WEAKLY STANDARD INTERFERENCE MAPPINGS: EXISTENCE OF FIXED POINTS AND APPLICATIONS TO POWER CONTROL IN WIRELESS NETWORKS

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ABSTRACT

We propose novel approaches to identify the existence of fixed points of the so-called weakly standard interference mappings, which include the well-known standard and general interference mappings as particular cases. The approaches are based on the concept of spectral radius of asymptotic mappings, a mathematical tool recently introduced to study the behavior of wireless networks. We show that, for arbitrary weakly standard interference mappings, knowledge of the spectral radius of an associated asymptotic mapping gives a sufficient condition to determine the existence of fixed points or their absence in the positive orthant. If the mapping has a fixed point, we further prove that the set of fixed points has a minimal element that can be easily computed with a simple fixed point algorithm. The theory developed here is applied to the problem of power control for load planning in LTE networks. Unlike previous approaches in the literature, the proposed solution takes into account the limited number of modulation and coding schemes of practical transceivers.

Index Terms— Interference mappings, power control, asymptotic mappings, fixed point algorithms

1. INTRODUCTION

The problem of determining rate regions of some wireless networks is known to be strongly related to the existence of fixed points of special classes of nonnegative mappings [1–3], typically standard interference mappings [4] or general interference mappings [1, 5] (see Definition 1 in the next section). If we further restrict the mappings to be affine, this existence problem is now well understood, and the solutions are deeply rooted in linear algebra. However, extending these existing results to more general problems involving nonlinear mappings, as required in modern communication systems [1, 6–11], has proven to be significantly difficult, as explicitly mentioned in [1, Ch. 4.2]. Therefore, previous studies have restricted the attention to particular cases of nonlinear mappings.

For example, in [1, Ch. 4.2] the existence problem has been studied for a class of log-concave mappings, and some results are based on the existence of a matrix for which no simple construction method is provided (see, for instance, [1, Theorem 4.22]). For a very specific interference mapping that models interference in LTE networks, the study in [10] has shown that the spectral radius of a matrix constructed from the mapping gives a necessary and sufficient condition for the existence of the fixed point of the mapping. This result has been shown in [11] to be a particular instance of a more general approach able to determine the existence of fixed points of a large subclass of positive concave mappings. More recently, the studies in [7, 8] have unified and generalized many of the schemes described in [2, 10, 11]. In particular, [7, Proposition 2] or [8, Proposition 4] shows a simple necessary and sufficient condition for a continuous standard interference mapping to have a fixed point, and the condition is often easy to verify in practice. However, the approaches proposed in [7, 8] are not applicable to some interference models in [1] involving general interference mappings. The reason is that, although related, the sets of standard interference mappings and general interference mappings are disjoint (see Definition 1 below), so results involving the former mappings do not necessarily carry over to the latter mappings (see Example 1 in Sect. 3).

Against this background, building upon the findings in [7, 8], we propose simple methods to determine the existence of fixed points of a class of mappings that include both standard and general interference mappings as particular cases. For this class of mappings, we further show a simple fixed point algorithm to compute the minimal element of the set of fixed points, if this set is nonempty. As a concrete application of these results, we solve the problem of downlink power control for load planning in OFDMA networks with rate constraints, and we note that the unconstrained version of this problem has been addressed in [12] and later in [11].

2. PRELIMINARIES

In this section we establish notation and show the mathematical tools used in the proof of the main results. In more detail, the sets of nonnegative and positive reals are denoted by, respectively, $\mathbb{R}_+$ and $\mathbb{R}_{++}$. Inequalities involving vectors should be understood coordinate-wise. We say that a sequence $(\boldsymbol{x}_n)_{n\in\mathbb{N}} \subset \mathbb{R}^N$ converges to $\boldsymbol{x}^\star$ if $\lim_{n\to\infty} \|\boldsymbol{x}_n - \boldsymbol{x}^\star\| = 0$ for some (and hence for every) norm $\|\cdot\|$ in $\mathbb{R}^N$, and in this case we also write $\boldsymbol{x}_n \to \boldsymbol{x}^\star$.

The focus of this study is on functions belonging to the following classes:

Definition 1. Consider the following possible statements for a *continuous* function $f : \mathbb{R}_+^N \to \mathbb{R}_+$:

[(i)] [monotonicity] $(\forall \boldsymbol{x} \in \mathbb{R}_+^N)(\forall \boldsymbol{y} \in \mathbb{R}_+^N)\ \boldsymbol{x} \geq \boldsymbol{y} \Rightarrow f(\boldsymbol{x}) \geq f(\boldsymbol{y})$

[(ii)] [nonnegative homogeneity] $(\forall \boldsymbol{x} \in \mathbb{R}_+^N)$ $(\forall \alpha \geq 0)$ $\alpha f(\boldsymbol{x}) = f(\alpha \boldsymbol{x})$

[(iii)] [scalability] $(\forall \boldsymbol{x} \in \mathbb{R}_+^N)$ $(\forall \alpha > 1)$ $\alpha f(\boldsymbol{x}) > f(\alpha \boldsymbol{x})$.

[(iv)] [weak scalability] $(\forall \boldsymbol{x} \in \mathbb{R}_+^N)$ $(\forall \alpha \geq 1)$ $\alpha f(\boldsymbol{x}) \geq f(\alpha \boldsymbol{x})$.

If (i) and (ii) are satisfied, then f is said to be a *general interference function* [1]. If f satisfies (i) and (iii), then f is called a *standard interference function* [4]. If f satisfies (i) and (iv), then f is called a *weakly standard interference function*.

A mapping $T : \mathbb{R}_+^N \to \mathbb{R}_+^N : \boldsymbol{x} \mapsto [f_1(\boldsymbol{x}), \ldots, f_N(\boldsymbol{x})]$ is said to be a standard interference (SI) mapping, a weakly standard interference (WSI) mapping, or a general interference (GI) mapping if each coordinate function $f_i : \mathbb{R}_+^N \to \mathbb{R}_+$ $(i = 1, \ldots, N)$

is, respectively, a standard function, a weakly standard interference function, or a general interference function. For convenience, we denote by $\mathcal{F}_{\mathrm{SI}}$, $\mathcal{F}_{\mathrm{WSI}}$, $\mathcal{F}_{\mathrm{GI}}$ the sets of, respectively, SI mappings, WSI mappings, and GI mappings. By definition, all these mappings are *monotonic*, in the sense that $T:\mathbb{R}_+^N\to\mathbb{R}_+^N\in\mathcal{F}_{\mathrm{WSI}}$ satisfies $(\forall \boldsymbol{x}_1\in\mathbb{R}_+^N)(\forall \boldsymbol{x}_2\in\mathbb{R}_+^N)\ \boldsymbol{x}_1\geq\boldsymbol{x}_2\Rightarrow T(\boldsymbol{x}_1)\geq T(\boldsymbol{x}_2)$. It follows directly from the definition of the mappings that the sets $\mathcal{F}_{\mathrm{SI}}$ and $\mathcal{F}_{\mathrm{GI}}$ are disjoint, and these two sets are proper subsets of $\mathcal{F}_{\mathrm{WSI}}$.

The set of fixed points of a mapping $T:\mathbb{R}_+^N\to\mathbb{R}_+^N$ is denoted by $\mathrm{Fix}(T):=\{\boldsymbol{x}\in\mathbb{R}_+^N\mid T(\boldsymbol{x})=\boldsymbol{x}\}$. If $T\in\mathcal{F}_{\mathrm{SI}}$, then $\mathrm{Fix}(T)$ is either a singleton or the empty set [4].

An important tool for the main results in this study is the notion of asymptotic mappings, which has been recently introduced in [7, 8] and slightly extended in [13]. These mappings are defined as follows:

Definition 2. Let $T:\mathbb{R}_+^N\to\mathbb{R}_+^N$ be a weakly standard interference mapping. Its associated asymptotic mapping is the continuous mapping given by $T_\infty:\mathbb{R}_+^N\to\mathbb{R}_+^N:\boldsymbol{x}\mapsto\lim_{p\to\infty}\dfrac{1}{p}T(p\boldsymbol{x})$. [1]

We can verify that asymptotic mappings are members of $\mathcal{F}_{\mathrm{GI}}$ [8]; i.e., $(\forall T\in\mathcal{F}_{\mathrm{WSI}})\ T_\infty\in\mathcal{F}_{\mathrm{GI}}$. Therefore, we can define a weak notion of spectral radius for an arbitrary asymptotic mapping:

Definition 3. [14] The spectral radius $\rho(T_\infty)$ of an asymptotic mapping T_∞ associated with $T\in\mathcal{F}_{\mathrm{WSI}}$ is defined by $\rho(T_\infty):=\sup\{\lambda\in\mathbb{R}_+\mid(\exists\boldsymbol{x}\in\mathbb{R}_+^N\backslash\{\boldsymbol{0}\})\ T_\infty(\boldsymbol{x})=\lambda\boldsymbol{x}\}\in\mathbb{R}_+$.

We recall that there always exists $\boldsymbol{x}\in\mathbb{R}_+^N$ satisfying $\rho(T_\infty)\boldsymbol{x}=T_\infty(\boldsymbol{x})$ (see [15, Proposition 5.3.2(ii)] and [15, Corollary 5.4.2]). Furthermore, the spectral radius of any asymptotic mapping T_∞ can be easily estimated with simple approaches [8, 13].

The next fact, which is crucial to the main contributions that follow, is one of the results in [7, 8] that has been used to characterize the existence of fixed points of mappings $T\in\mathcal{F}_{\mathrm{SI}}$.

Fact 1. [7, 8] Let $T:\mathbb{R}_+^N\to\mathbb{R}_+^N$ be a standard interference mapping. Then $\mathrm{Fix}(T)\neq\emptyset$ if and only if $\rho(T_\infty)<1$.

Note that Fact 1 is especially useful if T is nonlinear with a linear associated asymptotic mapping T_∞. In this particular case, to determine whether $\mathrm{Fix}(T)\neq\emptyset$, we only have to compute the spectral radius of a nonnegative matrix. The study in [7] (see also [8]) shows concrete problems in OFDMA networks in which this interesting case occurs.

3. FIXED POINTS OF WEAKLY STANDARD INTERFERENCE MAPPINGS

In this section, we provide partial generalizations of known results related to SI mappings. As shown later in this study, these generalizations are useful in applications involving interference models for which previous mathematical tools in the wireless literature are not applicable. To motivate the work, we start with simple examples illustrating that known properties of SI mappings do not necessarily carry over to arbitrary WSI mappings.

Example 1. Consider the WSI mappings given by $T':\mathbb{R}_+\to\mathbb{R}_+:x\mapsto x$ and $T'':\mathbb{R}_+\to\mathbb{R}_+:x\mapsto x+1$. Note that $\rho(T'_\infty)=\rho(T''_\infty)=1$. In addition, $\mathrm{Fix}(T')=\mathbb{R}_+$ and $\mathrm{Fix}(T'')=\emptyset$. This example illustrates that, unlike SI mappings, the set of fixed points of WSI mappings is not a singleton or the empty set in general. Furthermore, unlike the result in Fact 1 for SI mappings, knowledge of $\rho(T_\infty)$ does not give a sufficient and necessary condition for the existence of fixed points of general WSI mappings.

[1] Existence of the limit and continuity of asymptotic mappings associated with continuous WSI mappings have been proved in [8].

As it is now clear from Example 1, to gain information about the existence of fixed points from the spectral radius of asymptotic mappings, we need additional assumptions in the statement of Fact 1 if we replace SI mappings with the more general class of WSI mappings. In the remainder of this section, we show results that enable us to identify simple cases in which the spectral radius of asymptotic mappings provides us with information about the existence of fixed points of arbitrary WSI mappings. We also show that the standard fixed algorithm enables us to compute the minimal element of the set of fixed points, if this set is nonempty (see Proposition 3). To derive these results, we need the following two simple lemmas. [We omit the proof of Lemma 1 because it is analogous to similar results in the literature (see, for example, [4]).]

Lemma 1. *Let $T:\mathbb{R}_+^N\to\mathbb{R}_+^N$ be an arbitrary continuous monotonic mapping. Denote by $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ the sequence generated by $\boldsymbol{x}_{n+1}=T(\boldsymbol{x}_n)$, where $\boldsymbol{x}_1\in\mathbb{R}_+^N$ is arbitrary. Then each of the following holds:*

(i) If $T(\boldsymbol{x}_1)\geq\boldsymbol{x}_1$, then $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ is nondecreasing in each component. Furthermore, the sequence $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ either converges or increases unboundedly in norm (i.e., for any norm $\|\cdot\|$, we have $\lim_{n\to\infty}\|\boldsymbol{x}_n\|=\infty$).

(ii) If $T(\boldsymbol{x}_1)\leq\boldsymbol{x}_1$, then $\mathrm{Fix}(T)\neq\emptyset$ and $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ is nonincreasing in each component. In addition, the sequence $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ converges to a fixed point of T.

Lemma 2. *Let $T_1:\mathbb{R}_+^N\to\mathbb{R}_+^N$ and $T_2:\mathbb{R}_+^N\to\mathbb{R}_+^N$ be arbitrary continuous monotonic mappings. Assume that $(\forall\boldsymbol{x}\in\mathbb{R}_+^N)\ T_1(\boldsymbol{x})\leq T_2(\boldsymbol{x})$. If $\mathrm{Fix}(T_2)\neq\emptyset$, then $\mathrm{Fix}(T_1)\neq\emptyset$.*

Proof. Let $\boldsymbol{x}^\star\in\mathrm{Fix}(T_2)$. By assumption, we have $T_1(\boldsymbol{x}^\star)\leq T_2(\boldsymbol{x}^\star)=\boldsymbol{x}^\star$. The desired result $\mathrm{Fix}(T_1)\neq\emptyset$ now follows from Lemma 1(ii). ☐

We are now ready to prove that knowledge of $\rho(T_\infty)<1$ is a sufficient condition for a mapping $T\in\mathcal{F}_{\mathrm{WSI}}$ to have at least one fixed point.

Proposition 1. *Let $T:\mathbb{R}_+^N\to\mathbb{R}_+^N$ be a WSI mapping. If $\rho(T_\infty)<1$, then $\mathrm{Fix}(T)\neq\emptyset$.*

Proof. Define $T':\mathbb{R}_+^N\to\mathbb{R}_{++}^N:\boldsymbol{x}\mapsto T(\boldsymbol{x})+\boldsymbol{1}$, where $\boldsymbol{1}$ denotes the vector of ones. Given an arbitrary scalar $\alpha>1$ and a vector $\boldsymbol{x}\in\mathbb{R}_+^N$, we verify from the definition of WSI mappings that $\boldsymbol{0}<T'(\alpha\boldsymbol{x})=T(\alpha\boldsymbol{x})+\boldsymbol{1}\leq\alpha T(\boldsymbol{x})+\boldsymbol{1}<\alpha(T(\boldsymbol{x})+\boldsymbol{1})=\alpha T'(\boldsymbol{x})$, which shows that $T'\in\mathcal{F}_{\mathrm{SI}}$. We also have $(\forall\boldsymbol{x}\in\mathbb{R}_+^N)\ T_\infty(\boldsymbol{x})=T'_\infty(\boldsymbol{x})$. Therefore, $\rho(T'_\infty)=\rho(T_\infty)<1$, which by Fact 1 implies $\mathrm{Fix}(T')\neq\emptyset$. The desired result $\mathrm{Fix}(T)\neq\emptyset$ is now immediate from Lemma 2 because, by construction, $(\forall\boldsymbol{x}\in\mathbb{R}_+^N)\ T(\boldsymbol{x})\leq T'(\boldsymbol{x})$. ☐

Proposition 1 may not be always satisfactory because it does not rule out the possibility of fixed points on the boundary $\mathrm{bd}(\mathbb{R}_+^N):=\mathbb{R}_+^N\backslash\mathbb{R}_{++}^N$, which may be undesirable fixed points in some (but not all) applications. For example, if the fixed point is a power vector for radios that need to be active, then a fixed point on $\mathrm{bd}(\mathbb{R}_+^N)$ could imply that a transmitter should communicate with zero power, which is clearly not possible. To rule out the possibility of fixed points only on $\mathrm{bd}(\mathbb{R}_+^N)$, we need additional assumptions, and the following corollary of Proposition 1 will be useful in the application described in Sect. 4.

Corollary 1. *Let $T : \mathbb{R}_+^N \to \mathbb{R}_+^N$ be a WSI mapping satisfying $\rho(T_\infty) < 1$. If $(\forall \boldsymbol{x} \in \mathbb{R}_+^N)(\exists m \in \mathbb{N})\ T^m(\boldsymbol{x}) > \boldsymbol{0}$, then $\emptyset \neq \mathrm{Fix}(T) \subset \mathbb{R}_{++}^N$.*

Proof. It follows from Proposition 2 that T has at least one fixed point. From the assumptions of the corollary, for an arbitrary fixed point $\boldsymbol{x}^\star \in \mathrm{Fix}(T)$, we have $\boldsymbol{x}^\star = T^m(\boldsymbol{x}^\star) > \boldsymbol{0}$ for some $m \in \mathbb{N}$. This inequality completes the proof. □

For the case $\rho(T_\infty) > 1$, we have the following result.

Proposition 2. *If the asymptotic mapping $T_\infty : \mathbb{R}_+^N \to \mathbb{R}_+^N$ associated with a WSI mapping $T : \mathbb{R}_+^N \to \mathbb{R}_+^N$ satisfies $\rho(T_\infty) > 1$, then T has no fixed point in $\mathbb{R}_{++}^N$; i.e., $\mathrm{Fix}(T) \cap \mathbb{R}_{++} = \emptyset$.*

Proof. Assume for the sake of contradiction that $\rho(T_\infty) > 1$ and there exists $\boldsymbol{x}^\star \in \mathbb{R}_{++}^N$ such that $\boldsymbol{x}^\star = T(\boldsymbol{x}^\star)$. Then, by using Definition 2 and arguments similar to those in the proof of [16, Lemma 1(ii)], we can show that $T_\infty(\boldsymbol{x}^\star) \leq T(\boldsymbol{x}^\star) = \boldsymbol{x}^\star$, which implies $\rho(T_\infty) \leq 1$ by [14, Theorem 3.1(1)]. This inequality contradicts the assumption $\rho(T_\infty) > 1$, and the proof is complete. □

Although WSI mappings need not have a unique fixed point, we can show that there exists a minimal fixed point by considering the partial ordering induced by the nonnegative cone $\mathbb{R}_+^N$. This minimal point can be easily obtained with the standard fixed point iteration $\boldsymbol{x}_{n+1} = T(\boldsymbol{x}_n)$ by starting from $\boldsymbol{x}_1 = \boldsymbol{0}$. These results are formally stated in the next proposition.

Proposition 3. *Let $T : \mathbb{R}_+^N \to \mathbb{R}_+^N$ be a WSI mapping satisfying $\rho(T_\infty) < 1$. Then $(\exists \boldsymbol{x}^\star \in \mathrm{Fix}(T))(\forall \boldsymbol{y} \in \mathrm{Fix}(T))\ \boldsymbol{x}^\star \leq \boldsymbol{y}$. In addition, this minimal fixed point $\boldsymbol{x}^\star$ is the limit of the sequence $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ generated by $\boldsymbol{x}_{n+1} = T(\boldsymbol{x}_n)$ with $\boldsymbol{x}_1 = \boldsymbol{0}$.*

Proof. By Proposition 1, we know that $\mathrm{Fix}(T) \neq \emptyset$. Let the sequence $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ be generated as stated in the proposition, and let $\boldsymbol{y} \in \mathrm{Fix}(T)$ be arbitrary. By $\boldsymbol{0} \leq \boldsymbol{y}$, it follows from monotonicity of T that

$$(\forall n \in \mathbb{N})\ \boldsymbol{x}_{n+1} = T^n(\boldsymbol{0}) \leq T^n(\boldsymbol{y}) = \boldsymbol{y}, \tag{1}$$

which in particular implies boundedness of $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$. Furthermore, we have $\boldsymbol{x}_1 = \boldsymbol{0} \leq T(\boldsymbol{0})$, so $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ converges to a point $\boldsymbol{x}^\star \in \mathrm{Fix}(T) \subset \mathbb{R}_+^N$ as an implication of Lemma 1(i). Passing to the limit as $n \to \infty$ in (1), we obtain $\boldsymbol{x}^\star \leq \boldsymbol{y}$. Since the choice of $\boldsymbol{y} \in \mathrm{Fix}(T)$ is arbitrary, the fixed point $\boldsymbol{x}^\star$ is minimal, and the proof is complete. □

From a practical perspective, Proposition 3 is important for the following reason. In wireless resource allocation tasks, if the argument of a mapping $T \in \mathcal{F}_{\mathrm{WSI}}$ has the interpretation of power of base stations, and the fixed points determine feasible power configurations, then the minimal element of $\mathrm{Fix}(T)$ is the configuration able to satisfy the requirements of the network with the minimum total power. We formalize this simple fact in the next corollary, which is immediate from Proposition 3.

Corollary 2. *Let $T \in \mathcal{F}_{\mathrm{WSI}}$ and $f : \mathbb{R}_+^N \to \mathbb{R}$ be an arbitrary function satisfying $(\forall \boldsymbol{x} \in \mathbb{R}_+^N)(\forall \boldsymbol{y} \in \mathbb{R}_+^N)\ \boldsymbol{x} \geq \boldsymbol{y} \Rightarrow f(\boldsymbol{x}) \geq f(\boldsymbol{y})$. Then the minimal fixed point $\boldsymbol{x}^\star$ described in Proposition 3 is a solution to the following optimization problem:*

$$\begin{array}{ll} \mathrm{minimize}_{\boldsymbol{x}} & f(\boldsymbol{x}) \\ \mathrm{s.t.} & \boldsymbol{x} \in \mathrm{Fix}(T) \end{array} \tag{2}$$

4. POWER CONTROL IN OFDMA NETWORKS WITH LIMITED NUMBER OF MODULATION AND CODING SCHEMES

We now apply the results in the previous section to the problem of downlink power control for load planning in OFDMA networks. In Sect. 4.1 we describe an existing interference model with rate constraints. Sect. 4.2 formalizes the novel power control problem and describes the proposed algorithm. Sect. 4.3 shows the performance of the proposed approach in a particular scenario.

4.1. System model

We consider a scenario where N base stations transmit data to M users. The set of base stations and users are denoted by, respectively, $\mathcal{N} := \{1, \ldots, N\}$ and $\mathcal{M} := \{1, \ldots, M\}$, and each user requests the downlink rate d_j, $j \in \mathcal{M}$. As common in OFDMA networks, the time and frequency grid is divided into $K \in \mathbb{N}$ units called resource blocks. Each base station serves at least one user, and the set of users connected to base station $i \in \mathcal{N}$ is denoted by $\mathcal{M}_i \subset \mathcal{M}$. The effective gain of the wireless path between base station $i \in \mathcal{N}$ and user $j \in \mathcal{M}$ is denoted by $g_{i,j} \in \mathbb{R}_{++}$. Each base station $i \in \mathcal{N}$ transmits with power $p_i \in \mathbb{R}_{++}$ per resource block, and, using standard simplifications to make the mathematical problems tractable [9, 10, 12, 17–20], we assume that this power is the same across all resource blocks. We denote by $\boldsymbol{p} := [p_1, \ldots, p_N] \in \mathbb{R}_+^N$ the transmit power vector. The load $\rho_i \in\ ]0,1]$ of base station $i \in \mathcal{N}$ is defined as the fraction of the K resource blocks that the base station uses for data transmission, and we collect the load of each base station in the vector $\boldsymbol{\rho} = [\rho_1, \ldots, \rho_N] \in\ ]0,1]^N$. If the load $\boldsymbol{\rho}$ is known for a given power allocation $\boldsymbol{p} \in \mathbb{R}_{++}^N$, the achievable rate per resource block of a link connecting base station $i \in \mathcal{N}$ to user $j \in \mathcal{M}$ is assumed to be given by $\omega_{i,j}(\boldsymbol{p}) = \min\{c_{i,j}(\boldsymbol{p}), u\}$, where

$$c_{i,j}(\boldsymbol{p}) := B \log_2\left(1 + \frac{p_i g_{i,j}}{\sum_{k\in\mathcal{N}\setminus\{i\}} \rho_k p_k g_{k,j} + \sigma^2}\right)$$

is the achievable rate of the link between base station i and user j, $\sigma^2 \in \mathbb{R}_{++}$ is the noise per resource block, $B \in \mathbb{R}_{++}$ is the bandwidth of each resource block, and $u \in \mathbb{R}_{++}$ is the maximum rate that we can achieve in a resource block because of the limited number of modulation and coding schemes [7–9, 18]. If the power $\boldsymbol{p} \in \mathbb{R}_{++}^N$ induces the load $\boldsymbol{\rho} \in\ ]0,1]^M$ in the network, then it follows from the definition of load that the load ρ_i at base station $i \in \mathcal{N}$ is given by

$$\rho_i = \sum_{j\in\mathcal{M}_i} \frac{d_j}{K\omega_{i,j}(\boldsymbol{p})} > 0. \tag{3}$$

4.2. Problem statement and proposed solution

Having introduced the system model, we can now informally state the power estimation problem as follows: given a desired load allocation $\boldsymbol{\rho} \in\ ]0,1]^N$, we have to estimate the power vector $\boldsymbol{p} \in \mathbb{R}_{++}^N$ inducing this load, if it exists. In the remainder of this subsection, we show that this problem can be formally posed as a fixed point problem involving a WSI mapping.

In more detail, assuming that (3) holds, we can multiply both sides of (3) by p_i/ρ_i for each $i \in \mathcal{N}$ to obtain:

$$p_i = \sum_{j\in\mathcal{M}_i} \max\left\{g'_{i,j}(\boldsymbol{p}),\ \frac{p_i\ d_j}{\rho_i\ u}\right\} =: t'_{\boldsymbol{\rho},i}(\boldsymbol{p}), \tag{4}$$

where $g'_{i,j} : \mathbb{R}^N_{++} \to \mathbb{R}_+ : \boldsymbol{p} \mapsto p_i\, d_j/(\rho_i\, K c_{i,j}(\boldsymbol{p}))$ for each $i \in \mathcal{N}$ and $j \in \mathcal{M}$ is a positive concave function [19, 21, 22] (and thus a SI function [11, Proposition 1]). Equation (4) shows that, for a given load vector $\boldsymbol{\rho} \in\ ]0,1]^N$, the power inducing this load, if it exists, is the fixed point of the nonlinear mapping given by $T'_{\boldsymbol{\rho}} : \mathbb{R}^N_{++} \to \mathbb{R}^N_+ : \boldsymbol{p} \mapsto [t'_{\boldsymbol{\rho},1}(\boldsymbol{p}) \cdots t'_{\boldsymbol{\rho},N}(\boldsymbol{p})]$.

We can show that, in the positive cone $\mathbb{R}^N_{++}$, the mapping $T'_{\boldsymbol{\rho}}$ is monotonic and weakly scalable because these properties are preserved by taking positive sums and coordinate-wise maximum of mappings with these properties. Furthermore, for any $\boldsymbol{\rho} \in\ ]0,1]^N$, the mapping $T'_{\boldsymbol{\rho}} : \mathbb{R}^N_{++} \to \mathbb{R}^N_+$ is continuous, so, to apply the theory developed in the previous sections, we only need to extend $T'_{\boldsymbol{\rho}}$ continuously to the domain $\mathbb{R}^N_+$. The existence of this continuous extension, which we denote by $T_{\boldsymbol{\rho}} : \mathbb{R}^N_+ \to \mathbb{R}^N_+ \in \mathcal{F}_{\mathrm{WSI}}$, follows from [15, Theorem 5.1.5]. To obtain the values $T_{\boldsymbol{\rho}}(\boldsymbol{p})$ for $\boldsymbol{p} \in \mathrm{bd}(\mathbb{R}^N_+)$, we can use the same approach used in the proof of [22, Lemma 3], but we omit the details because of the space limitation. By doing so, we verify that $(\forall \boldsymbol{p} \in \mathbb{R}^N_+)(\forall \boldsymbol{\rho} \in \mathbb{R}^N_+) T_{\boldsymbol{\rho}}(\boldsymbol{p}) > \boldsymbol{0}$, so, by Corollary 1, we have $\emptyset \neq \mathrm{Fix}(T) \subset \mathbb{R}^N_{++}$ if $\rho((T_{\boldsymbol{\rho}})_\infty) < 1$. Moreover, in light of Proposition 2, we also know that $\mathrm{Fix}(T) = \emptyset$ if $\rho((T_{\boldsymbol{\rho}})_\infty) > 1$.

The above shows that the spectral radius $\rho((T_{\boldsymbol{\rho}})_\infty)$ of $(T_{\boldsymbol{\rho}})_\infty$ is a convenient means of identifying feasibility in many practical cases. Once we certify that $\mathrm{Fix}(T_{\boldsymbol{\rho}}) \neq \emptyset$ for the desired load vector $\boldsymbol{\rho} \in\ ]0,1]^N$, we can proceed to compute the power allocation $\boldsymbol{p}^\star \in \mathrm{Fix}(T_{\boldsymbol{\rho}})$ with the minimum total transmit power $\|\boldsymbol{p}^\star\|_1$; i.e., compute the solution to the optimization problem in (2) with the l_1 norm as the objective function $f : \mathbb{R}^N_+ \to \mathbb{R}_+ : \boldsymbol{x} \mapsto \|\boldsymbol{x}\|_1$. By Corollary 2, this power allocation is the limit of the sequence $(\boldsymbol{p}_n)_{n\in\mathbb{N}}$ generated with the fixed point algorithm

$$\boldsymbol{p}_{n+1} = T_{\boldsymbol{\rho}}(\boldsymbol{p}_n), \quad \boldsymbol{p}_1 = \boldsymbol{0}. \tag{5}$$

Before we proceed with a numerical example, we highlight some key points that distinguish the results shown here from those in previous studies. First, unlike the mappings in [12, 19, 22, 23], the mapping used here can have multiple fixed points. Second, the proposed iteration highlights the need to start the iterations in (5) from $\boldsymbol{p}_1 = \boldsymbol{0}$, which is possible because we have extended the mapping $T'_{\boldsymbol{\rho}}$ continuously to the boundary of its domain. In particular, we do not necessarily obtain the minimal power allocation by starting the fixed point algorithm from arbitrary points $\boldsymbol{p}_1 \in \mathbb{R}^N_+$. Third, in the application described above, we cannot apply the mathematical tools for feasibility assessment described in [7, 8], which justifies the need for the results in Sect. 3.

4.3. Simulations

In the numerical example, unless otherwise explicitly stated, we used the same parameters of the network described in [11, Sect. V-B] (see, in particular, [11, Table I]). The main differences in the numerical example shown here from that in [11, Sect. V-B] are (i) the number of users and their rate requirements (here we simulated $M = 400$ users requesting 100 kbits/s), and (ii) in the current simulations each resource block was limited to a maximum of $u = 200$ kbits/s. To guarantee the existence of the fixed point of $T_{\boldsymbol{\rho}}$ with $\boldsymbol{\rho} = \boldsymbol{1}$ ($\boldsymbol{1}$ denotes the vector of ones), we sampled the random variables of the model described in [11, Sect. V-B] until $\rho((T_{\boldsymbol{\rho}})_\infty) < 1$. The spectral radius of $(T_{\boldsymbol{\rho}})_\infty$ was computed with the scheme in [13, Sect. 3.1].

In Fig. 1 we show the power obtained with the fixed point algorithm in (5) and the power that we obtain with the fixed point algorithms in [11, 12, 22] to induce the load $\boldsymbol{\rho} = \boldsymbol{1}$. We show only one curve for the approaches in [11, 12, 22] because these existing methods compute the same power allocation, and they differ in their computational complexity and convergence speed. However, they do not consider rate constraints, so the transmit power computed with these previous schemes is lower than that obtained with iteration in (5), as verified in Fig. 1. By using the power allocation computed with the fixed point algorithms in a network with bounds on the rate per resource block, we verify in Fig. 2 that the load obtained with the approaches in [11, 12, 22] is greater than one at every base station. This fact shows that the network configuration obtained with the previous algorithms is unable to support the traffic demand [9, 10, 17, 18]. In contrast, the power allocation obtained with the iteration in (5), which takes into account rate constraints, induces the desired load $\boldsymbol{\rho} = \boldsymbol{1}$ at every base station, as expected.

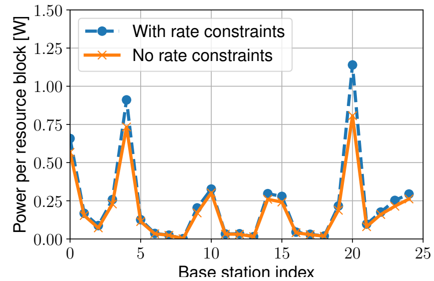

Fig. 1. Power per resource block obtained by solving the fixed point algorithm with and without maximum rate constraints.

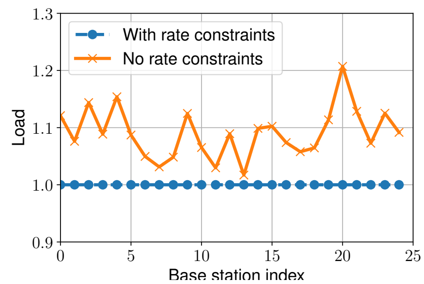

Fig. 2. Load induced with the power allocation computed with the fixed point algorithms. Maximum rate per resource block bounded by 200 kbits/s.

5. SUMMARY AND FINAL REMARKS

We have shown that the concept of asymptotic mappings, which has already been successfully used to characterized the existence of fixed points of SI mappings, can also be used to verify the existence of fixed points of the more general class of WSI mappings. The theory developed here has been applied to the problem of power allocation in OFDMA networks with maximum rate constraints on the resource blocks. In particular, these constraints model the limited number of modulation and coding schemes of practical networks. Previous results in the literature are not applicable to the resulting interference model, and we have shown that these previous methods underestimate the power of base stations. As a consequence, they obtain network configurations that may not be able to support the traffic demand, unlike the configurations obtained with the proposed algorithm.

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