recovery

Following similar arguments as given in subsection 3.1.1, we can identify the support using

$$S = \{j : |supp(\Phi(:,i)) \cap supp(y)| > \frac{q_{\min}}{2}\}.$$

4.1.2. Nonzero coefficient recovery

Since Φ is t^e-disjunct as given in Theorem 6, it is also $\lfloor \frac{q_{\min}}{r_{\max}} \rfloor^{\frac{q_{\min}}{2}}$-disjunct. Hence, whenever $s \in S$, for $\Phi_S(:,s)$ there exist $j_s^1, \ldots, j_s^{e+1}$ rows such that $\Phi_S(j_s^d, s) \neq 0$ and $\Phi_S(j_s^d, l) = 0$ for $l \in S \setminus s$ and $d = 1, \ldots, e+1$. We can therefore recover the nonzero coefficients as

$$x_s = \begin{cases} \frac{y_{j_s^d}}{\Phi_S(j_s^d, s)}, \ d = 1, \ldots, e+1 & \text{if } i \in S \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

4.2. Recovery of almost all sparse signals

The sparse recovery algorithm for disjunct matrices discussed in section 3.2 can be generalized to sparse matrices. Let Φ be a t-disjunct $m \times M$ sparse matrix. Suppose the sparse vector x has support S, with $|S| = k \leq t+1$, and $y = \Phi x$. Then,

$$y_j = \sum_{l \in supp(\Phi_S(j,:))} \Phi(j,l)x_l, \ \forall \ j \in [m].$$

We assume that $y_j \neq 0$ whenever $supp(\Phi_S(j,:))$ is non empty. This holds (a) with probability 1 if x is drawn from a generic random model, and (b) when x is a non negative sparse vector. Again, we note that these assumptions are mild.

4.2.1. *Support Recovery*

As Φ is t-disjunct and $|S| = k \leq t+1$, we can find the support of x as $S = \{i : supp(\Phi(:,i)) \subseteq supp(y)\}$.

4.2.2. *Non-zero coefficient recovery*

Similar to binary matrices, we make use of the disjunct property of sparse matrices for finding the nonzero coefficients of x. For $i \in [k]$, there exists j_i such that $\Phi_S(j_i, i) \neq 0$ and $\Phi_S(j_i, l) = 0$ for all $l \in [k] \setminus i$. Now set

$$x_i = \begin{cases} \frac{y_{j_i}}{\Phi(j_i,i)}, & \text{if } i \in S \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

In this case also, the recovery of x is non-iterative.

5. SIMULATION RESULTS

In this section, we present our numerical observations on sparse signal recovery abilities of distunct matrices via our proposed non-iterative algorithms. We also compare our solution with the recently proposed non-iterative algorithms presented in [11]. We use the binary sensing matrix Φ of size $q^2 \times q^{r+1}$ constructed in [12] with q being prime power and $r > 1$. by construction, every column of Φ has q ones and the overlap between any two distinct columns is at most r. The mutual coherence μ_Φ of Φ is $\frac{r}{q}$ [12]. From Theorem 1, Φ is $\lfloor \frac{q-1}{r} \rfloor$-disjunct. By Theorem 4, Φ is also t^e-disjunct with $t < \lfloor \frac{q}{r} \rfloor$ and $e + 1 \geq q - tr$.

As an example, we take Φ of size $(29)^2 \times (29)^3$. Therefore, Φ is $14-$disjunct and also 7^{14}-disjunct (i.e., $t = 7, e = 14$) and $\mu_\Phi \leq \frac{2}{29}$. We consider the sparsity $k \leq 33$. For a fixed k, we generate 1000 different $k-$sparse signals with both support set of size k and the non-zero values of sparse signal x generated at random. Our algorithm recovers the unknown sparse vector x up to sparsity 15 exactly in all 1000 trials, as expected from the sufficient condition in Theorem 1. Further, our algorithm can exactly recover x with sparsity 33 in all 1000 trials, i.e., it can recover much denser vectors. In contrast, the sparse recovery algorithm proposed in [11] can recover the unknown sparse vector x only up to sparsity 7 exactly in all 1000 trials. Beyond a sparsity level 9, it fails to recover even a single unknown sparse vector x. This is because the algorithm in [11] requires $4k < q$, i.e., $k < 8$, in order to ensure that each nonzero entry in x occurs at least $q/2$ times in y. This clearly demonstrates the better recovery performance of our algorithm over the algorithm in [11].

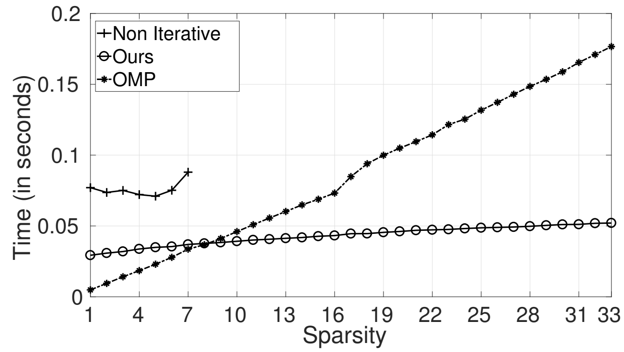

Fig. 1. Avarage runtime comparison between Our proposed method, OMP and existing non-iterative algorithm for matrix size $(29)^2 \times (29)^3$. The plot suggests that our proposed algorithm takes the least time among the three.

Figure 1 illustrates that the run time of our proposed algorithm is far lower than OMP and the non-iterative algorithm in [11]. While the runtime of the non-iterative algorithm is roughly constant, it cannot go beyond a sparsity level of 7. On the other hand, OMP's complexity is linear in the sparsity level and far exceeds the complexity of our non-iterative algorithm at higher sparsity levels. The complexity of our algorithm only increases marginally with the sparsity level.

6. CONCLUSIONS AND FUTURE DIRECTIONS

We showed that disjunctness property of a binary matrix is a crucial parameter in compressed sensing. For disjunct binary matrices, we provided theoretical guarantees in recovering sparse signals. Then we generalized the disjunctness from binary matrices to sparse matrices. We used disjuntness of sparse matrices to provide theoretical guarantees for recovering sparse signals using sparse sensing matrices. Numerical results suggest that our proposed sparse signal recovery algorithms can recover signals with higher sparsity than the recently proposed non-iterative sparse signal recovery algorithms. Our future work will consider bounds on the number of rows required for the measurement matrix to satisfy t-disjunctness and sparse signal recovery guarantees for disjunct matrices in noisy measurement settings.

7. REFERENCES

[1] D. L. Donoho and M. Elad, "Optimally sparse representation in general (nonorthogonal) dictionaries via l1 minimization," *Proc. National Academy of Sciences*, vol. 100, no. 5, pp. 2197–2202, 2003.

[2] E. Candes and T. Tao, "Decoding by linear programming," *IEEE Trans. Inform. Theory*, vol. 51, no. 12, pp. 4203–4215, Dec. 2005.

[3] D. L. Donoho, "Compressed sensing," *IEEE Trans. Inform. Theory*, vol. 52, pp. 1289–1306, 2006.

[4] J. Bourgain, S. J. Dilworth, K. Ford, S. Konyagin, and D. Kutzarova, "Explicit constructions of RIP matrices and related problems," *Duke Math. J.*, vol. 159, no. 1, pp. 145–185, 2011.

[5] J. A. Tropp, "Greed is good: algorithmic results for sparse approximation," *IEEE Trans. Inform. Theory*, vol. 50, no. 10, pp. 2231–2242, 2004.

[6] E. Candès, "The restricted isometry property and its implications for compressed sensing," *Comptes Rendus Mathematique*, vol. 346, no. 9-10, pp. 589–592, 2008.

[7] J. A. Tropp and S. J. Wright, "Computational methods for sparse solution of linear inverse problems," *Proc. IEEE*, vol. 98, no. 6, pp. 948–958, 2010.

[8] A. Barg, A. Mazumdar, and R. Wang, "Restricted isometry property of random subdictionaries," *IEEE Trans. Inform. Theory*, vol. 61, no. 8, pp. 4440–4450, 2015.

[9] C. L. Chan, S. Jaggi, V. Saligrama, and S. Agnihotri, "Non-adaptive group testing: Explicit bounds and novel algorithms," *IEEE Trans. Inform. Theory*, vol. 60, no. 5, pp. 3019–3035, 2014.

[10] A. J. Macula, "Error-correcting nonadaptive group testing with de-disjunct matrices," *Discrete Applied Mathematics*, vol. 80, no. 2-3, pp. 217–222, 1997.

[11] M. Lotfi and M. Vidyasagar, "A fast noniterative algorithm for compressive sensing using binary measurement matrices," *IEEE Trans. Sig. Process.*, vol. 66, pp. 4079–4089, Aug. 2018.

[12] R. A. DeVore, "Deterministic constructions of compressed sensing matrices," *Journal of Complexity*, vol. 23, no. 4, pp. 918–925, 2007.