QUANTIZED EVENT-TRIGGERED SAMPLED-DATA AVERAGE CONSENSUS WITH GUARANTEED RATE OF CONVERGENCE

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ABSTRACT

The paper proposes a novel distributed, sampled-data, event-triggered algorithm with quantized information exchange for average consensus (Q-CEASE) in multi-agent/multi-sensor networks. Q-CEASE communicates quantized information with its neighbouring nodes only if a discretized event-triggering condition is satisfied. Both design and implementation of Q-CEASE are distributed and do not require a fusion center. The design stage determines its operating region in terms of the sampling period and transmission thresholds for the constituent nodes. A minimum exponential rate for consensus convergence is guaranteed using the Lyapunov stability theorem. The performance of the Q-CEASE algorithm is quantified through Monte-Carlo simulations on randomized networks.

Index Terms–Distributed average consensus, Event-triggered transmission, Quantized information.

1. INTRODUCTION

Distributed average consensus is a major step in collaborative signal processing applications, including gossip algorithms [1], distributed inference [2], multi-agent coordination [3], sensor calibration [4], and node counting in multi-agent networks [5, Section 3.3]. While traditional distributed consensus implementations are often based on continuous information exchange between the neighbouring nodes, in recent years there has been an overwhelming interest in periodic and/or event-triggered schemes that are able to reduce the transmission load and save scarce energy of the nodes. Numerous event-triggered average consensus (ETAC) schemes have recently been developed based on predefined objectives [6–10]. Despite offering some advantages, existing event-triggered consensus strategies face at least one of the following three shortcomings.

First, the event-triggering condition in many ETAC schemes should be *continuously* monitored during the average consensus process. Along with the implementation difficulties of such approaches, constant measurement and monitoring of the event-triggering condition waste valuable energy resources available at the nodes [11]. In order to relax this limitation, sampled-data approaches have been proposed to enable the event-triggering condition to be monitored only at periodic samples of the system [12]. Introducing a sampler to the scheme, however, makes convergence analysis of the closed-loop system difficult.

Secondly, in the context of ETAC it is often assumed that the information can be transmitted with infinite precision. In practice, however, the information is quantized to a finite number of levels [13]. It is, therefore, necessary to study the effect of quantization error on average consensus.

Thirdly, an important design criteria in distributed networks is the *convergence rate* of the proposed implementation [14]. In this regard, a minimum convergence rate guarantee is of great value. Most existing sampled-data ETAC approaches, however, are only designed with an asymptotic convergence rate. The rate of convergence remains unknown [9]. In general, one expects that higher the connectivity in the network faster should be the convergence. A mathematical characterization of this expectation along with the determination of the sampling period and event-triggering threshold is an interesting and yet a disregarded challenge.

To the best of our knowledge, there exists no implementation for sampled-data event-triggered average consensus that uses quantized data and guarantees a rate of convergence. To address this gap, we extend our CEASE framework [15] by incorporating a quantizer in the transmission scheme. It is worth mentioning that many distributed implementations of different signal processing applications [16–19] require a fusion center for the parameter design stage and/or the termination step. However, the implementation of the proposed Q-CEASE (quantized collaborative event-triggered average consensus sampled-data) algorithm in this paper is distributed even for the design stage. A fusion center is not needed.

2. NOTATION AND PROBLEM STATEMENT

Matrices and vectors are denoted by bold alphabets, and scalars by normal fonts. Notation $(\cdot)^{\mathcal{T}}$ represents transpose of the argument. $\mathbf{1}_N$ and $\mathbf{0}_N$ are column vectors of order N with, respectively, one and zero entries. For a network with N nodes, $\boldsymbol{\mathcal{A}} = \{a_{i,j}\}_{N\times N}$ is the weighted adjacency matrix; $\boldsymbol{L}$ is the Laplacian matrix; λ_2 and λ_N are, respectively, the second smallest and the largest eigenvalues of $\boldsymbol{L}$; and $\mathcal{N}_i$ is the neighbouring set for node i. A uniform quantizer $q(\cdot):\mathbb{R}\rightarrow\delta\mathbb{Z}$ with a quantization level $\delta>0$ is defined by $q(x)=\lfloor x\delta^{-1}+0.5\rfloor\delta$, where the operation $\lfloor\cdot\rfloor$ is the greatest integer less than or equal to the argument. For a uniform quantizer with quantization level δ, it holds that $|\,q(x)-x\,|\leq 0.5\,\delta$, [9].

We use the first-order multi-agent system given below to reach distributed average consensus on an initially localized parameter $x_i(0)$, [20],

$$\dot{x}_i(t)=u_i(t),\quad (1\leq i\leq N), \tag{1}$$

where $u_i(t)\in\mathbb{R}$ is the proposed distributed control signal (to be introduced later), which forces the agents to reach the

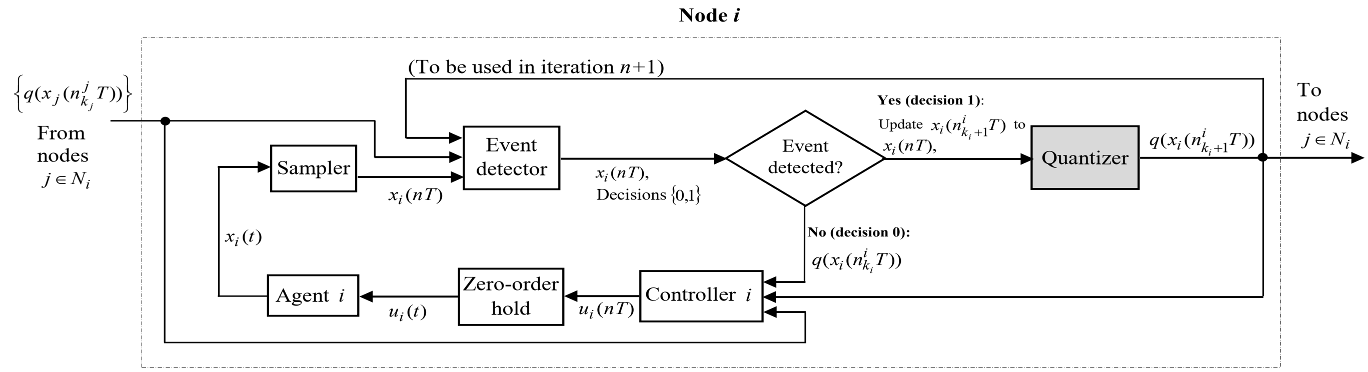

Fig. 1: Proposed Q-CEASE for node i, iteration n, and the previously quantized event $q(x_i(n^i_{k_i}T))$. Compared to implementation [15], the shaded quantization block adds an additional level of complexity.

average of $x_i(0)$, for $(1 \leq i \leq N)$, i.e.,

$$\lim_{t\to\infty} |x_i(t) - \bar{x}(0)\,| = 0, \quad (1 \leq i \leq N), \tag{2}$$

where $\bar{x}(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)$.

Assumption 1. The multi-agent system (1) is configured as an undirected (symmetric) connected network.

To reach average consensus, node i shares its state information within its neighbourhood $\mathcal{N}_i$. In order to reduce the amount of information exchange and preserve the allocated on-board energy available at each agent, a localized event-detector is incorporated at each node. The event-detector is responsible to monitor an 'event-triggering condition' (to be introduced later) to determine whether or not to transmit the state of agent i. Only if the event-triggering condition is satisfied, an event is triggered and node i transmits its state value. Since constant-time monitoring of the event-triggering condition is energy consuming and often not practical, a local sampler (with sampling period T) is used to sample the state value $x_i(t)$ and provides discrete samples $x_i(nT)$, $(n \in \mathbb{N})$. Therefore, the event-detector only monitors a discretized condition with a frequency of T^{-1} Hz. Due to the finite word length limitation, it is not possible to transmit the precise values of the states. Therefore, a uniform quantizer (with quantization level δ) is incorporated at each node to quantize the state value before being transmitted. Let $n^i_{k_0}T, n^i_{k_1}T, \ldots$ denote the time sequence at which events are triggered at node i. We denote the most recent event of node i by $\hat{x}_i(nT) \triangleq x_i(n^i_{k_i}T)$, $t \in [n^i_{k_i}T, n^i_{k_i+1}T)$. Using the quantized values of the most recent events, the following control signal is generated to enable consensus at node i

$$u_i(t) = -\mathbb{X}_i(nT), \;\; (1 \leq i \leq N), \tag{3}$$

where $\mathbb{X}_i(nT) = \sum_{j\in\mathcal{N}_i} a_{ij}\big(\,q(\hat{x}_i(nT)) - q(\hat{x}_j(nT))\,\big)$ is the disagreement value for node i. Based on the definition of the uniform quantizer, it holds that $q\,(\,\hat{x}_i(nT)\,) = \hat{x}_i(nT) + \tilde{\delta}_i(nT)$, where $|\,\tilde{\delta}_i(nT)\,| \leq 0.5\,\delta$. The disagreement value $\mathbb{X}_i(nT)$ is used by the event detector. More precisely, for a given event instant $t = n^i_{k_i}T$, the next event for agent i is triggered at $t = n^i_{k_i+1}T$, with $n^i_{k_i+1}$ satisfying the following event-triggering condition

$$n^i_{k_i+1} = \min_n \{n \mid n > n^i_{k_i}, |e_i(nT)| - \phi\,|\mathbb{X}_i(nT)| \geq 0\}, \tag{4}$$

where $e_i(nT) = \hat{x}_i(nT) - x_i(nT)$ is the transmission error for node i. Scalar $\phi > 0$ is the transmission threshold to be designed. The block diagram for the proposed Q-CEASE algorithm is shown in Fig 1.

Design objectives: The design objectives to be investigated in this paper are fourfold: **(i)** A sampled-data scheme is often translated to a time-varying time-delay in the multi-agent system [11]. But a large sampling period T (or, equivalently a large time-delay) can endanger stability of the closed-loop system. Therefore, the maximum allowable sampling period T that guarantees average consensus is of great interest. **(ii)** From (4), it is clear that if $\phi \to 0$, the event-triggered sampled-data scheme reduces to a sampled-data alone system studied previously in [21]. On the other hand, an extremely large value for ϕ may prevent average consensus due to insufficient information exchanges in the network. Hence, the maximum allowable value for ϕ (that reduces frequency of the events as much as possible) is of paramount interest. **(iii)** Since the state value $\hat{x}_i(nT)$, $(1 \leq i \leq N)$, is quantized before transmission, exact average value cannot be reached [9]. The upper-bound error for the proposed scheme should, therefore, be known precisely. **(iv)** In contrast to [22] where an asymptotic consensus convergence rate is studied, in this paper we analyze the *convergence rate* of the proposed scheme. In particular, we are interested in determining the relationship between the minimum consensus convergence rate, sampling period T, and transmission threshold ϕ.

We note that the upped-bounds for T and ϕ must only contain distributed network parameters. Otherwise, a fusion center is required to compute T and ϕ which vastly degrades practicability the implementation. In addition, unlike [23], which assumes a discrete-time model for the agents and ignores system dynamics during the sampling intervals, this paper presents an analysis for the sampling rate and its impact on stability of the closed-loop system.

3. THE PROPOSED Q-CEASE ALGORITHM

Let $\boldsymbol{x}(t) = [\,x_1(t), \ldots, x_N(t)\,]^{\mathcal{T}}$, $\boldsymbol{e}(t) = [e_1(t), \ldots, e_N(t)]^{\mathcal{T}}$, $\boldsymbol{u}(t) = [\,u_1(t), \ldots, u_N(t)\,]^{\mathcal{T}}$, $\hat{\boldsymbol{x}}(t) = [\,\hat{x}_1(t), \ldots, \hat{x}_N(t)\,]^{\mathcal{T}}$, $\tilde{\boldsymbol{\delta}}(t) = [\,\tilde{\delta}_1(t), \ldots, \tilde{\delta}_N(t)\,]^{\mathcal{T}}$. Then, it also holds that $\boldsymbol{e}(t) = \hat{\boldsymbol{x}}(t) - \boldsymbol{x}(t)$.
Moreover, $\bar{x}(t) = \frac{1}{N}\mathbf{1}_N^{\mathcal{T}}\boldsymbol{x}(t)$. Combining (1) with (3) leads to the following close-loop system

$$\dot{\boldsymbol{x}}(t) = -\boldsymbol{L}\,q\,(\,\hat{\boldsymbol{x}}(nT)\,), \tag{5}$$

where $\boldsymbol{L}$ is the Laplacian matrix. From Assumption 1, it holds that $\boldsymbol{L}\mathbf{1}_N = \mathbf{0}_N$ and $\mathbf{1}_N^{\mathcal{T}}\boldsymbol{L} = \mathbf{0}_N^{\mathcal{T}}$. Therefore, $\dot{\bar{x}}(t) =$

Algorithm 1 . The proposed Q-CEASE Algorithm.

I. Parameter Design: (D1 -D2)

D1. Use distributed approaches [24, 25] to locally estimate λ_2, and λ_N. Run an average consensus to agree on ζ.

D2. Each node chooses a sampling period T and a transmission threshold ϕ from (10).

II. Execution: (E1 and E2)

E1. All agents transmit their quantized initial state values $q(x_i(0))$ to the neighbours.

E2. Using T and ϕ selected within (10), the states of the agents approach average consensus using implementation in Fig. 1 with a maximum quantization error given in (11).

$\frac{1}{N}\mathbf{1}_N^{\mathsf{T}}\dot{\boldsymbol{x}}(t) = -\frac{1}{N}\mathbf{1}_N^{\mathsf{T}}\boldsymbol{L}\, q\,(\,\hat{\boldsymbol{x}}(nT)\,) = 0$. Since $\dot{\bar{x}}(t)=0$, we conclude that $\bar{x}(t)=\bar{x}(0)$. Based on this fact, we define the following variable

$$\boldsymbol{r}(t) = \boldsymbol{x}(t) - \bar{x}(t)\mathbf{1}_N. \tag{6}$$

From (6), if $\|\boldsymbol{r}(t)\| \to 0$, then $\|\boldsymbol{x}(t) - \bar{x}(t)\mathbf{1}_N\| \to 0$ which leads to average consensus. Therefore, the average consensus problem for system (5) is equivalent to the stability problem of the system expresses as $\boldsymbol{r}(t)$. To this end, one begins with $\dot{\boldsymbol{r}}(t) = \dot{\boldsymbol{x}}(t) = -\boldsymbol{L}\, q\,(\,\hat{\boldsymbol{x}}(nT)\,)$. Therefore, $\dot{\boldsymbol{r}}(t) = -\boldsymbol{L}q\,(\boldsymbol{x}(nT) + \boldsymbol{e}(nT)) = -\boldsymbol{L}\left(\boldsymbol{x}(nT) + \boldsymbol{e}(nT) + \tilde{\boldsymbol{\delta}}(nT)\right)$. Using (6), $\dot{\boldsymbol{r}}(t) = -\boldsymbol{L}\,(\,\boldsymbol{r}(nT) + \bar{x}(0)\mathbf{1}_N + \boldsymbol{e}(nT) + \tilde{\boldsymbol{\delta}}(nT)\,)$, which leads to the following expression

$$\dot{\boldsymbol{r}}(t) = -\,\boldsymbol{L}\,(\,\boldsymbol{r}(nT) + \boldsymbol{e}(nT) + \tilde{\boldsymbol{\delta}}(nT)\,). \tag{7}$$

For the uniform quantizers, it holds that

$$\|\tilde{\boldsymbol{\delta}}(nT)\| \le 0.5\sqrt{N}\delta. \tag{8}$$

On the other hand, between two consecutive events, one can obtain $|e_i(nT)| \le \phi\,|\mathbb{X}_i(nT)|$. We can further revise this inequality in the global sense as $\|\boldsymbol{e}(nT)\| \le \phi\,\|\boldsymbol{L}q(\hat{\boldsymbol{x}}(nT))\|$, or equivalently $\|\boldsymbol{e}(nT)\| \le \phi\,\|\boldsymbol{L}\boldsymbol{r}(nT) + \boldsymbol{L}\boldsymbol{e}(nT) + \boldsymbol{L}\tilde{\boldsymbol{\delta}}(nT)\|$. Under constraint $\phi \le \frac{1}{\|\boldsymbol{L}\|}$, the former condition on $\boldsymbol{e}(nT)$ leads to the following inequality

$$\|\boldsymbol{e}(nT)\| \le \alpha\,\|\boldsymbol{r}(nT)\| + 0.5\,\alpha\,\sqrt{N}\,\delta, \tag{9}$$

where $\alpha = \frac{\phi\|\boldsymbol{L}\|}{1-\phi\|\boldsymbol{L}\|}$. Based on the following definition, we study the minimum convergence rate of the proposed scheme.

Definition 1. *Exponential Stability* [26]: Given a convergence rate $\zeta > 0$, system (7) is said to be ζ-exponentially stable if there exists a positive scalar η such that $\boldsymbol{r}(t)$ satisfies $\|\boldsymbol{r}(t)\| \le \eta\, e^{-\zeta t}\|\boldsymbol{r}(0)\|$, $t \ge 0$ for any initial vector $\|\boldsymbol{r}(0)\|$.

Next, we compute the operating regions for T and ϕ that collectively guarantee a minimum convergence rate ζ for $\|\boldsymbol{r}(nT)\|$. Moreover, the worst-case error from $\bar{x}(0)$ is calculated. The proposed approach requires λ_2 and λ_N, which can be estimated in a distributed manner based on [24, 25].

Theorem 1. If the sampling period T and transmission threshold ϕ satisfy $T < T_{\max}$ and $\phi < \phi_{\max}$, where

$$T_{\max} = \frac{\lambda_2 - \zeta}{\lambda_N^2}, \qquad \phi_{\max} = \frac{c_1 - \sqrt{c_2}}{2\lambda_N c_3}, \tag{10}$$

with $c_1 = 2\lambda_2 + \lambda_N + 2\zeta T\lambda_N - 2\zeta$, $c_2 = 8T^2\zeta\lambda_N^3 + 4T^2\zeta^2\lambda_N^2 + 4T\lambda_N^3 + 4T\lambda_2\lambda_N^2 + \lambda_N^2$, and $c_3 = \lambda_2 + \lambda_N - \zeta + 2T\zeta\lambda_N$, then, all agents are guaranteed to converge to the set defined by $\{\boldsymbol{x}(t)\,|\,\|\boldsymbol{x}(nT) - \bar{x}(0)\mathbf{1}_N\| \le M\}$ with a least convergence rate ζ, where M is the maximum quantization error given below

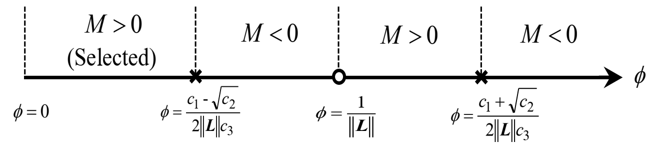

Fig. 2: Operational range for ϕ for guaranteed convergence. '×' and '∘', respectively, denote the poles and zeros of M .

$$M = \frac{0.5\sqrt{N}(1 - \phi\|\boldsymbol{L}\|)\lambda_N}{c_3\phi^2\|\boldsymbol{L}\|^2 - c_1\phi\|\boldsymbol{L}\| + (\lambda_2 - \zeta - T\lambda_N^2)}\delta. \tag{11}$$

Proof. To develop the ζ-exponential stability conditions for system (7), we consider the following inequality

$$\dot{V}(t) + 2\zeta V(t) < 0, \tag{12}$$

where $V(t) = 0.5\,\boldsymbol{r}^{\mathsf{T}}(t)\,\boldsymbol{r}(t)$ is a Lyapunov candidate [27]. If (12) is guaranteed, then $0.5\|\boldsymbol{r}(t)\|^2 \le V(t) < V(0)\,e^{-2\zeta t} \le 0.5\,e^{-2\zeta t}\|\boldsymbol{r}(0)\|^2$. Thus, $\|\boldsymbol{r}(t)\| \le e^{-\zeta t}\|\boldsymbol{r}(0)\|$ is guaranteed if inequality (12) holds. We consider the time evolution of $\boldsymbol{r}(t)$ in the interval $nT \le t < (n+1)T$ which is generated from (7)

$$\boldsymbol{r}(t) = -(t-nT)\boldsymbol{L}\left(\boldsymbol{r}(nT) + \boldsymbol{e}(nT) + \tilde{\boldsymbol{\delta}}(nT)\right) + \boldsymbol{r}(nT), \tag{13}$$

From (13), we expand $\dot{V}(t)$ in what follows

$$\begin{aligned}
\dot{V}(t) &= \boldsymbol{r}^{\mathsf{T}}(t)\dot{\boldsymbol{r}}(t) = -\boldsymbol{r}^{\mathsf{T}}(nT)\boldsymbol{L}\left(\boldsymbol{r}(nT) + \boldsymbol{e}(nT) + \tilde{\boldsymbol{\delta}}(nT)\right)\\
&+(t-nT)\left(\boldsymbol{r}(nT)+\boldsymbol{e}(nT)+\tilde{\boldsymbol{\delta}}(nT)\right)^{\mathsf{T}}\boldsymbol{L}^{\mathsf{T}}\boldsymbol{L}\left(\boldsymbol{r}(nT)+\boldsymbol{e}(nT)+\tilde{\boldsymbol{\delta}}(nT)\right)\\
&\le T\lambda_N^2\left(\|\boldsymbol{r}(nT)\| + \|\boldsymbol{e}(nT)\| + \|\tilde{\boldsymbol{\delta}}(nT)\|\right)^2 - \lambda_2\|\boldsymbol{r}(nT)\|^2\\
&\quad + \lambda_N\|\boldsymbol{r}(nT)\|\|\boldsymbol{e}(nT)\| + \lambda_N\|\boldsymbol{r}(nT)\|\|\tilde{\boldsymbol{\delta}}(nT)\|.
\end{aligned} \tag{14}$$

Using (8), (9), and ignoring negligible terms that include T^2, δ^2, $T\delta$, and their higher order terms, expression (14) is upper bounded by the following terms

$$\begin{aligned}
\dot{V}(t) &\le \left(T\lambda_N^2 + 2\alpha T\lambda_N^2 + \alpha^2 T\lambda_N^2 - \lambda_2 + \alpha\lambda_N\right)\|\boldsymbol{r}(nT)\|^2\\
&\quad + 0.5\sqrt{N}\,(\alpha + 1)\,\lambda_N\delta\,\|\boldsymbol{r}(nT)\|.
\end{aligned} \tag{15}$$

Likewise, $2\zeta V(t)$ is expanded and upper-bounded as below

$$2\zeta V(t) = \zeta\,\boldsymbol{r}^{\mathsf{T}}(t)\,\boldsymbol{r}(t) \le (2\alpha\zeta T\lambda_N + \zeta)\,\|\boldsymbol{r}(nT)\|^2. \tag{16}$$

From (15) and (16), the following inequality holds for $\boldsymbol{r}(nT) \neq 0$

$$\begin{aligned}
\dot{V}(t) + 2\zeta V(t) &\le 0.5\sqrt{N}\,(\alpha + 1)\,\lambda_N\delta\\
&+ \left(\alpha^2 T\lambda_N^2 + \alpha(2T\lambda_N^2 + \lambda_N + 2\zeta T\lambda_N) + \zeta + T\lambda_N^2 - \lambda_2\right)\|\boldsymbol{r}(nT)\|.
\end{aligned} \tag{17}$$

Next, we replace α in (17). Inequality (12) is guaranteed if the constraint $\|\boldsymbol{r}(nT)\| < M$ is guaranteed, where M is given in (11). It is obvious that M must be a positive scalar. With respect to ϕ, M has one zero at $\phi = \frac{1}{\|\boldsymbol{L}\|}$ and tow poles at $\phi = \frac{c_1 \pm \sqrt{c_2}}{2c_3\|\boldsymbol{L}\|}$. Since $c_2 > 0$, both poles are real values. It can be verified that if $T < (\lambda_2 - \zeta)\lambda_N^{-2}$, then $c_1 > \sqrt{c2}$ and the poles of M remain positive. Assuming $T < (\lambda_2 - \zeta)\lambda_N^{-2}$, in Fig. 2 we show the sign of M with respect to different ranges of ϕ. According to Fig. 2, if $0 \le \phi < \frac{c_1 - \sqrt{c_2}}{2c_3\|\boldsymbol{L}\|}$ and $T < (\lambda_2 - \zeta)\lambda_N^{-2}$, then M is guaranteed to remain positive for all $t > 0$. Since, the upper-bound for ϕ, i.e., $\frac{c_1 - \sqrt{c_2}}{2c_3\|\boldsymbol{L}\|}$, depends on the global information $\|\boldsymbol{L}\|$, the largest eigenvalue (λ_N) should be used instead, and that completes the proof. □

The Q-CEASE algorithm is summarized in Algorithm 1.

Remarks. The following features of Q-CEASE are worth mentioning: **(i)** If $T \to T_{\max}$, then $\phi_{\max} \to 0$ and $M \to \infty$. On the

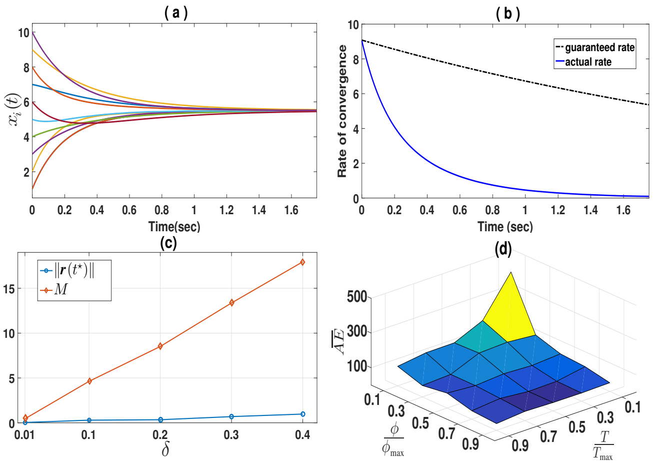

Fig. 3: (a): Average consensus for $x_i(t)$, (b): Convergence rate, (c): Actual ($\|\boldsymbol{r}(t^\star)\|$) and max. quantization error (M), (d): Average no. of events $\overline{\text{AE}}$ versus $\{T/T_{\max}, \phi/\phi_{\max}\}$.

other hand, for a given T, if $\phi \to \phi_{\max}$, then $M \to \infty$. Therefore, there is a trade-off between the selected values for T and ϕ, and the maximum quantization error M. In general, parameter M in the selected region shown in Fig. 2 is a monotonically increasing function with respect to all parameters T, ϕ, ζ and δ. **(ii)** If $T \to 0$ and $\zeta \to 0$, the proposed Q-CEASE algorithm reduces to [9] where an asymptotic convergence without sampling is studied. In this case, i.e., $\{T, \zeta\} \to 0$, the operating region for ϕ would be $\phi < \frac{\lambda_2}{\lambda_N(\lambda_2+\lambda_N)}$. **(iii)** The desired consensus convergence rate must satisfy $\zeta < \lambda_2$. Otherwise, T would be negative (see (10)). This is consistent with the widely known fact that the consensus convergence rate is higher than or equal to λ_2, [20]. **(iv)** In contrast to [22], where the obtained regions for T and ϕ depend on a global transformation matrix, in this paper no global information is required. Moreover, the convergence rate in [22] is unknown. **(v)** For a finite combination of networks with fixed N, the Q-CEASE algorithm can be operated based on a switching network topologies. In this case, λ_2 should be chosen less than or equal to the minimum of the second smallest eigenvalues, and λ_N greater than or equal to the maximum of maximum eigenvalues. The design parameters need not to be recomputed for networks with switching topologies.

4. SIMULATION

In this section, we run Monte-Carlo experiments on randomized networks. Non-zero elements in the adjacency matrix are set to 1. To explain Algorithm 1 in details, we randomly select one network realization with $N = 10$. Non-zero edges in the upper triangle of the adjacency matrix $\mathcal{A}$ are as follows: $\{a_{1,2},\ a_{1,3},\ a_{1,4},\ a_{1,6},\ a_{1,7},\ a_{2,5},\ a_{2,8},\ a_{2,9},\ a_{2,10},\ a_{3,4},\ a_{3,8},\ a_{3,9},\ a_{4,5},\ a_{4,9},\ a_{4,10},\ a_{5,6},\ a_{5,7},\ a_{5,9},\ a_{6,10},\ a_{7,8},\ a_{7,9},\ a_{7,10},\ a_{9,10}\}$. It can be verified that $\lambda_2 = 1.9908$ and $\lambda_{10} = 8.1941$ for this network. Let $\delta = 0.05$ and $\zeta = 0.3$. From (10), $T_{\max} = 0.0252$. We select $T \approx 0.1 T_{\max} = 0.003$, which leads to $\phi_{\max} = 0.0179$. To keep M small enough, we select $\phi = 0.9\phi_{\max}$. For these values, $M = 4.3340$. Starting from initial values $x_i(0) = i$, $(1 \le i \le 10)$, we run the Q-CEASE algorithm until $t^\star = nT$, where $t^\star = \min\limits_{t=nT}\{\, t \mid |\, \|\boldsymbol{r}(t+T)\| - \|\boldsymbol{r}(t)\| \,| \le 0.0001\}$. The evolution of the states $x_i(t)$ for the ten nodes is shown in Fig. 3(a). For this setting, $t^\star = 1.711$ sec, which is equivalent to a total number of $1.711/0.003 \approx 570$ samples. However, the ten nodes, respectively, trigger 108, 103, 96, 100, 97, 94, 95, 108, 95, and 110, events, leading to an average event ($\overline{\text{AE}}$) of 100.60 per node. The ratio of the average events to total samples for this setting is, therefore, $100.6/570 = 0.1765$. The norm of $\boldsymbol{r}(t^\star)$ which shows how close the nodes have reached $\bar{x}(0)$ is calculated as $\|\boldsymbol{r}(t^\star)\| = 0.1015$. As expected, $\|\boldsymbol{r}(t^\star)\| \le M$. In Fig. 3(b), we compare the guaranteed rate $e^{-0.3t}\|\boldsymbol{r}(0)\|$ with the actual rate $\|\boldsymbol{r}(t)\|$, which satisfies $\|\boldsymbol{r}(t)\| \le e^{-0.3t}\|\boldsymbol{r}(0)\|$ for all $t > 0$.

Table 1: Q-CEASE performance for differnet $\{T, \phi, \delta\}$.

$T/T_{\max}$	$\phi/\phi_{\max}$	δ	M	$t^\star$	$\|\boldsymbol{r}(t^\star)\|$	$\overline{\text{AE}}$
0.1	0.5	0.05	0.93	2.01	0.1145	202.91
0.5	0.5	0.05	1.34	1.99	0.1521	238.78
0.9	0.5	0.05	1.95	2.00	0.1649	164.76
0.5	0.1	0.05	1.15	1.94	0.1205	204.25
0.5	0.5	0.05	2.04	1.95	0.1354	206.84
0.5	0.9	0.05	9.40	2.03	0.1414	196.15
0.5	0.5	0.01	0.41	1.98	0.0845	212.01
0.5	0.5	0.10	3.95	1.83	0.1754	197.93
0.5	0.5	0.40	21.41	1.63	0.4492	219.63

Next, we investigate the scalability of the Q-CEASE algorithm for random networks with $N = 10$ and varying parameters T, ϕ, and δ. Networks are generated with $\mathbb{E}\{\lambda_2\} = 2$ and $\mathbb{E}\{\lambda_{10}\} = 8$, where $\mathbb{E}$ returns the expected value. For given values of $T/T_{\max}$, $\phi/\phi_{\max}$, and δ in each row of Table 1, we run the Q-CEASE algorithm with $\zeta = 0.3$ over 25 random networks. The results for M, $t^\star$, $\|\boldsymbol{r}(t^\star)\|$, and $\overline{\text{AE}}$ are calculated based on the average values for all 25 networks. According to Table 1, we conclude that: (i) For fixed $\phi/\phi_{\max}$ and δ, increasing $T/T_{\max}$ results in smaller values for $\overline{\text{AE}}$ at the expense of a higher quantization error (larger $\|\boldsymbol{r}(t^\star)\|$); (ii) For fixed $T/T_{\max}$ and δ, increasing $\phi/\phi_{\max}$ decreases the $\overline{\text{AE}}$, however quantization error is slightly increased; (iii) As expected, increasing δ results in higher values for $\|\boldsymbol{r}(t^\star)\|$ which is translated to a higher error in average consensus.

In this section, we consider larger random networks with $N = 50$, $\mathbb{E}\{\lambda_2\} = 9$, and $\mathbb{E}\{\lambda_{50}\} = 27$. Let $\zeta = 0.3$. In Fig. 3(c), we show $\|\boldsymbol{r}(t^\star)\|$ and M with respect to δ for fixed $T/T_{\max} = \phi/\phi_{\max} = 0.5$. According to Fig. 3(c), the actual quantization error $\|\boldsymbol{r}(t^\star)\|$ is much lower than the worst-case error (M), especially for larger δ. In Fig. 3(d), the 3D graph for $\overline{\text{AE}}$ is shown with respect to different ratios of $T/T_{\max}$ and $\phi/\phi_{\max}$, ($\delta = 0.05$). Based on Fig. 3(d), the least amount for $\overline{\text{AE}}$ happens when both ϕ and T are chosen close to their maximum allowable values from (10), i.e., $\phi \to \phi_{\max}$ and $T \to T_{\max}$, which is at the expense of a higher quantization error. These observations quantify that the Q-CEASE implementation provides an efficient framework for quantized average consensus with transmission savings.

5. CONCLUSION

This paper proposes a sampled-data event-triggered average consensus algorithm for *quantized* information exchange with a guaranteed rate of convergence. The Lyapunov stability theorem is used to compute design parameters, namely, the sampling period and transmission threshold in a distributed manner. The relations between eigenvalues of the Laplacian matrix, sampling period, transmission threshold, maximum quantization error, and desired convergence rate are derived analytically. The effectiveness of the proposed Q-CEASE is validated using Monte-Carlo experiments. In future, we extend Q-CEASE to networks under denial of service attacks.

6. REFERENCES

[1] Y. Cao, W. Yu, W. Ren, and G. Chen, "An overview of recent progress in the study of distributed multi-agent coordination," *IEEE Transactions on Industrial informatics*, vol. 9, no. 1, pp. 427–438, 2013.

[2] S. Kar, S. Aldosari, and J. M. Moura, "Topology for distributed inference on graphs," *IEEE Transactions on Signal Processing*, vol. 56, no. 6, pp. 2609–2613, 2008.

[3] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Transactions on Automatic control*, vol. 50, no. 5, pp. 655–661, 2005.

[4] S. Kar and J. M. Moura, "Distributed consensus algorithms in sensor networks with imperfect communication: Link failures and channel noise," *IEEE Transactions on Signal Processing*, vol. 57, no. 1, pp. 355–369, 2009.

[5] F. Garin and L. Schenato, "A survey on distributed estimation and control applications using linear consensus algorithms," in *Networked control systems.* Springer, 2010, pp. 75–107.

[6] L. Ding and G. Guo, "Distributed event-triggered hâĹđ consensus filtering in sensor networks," *Signal Processing*, vol. 108, pp. 365–375, 2015.

[7] G. S. Seyboth, D. V. Dimarogonas, and K. H. Johansson, "Event-based broadcasting for multi-agent average consensus," *Automatica*, vol. 49, no. 1, pp. 245–252, 2013.

[8] L. Ding, Q.-L. Han, X. Ge, and X.-M. Zhang, "An overview of recent advances in event-triggered consensus of multiagent systems," *IEEE transactions on cybernetics*, vol. 48, no. 4, pp. 1110–1123, 2018.

[9] Z. Zhang, L. Zhang, F. Hao, and L. Wang, "Distributed event-triggered consensus for multi-agent systems with quantisation," *International Journal of Control*, vol. 88, no. 6, pp. 1112–1122, 2015.

[10] F. Xiao, X. Meng, and T. Chen, "Sampled-data consensus in switching networks of integrators based on edge events," *International Journal of Control*, vol. 88, no. 2, pp. 391–402, 2015.

[11] G. Guo, L. Ding, and Q.-L. Han, "A distributed event-triggered transmission strategy for sampled-data consensus of multi-agent systems," *Automatica*, vol. 50, no. 5, pp. 1489–1496, 2014.

[12] W. H. Heemels, M. Donkers, and A. R. Teel, "Periodic event-triggered control for linear systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 4, pp. 847–861, 2013.

[13] T. C. Aysal, M. J. Coates, and M. G. Rabbat, "Distributed average consensus with dithered quantization," *IEEE Transactions on Signal Processing*, vol. 56, no. 10, pp. 4905–4918, 2008.

[14] R. Carli, F. Fagnani, A. Speranzon, and S. Zampieri, "Communication constraints in the average consensus problem," *Automatica*, vol. 44, no. 3, pp. 671–684, 2008.

[15] A. Amini, A. Asif, and A. Mohammadi, "CEASE: A collaborative event-triggered average-consensus sampled-data framework with performance guarantees for multi-agent systems," *IEEE Transactions on Signal Processing*, vol. 66, no. 23, 2018.

[16] S. Zhang, S. Liu, V. Sharma, and P. K. Varshney, "Optimal sensor collaboration for parameter tracking using energy harvesting sensors," *IEEE Trans. Signal Process.*, vol. 66, no. 12, pp. 3339–3353, 2018.

[17] B. Kailkhura, T. Wimalajeewa, and P. K. Varshney, "Collaborative compressive detection with physical layer secrecy constraints." *IEEE Trans. Signal Process.*, vol. 65, no. 4, pp. 1013–1025, 2017.

[18] M. El Gamal and L. Lai, "On rate requirements for achieving the centralized performance in distributed estimation," *IEEE Trans. Signal Process.*, vol. 65, no. 8, pp. 2020–2032, 2017.

[19] A. Mohammadi and K. N. Plataniotis, "Event-based estimation with information-based triggering and adaptive update," *IEEE Trans. Signal Process.*, vol. 65, no. 18, pp. 4924–4939, 2017.

[20] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[21] G. Xie, H. Liu, L. Wang, and Y. Jia, "Consensus in networked multi-agent systems via sampled control: fixed topology case," in *American Control Conference, 2009. ACC'09.* IEEE, 2009, pp. 3902–3907.

[22] Z. Zhang, L. Zhang, F. Hao, and L. Wang, "Periodic event-triggered consensus with quantization," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 63, no. 4, pp. 406–410, 2016.

[23] H. Li, G. Chen, T. Huang, Z. Dong, W. Zhu, and L. Gao, "Event-triggered distributed average consensus over directed digital networks with limited communication bandwidth," *IEEE transactions on cybernetics*, vol. 46, no. 12, pp. 3098–3110, 2016.

[24] A. Bertrand and M. Moonen, "Distributed computation of the fiedler vector with application to topology inference in ad hoc networks," *Signal Processing*, vol. 93, no. 5, pp. 1106–1117, 2013.

[25] ——, "Topology-aware distributed adaptation of laplacian weights for in-network averaging," in *Proc. of the 21ste European Signal Processing Conference, 2013*, 2013, pp. 1–5.

[26] V. Phat, Y. Khongtham, and K. Ratchagit, "Lmi approach to exponential stability of linear systems with interval time-varying delays," *Linear Algebra and its Applications*, vol. 436, no. 1, pp. 243–251, 2012.

[27] E. Fridman, "Tutorial on lyapunov-based methods for time-delay systems," *European Journal of Control*, vol. 20, no. 6, pp. 271–283, 2014.