WHEN CAN A SYSTEM OF SUBNETWORKS BE REGISTERED UNIQUELY?

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ABSTRACT

Consider a network with N nodes in d dimensions, and M overlapping subsets $P_1, \cdots, P_M$ (subnetworks). Assume that the nodes in a given P_i are observed in a local coordinate system. We wish to *register* the subnetworks using the knowledge of the observed coordinates. More precisely, we want to compute the positions of the N nodes in a global coordinate system, given $P_1, \cdots, P_M$ and the corresponding local coordinates. Among other applications, this problem arises in divide-and-conquer algorithms for localization of adhoc sensor networks. The network is said to be *uniquely registrable* if the global coordinates can be computed uniquely (up to a rigid transform). Clearly, if the network is not uniquely registrable, then *any* registration algorithm whatsoever is bound to fail. We formulate a necessary and sufficient condition for uniquely registrability in arbitrary dimensions. This condition leads to a randomized polynomial-time test for unique registrability in arbitrary dimensions, and a combinatorial linear-time test in two dimensions.

Index Terms— sensor network localization, registration, rigid transform, uniqueness, rigidity theory.

1. INTRODUCTION

Consider an adhoc wireless network comprising of geographically distributed sensor nodes with limited radio range. To make sense of the data collected from the sensors, one requires the positions of the individual sensors. It is often not feasible to equip each sensor with a GPS following cost, power, and weight considerations. On the other hand, we can estimate (e.g. using time-of-arrival) the distances between sensors that are within the radio range of each other [1]. The problem of estimating the sensor locations from the available inter-sensor distances is referred to as sensor network localization (SNL) [1, 2]. Recently, scalable divide-and-conquer approaches for SNL were proposed in [3, 4, 5], where the network is first subdivided into smaller subnetworks that can be efficiently localized. The localized subnetworks are then *registered* in a global coordinate system to obtain the positions of the nodes in the original network. More specifically, the authors in [5] proposed to efficiently localize each subnetwork using classical multidimensional scaling (cMDS) [2, 6]. However, since the computation is based on the inter-node distances, the coordinates returned by cMDS will in general be arbitrarily rotated, flipped, and translated version of the ground-truth coordinates (unless the subnetwork has sufficiently many GPS-enabled sensors in it). The network is thus divided into multiple patches (following [5], we call each subnetwork a *patch*), where each patch can be regarded as constituting a local coordinate system. The local coordinates of nodes in a patch are related to the original coordinates via a rigid

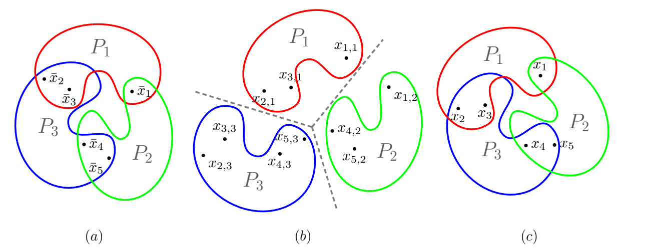

Fig. 1. A network registration scenario. (a) Ground-truth network, (b) Three local coordinate systems (patches), where $x_{k,i}$ denotes the coordinate of the k-th node in the i-th patch (based on this information, we wish to recover the ground-truth network), (c) Registered network. Note that the (a) and (c) are related via a rigid transform, which is the best we can hope to do with the given information.

transform (rotation, reflection and translation). The *registration problem* in question is to assign coordinates to every node in a global coordinate system based on these patch-specific local coordinates. Since the local coordinate systems are related to the ground-truth coordinate system via unknown rigid transforms, solving the registration problem involves estimating these rigid transforms and "undoing" them to obtain the global coordinates of the nodes (see Fig.1). We note that similar registration problems arise in manifold learning, computer vision, and molecular reconstruction [7, 8, 9, 10, 11]. A fundamental question that one is faced with when coming up with an algorithmic solution to this problem [5, 7] is that of uniqueness: Can we uniquely identify the global coordinates which are consistent with the observed local coordinates? Stated differently, can two different assignments of global coordinates (not related via a rigid transform) lead to the same observed local coordinates? We call the network *uniquely registrable* if the global coordinates are guaranteed to be unique. Additionally, can we come up with efficient tests to determine if the network is uniquely registrable? The practical significance of uniquely registrability is clear: One cannot hope *any* algorithm whatsoever to recover the ground-truth coordinates if the network is not uniquely registrable.

In this paper, we address the above questions using results from rigidity theory [12, 13]. Rigidity theory has been used in the literature to address the uniqueness question associated with SNL problem [14]. Namely, can a network be uniquely localized (positioned) from the available inter-sensor distances? Again, the uniqueness in question is up to a rigid transform (which preserves inter-sensor distances). For the registration problem in question, the authors in [7] proposed a simple *lateration* criterion which guarantees unique registrability. However, lateration cannot be tested efficiently, and is not a necessary condition for unique registrability. For instance, while the network

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in Fig. 5 is not laterated, it is uniquely registrable. More recently, a necessary condition for unique registrability was provided in [5].

In Section 2, we formalize the registration problem and the notion of unique registrability. In Section 3, we review relevant definitions and results from rigidity theory. In Section 4, we introduce the notion of a *body graph* (introduced in [15] in the context of affine rigidity), and establish an equivalence between unique registrability and global rigidity of the body graph. As a corollary, we obtain the aforementioned simple test for unique registrability of two-dimensional networks. We conclude with a summary of the results in Section 5. The technical proofs are deferred to Section 6 to improve readability.

2. PROBLEM STATEMENT

Suppose a network has N nodes in $\mathbb{R}^d$ which we label using[1] $\mathcal{S} = [1:N]$. Let $P_1, \cdots, P_M$ be subsets of $\mathcal{S}$, where we call each P_i a *patch*. Let $\mathcal{P} = \{P_1, \cdots, P_M\}$ be the collection of patches. A natural way to represent the node-patch correspondence is using the bipartite graph $\Gamma_C = (\mathcal{S}, \mathcal{P}, \mathcal{E})$, where $(k, P_i) \in \mathcal{E}$ if $k \in P_i$; with a slight abuse of notation, we will use (k, i) in place of (k, P_i). We will call Γ_C the *correspondence graph*. Let $\bar{x}_1, \ldots, \bar{x}_N \in \mathbb{R}^d$ be the ground-truth coordinates of the N nodes. We associate with each patch a local coordinate system. If $(k, i) \in \mathcal{E}$, let $x_{k,i} \in \mathbb{R}^d$ be the local coordinates of node k in patch P_i. More precisely, if $\bar{\mathcal{R}}_i$ is the rigid transform (defined with respect to the ground-truth coordinate system) associated with patch P_i, then

$$\bar{x}_k = \bar{\mathcal{R}}_i(x_{k,i}), \qquad (k, i) \in \mathcal{E}. \tag{1}$$

We will refer to $\bar{\mathcal{R}}_i$ as the *patch transform* associated with P_i. We are now ready to give a precise statement of the registration problem.

Problem. Given the correspondence graph $\Gamma_C = (\mathcal{S}, \mathcal{P}, \mathcal{E})$ and the local coordinates $\{x_{k,i} : (k, i) \in \mathcal{E}\}$, determine $\mathbf{X} = (x_k)_{k=1}^N$ and the patch transforms $\boldsymbol{\mathcal{R}} = (\mathcal{R}_i)_{i=1}^M$, such that

$$x_k = \mathcal{R}_i(x_{k,i}), \qquad (k, i) \in \mathcal{E}. \tag{REG}$$

Obviously, the true global coordinates $(\bar{x}_k)_{k=1}^N$ and the patch transforms $(\bar{\mathcal{R}}_i)_{i=1}^M$ satisfy REG. However, is this solution unique? Of course, by uniqueness, we mean uniqueness up to *congruence*, i.e., any two solutions that are related through a rigid transform are considered identical. Note that a solution to REG has two components: the global coordinates $\mathbf{X}$ and the patch transforms $\boldsymbol{\mathcal{R}}$. We will define uniqueness for each of these components. Suppose $(\mathbf{X}, \boldsymbol{\mathcal{R}})$ is a solution to REG. By *uniqueness of global coordinates*, we mean that given any other solution $(\mathbf{Y}, \boldsymbol{\mathcal{T}})$ to REG, there exists a rigid transform $\mathcal{Q}$ such that $y_k = \mathcal{Q}(x_k), k \in \mathcal{S}$. Similarly, by *uniqueness of patch transforms*, we mean that there exists a rigid transform $\mathcal{U}$ such that $\mathcal{T}_i = \mathcal{U} \circ \mathcal{R}_i, i \in [1:M]$, where $\circ$ denotes the composition of transforms.

It is clear that uniqueness of patch transforms implies uniqueness of global coordinates. That is, given solutions $(\mathbf{X}, \boldsymbol{\mathcal{R}})$ and $(\mathbf{Y}, \boldsymbol{\mathcal{T}})$ to REG, if there exists a rigid transform $\mathcal{U}$, such that $\mathcal{T}_i = \mathcal{U} \circ \mathcal{R}_i, i \in [1:M]$, then there exists a rigid transform $\mathcal{Q}$, such that $y_k = \mathcal{Q}(x_k), k \in \mathcal{S}$. However, uniqueness of global coordinates does not imply uniqueness of patch transforms. That is, given two solutions $(\mathbf{X}, \boldsymbol{\mathcal{R}})$ and $(\mathbf{Y}, \boldsymbol{\mathcal{T}})$ to REG, there may not exist a rigid transform $\mathcal{U}$, such that $\mathcal{T}_i = \mathcal{U} \circ \mathcal{R}_i, i \in [1:M]$, even if there exists a rigid transform $\mathcal{Q}$, such that $y_k = \mathcal{Q}(x_k), k \in \mathcal{S}$. This is explained with an example in Fig. 2.

[1] we use $[m:n]$ to denote the set of integers $\{m, \ldots, n\}$.

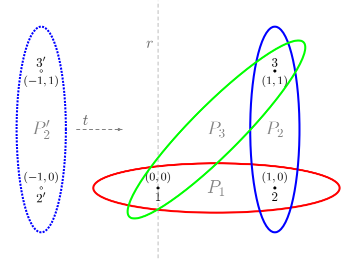

Fig. 2. In this example, $\mathcal{S} = \{1, 2, 3\}$ and $\mathcal{P} = \{P_1, P_2, P_3\}$, where $P_1 = \{1, 2\}, P_2 = \{2, 3\}$ and $P_3 = \{1, 3\}$. The true global coordinates are $\bar{\mathbf{X}} = ((0,0), (1,0), (1,1))$, and the patch transforms are $\bar{\boldsymbol{\mathcal{R}}} = (\mathcal{I}_d, \mathcal{I}_d, \mathcal{I}_d)$, where $\mathcal{I}_d$ is the identity transform. Let $\mathcal{T}$ be a reflection (along the dotted line marked r), followed by a translation of 2 units (along the dotted ray marked with t). Let $\boldsymbol{\mathcal{R}} = (\mathcal{I}_d, \mathcal{T}, \mathcal{I}_d)$. Notice that though both $(\bar{\mathbf{X}}, \boldsymbol{\mathcal{R}})$ and $(\bar{\mathbf{X}}, \bar{\boldsymbol{\mathcal{R}}})$ are solutions of REG, $\boldsymbol{\mathcal{R}}$ is not congruent to $\bar{\boldsymbol{\mathcal{R}}}$.

Notice that each patch has just two nodes in the example in Fig. 2. However, we know that a rigid transform in $\mathbb{R}^d$ is completely specified by its action on a set of $d+1$ non-degenerate points[2]. In particular, if $d+1$ or more non-degenerate points are left fixed by a rigid transform, then the transform must be identity. This leads to the following proposition.

Proposition 1. *If every patch contains at least $d+1$ non-degenerate nodes, then uniqueness of global coordinates is equivalent to uniqueness of patch transforms.*

That is, if every patch contains at least $d+1$ non-degenerate nodes, we need not distinguish between *uniqueness of global coordinates* and *uniqueness of patch transforms*, and we can generally talk about the *uniqueness of solution* to REG (and hence unique registrability of the network) without any ambiguity.

3. RIGIDITY THEORY

Before moving on to our results, we recall some definitions and results from rigidity theory [12, 13, 16, 17, 18]. Given a graph $G = (V, E)$, a d-dimensional *configuration* is a map $\mathbf{p} : V \to \mathbb{R}^d$ (which assigns coordinates to vertices of G). The pair $(G, \mathbf{p})$ is called a *framework*. Throughout this paper, $\|\cdot\|$ denotes the Euclidean norm.

Definition 2 (Equivalent frameworks)**.** *Two frameworks $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are said to be equivalent, denoted by $(G, \mathbf{p}) \sim (G, \mathbf{q})$, if $\|\mathbf{p}(u) - \mathbf{p}(v)\| = \|\mathbf{q}(u) - \mathbf{q}(v)\|$, for every $(u, v) \in E$.*

On the other hand, we require the distances between every pair of vertices to be equal for congruent frameworks, which we define next.

Definition 3 (Congruent frameworks)**.** *Two frameworks $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are said to be congruent, denoted by $(G, \mathbf{p}) \equiv (G, \mathbf{q})$, if $\|\mathbf{p}(u) - \mathbf{p}(v)\| = \|\mathbf{q}(u) - \mathbf{q}(v)\|$ for every $u, v \in V$.*

In other words, congruent frameworks are related through a rigid transform. Clearly, congruence implies equivalence, but the converse is generally not true (see Fig. 3). This leads to the concept of rigidity.

Definition 4 (Globally rigidity)**.** *A framework $(G, \mathbf{p})$ is globally rigid if $(G, \mathbf{q}) \sim (G, \mathbf{p})$ implies that $(G, \mathbf{q}) \equiv (G, \mathbf{p})$.*

[2] A set of points in $\mathbb{R}^d$ is *non-degenerate* if their affine span is $\mathbb{R}^d$.

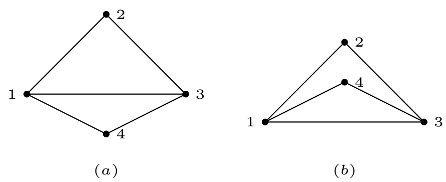

Fig. 3. The frameworks in (a) and (b) are equivalent because the corresponding edge lengths are equal; however, they are not congruent because the distance between vertices 2 and 4 is not equal in the two frameworks. Thus, the framework in (a) is not globally rigid in $\mathbb{R}^2$. On the other hand, it is not difficult to see that the framework is locally rigid in $\mathbb{R}^2$, i.e., we cannot take (a) to (b) via a continuous motion in $\mathbb{R}^2$ (the only way to do so is via a reflection about the line 1-3).

In other words, if $(G, \mathbf{q}) \sim (G, \mathbf{p})$ and $(G, \mathbf{p})$ is globally rigid, then $\mathbf{p}$ and $\mathbf{q}$ must be related via a rigid transform. Local rigidity of a framework, on the other hand, just means that the framework cannot be continuously deformed into an equivalent framework.

Definition 5 (Locally rigidity)**.** *A framework $(G, \mathbf{p})$ is locally rigid if there exists $\epsilon > 0$ such that any $(G, \mathbf{q}) \sim (G, \mathbf{p})$ satisfying $\|\mathbf{p}(v) - \mathbf{q}(v)\| \leq \epsilon, v \in V$, is congruent to $(G, \mathbf{p})$.*

In other words, $(G, \mathbf{q}) \sim (G, \mathbf{p})$ implies that $(G, \mathbf{q}) \equiv (G, \mathbf{p})$, for all $\mathbf{q}$ "sufficiently close" to $\mathbf{p}$. Clearly, global rigidity implies local rigidity, but the converse is not true (see Fig. 3).

A fundamental problem in rigidity theory is the following: *Given a d-dimensional framework $(G, \mathbf{p})$, decide whether it is (locally or globally) rigid in $\mathbb{R}^d$.* In general, the notions of local and global rigidity depend not only on the graph, but also on the configuration (see Fig. 4). This makes testing of rigidity computationally intractable [19, 20]. A standard way of getting around this is to make an additional assumption of *genericity*. A framework (or configuration) is said to be *generic* if there are no algebraic dependencies among the coordinates of the configuration, i.e., the coordinates of the configuration do not satisfy any non-trivial algebraic equation with rational coefficients. For a given graph, the set of non-generic configurations is a measure-zero set in the space of all possible configurations [21], and hence *almost every* configuration is generic. The genericity assumption makes local and global rigidity a property of the graph, independent of any particular configuration. We record this important fact as a proposition.

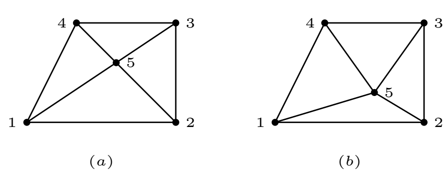

Fig. 4. Frameworks (a) and (b) with the same underlying graph. Framework (a) is not globally rigid because vertex 4 can be reflected along the line 1-5-3, which results in an equivalent but non-congruent framework. Such an edge-length-preserving reflection is not possible in (b), which is globally rigid.

Proposition 6 ([12, 13, 17])**.** *Global (respectively, local) rigidity is a generic property, i.e., either all or none of the generic configurations of a graph form a globally (respectively, locally) rigid framework.*

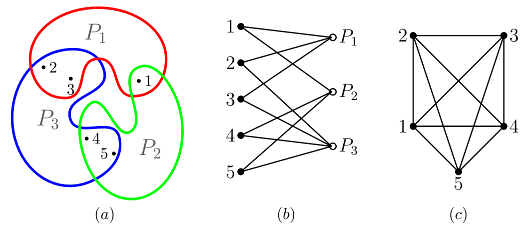

Fig. 5. In this example, $\mathcal{S} = [1:5]$ and $\mathcal{P} = \{P_1, P_2, P_3\}$, where $P_1 = \{1,2,3\}$, $P_2 = \{1,4,5\}$ and $P_3 = \{2,3,4,5\}$. (a) Visualization of the node-patch correspondence, (b) Correspondence graph $\Gamma_C = (\mathcal{S}, \mathcal{P}, \mathcal{E})$, (c) Body graph Γ_B.

Thus, under the assumption of genericity, global rigidity becomes a property of the *graph*, and we can talk of a graph being *generically globally rigid*. This also tells us that there ought to be a characterization of generic global rigidity solely in terms of the graph properties. In fact, we have a complete combinatorial characterization of generic global rigidity in two-dimensions (finding such a characterization for three and higher dimensions is an open problem).

Theorem 7 ([18, 22])**.** *A graph G is generically globally rigid in $\mathbb{R}^2$ if and only if either G is a triangle, or G is 3-connected and redundantly rigid in $\mathbb{R}^2$.*

For completeness, we define the terms in Theorem 7. A graph is said to be *k-connected* (or, k-vertex-connected) if it has more than k vertices and any subgraph obtained after removing fewer than k vertices remains connected [23]. A graph is said to be *redundantly rigid* in $\mathbb{R}^d$ if it is generically locally rigid in $\mathbb{R}^d$, and remains generically locally rigid in $\mathbb{R}^d$ after removal of any edge.

4. UNIQUE REGISTRABILITY

We now formulate a necessary and sufficient condition for unique registrability under the following assumptions:

(A1) Each patch contains at least $d+1$ non-degenerate nodes.

(A2) The nodes are in generic positions.

We briefly recall the rationale behind the assumptions. Under Assumption (A1), uniqueness of the global coordinates and uniqueness of the patch transforms become equivalent, making unique registrability a well-defined notion (see Proposition 1). Moreover, we can easily force this assumption for divide-and-conquer algorithms [24, 3, 5]. Assumption (A2) allows us to formulate conditions for unique registrability for *almost every* problem instance based solely on the combinatorial structure of the problem (see Proposition 6).

We now introduce the notion of a body graph, which will help us tie unique registrability to rigidity theory. For a network with correspondence graph $\Gamma_C = (\mathcal{S}, \mathcal{P}, \mathcal{E})$, consider a graph $\Gamma_B = (V, E)$, where $V = \mathcal{S}$, and $E = \{(k_1, k_2) : k_1, k_2 \in P_i \text{ for some } i \in [1 : M]\}$. In other words, the nodes of Γ_B are simply the nodes of Γ_C,and we connect two nodes by an edge if they belong to a common patch (see Fig. 5). We will call Γ_B the body graph of Γ_C. We derive the term body graph from [15], where a similar notion was introduced in the context of affine rigidity. Using the body graph, we can now state our main result.

Theorem 8. *Under Assumptions (A1) and (A2), the ground-truth $(\bar{\mathbf{X}}, \bar{\boldsymbol{\mathcal{R}}})$ is a unique solution of REG if and only if the body graph Γ_B is generically globally rigid.*

The import of Theorem 8 lies in the fact that generic global rigidity in an arbitrary dimension can be tested using a randomized polynomial-time algorithm [17]. Moreover, combining Theorem 8 with the combinatorial characterization of generic global rigidity in Theorem 7, and using further results from rigidity theory, we get the following characterization of unique registrability in two-dimensions (we skip the proof due to space constraints).

Corollary 9. *Under Assumptions (A1) and (A2), a network is uniquely registrable in $\mathbb{R}^2$ if and only if Γ_B is 3-connected.*

The implication of Corollary 9 is that (assuming each patch has at least 3 nodes) we need only test for 3-connectivity to establish generic global rigidity of the body graph in $\mathbb{R}^2$. We need not perform an additional check for redundant rigidity, as required by Theorem 7. As is well-known, 3-connectivity can be tested efficiently using linear-time algorithms [23].

5. DISCUSSION

In this paper, we looked at the notion of unique registrability of a network through the lens of rigidity theory. Given that there are two families of unknowns inherent in the problem—the global coordinates and the patch transforms—we first addressed the question as to what uniqueness exactly means for the registration problem. We ended up making a mild assumption of non-degeneracy that makes the notion of uniqueness equivalent for both families of unknowns. We then introduced the notion of body graph which allowed us to turn the question of unique registrability into a question about graph rigidity. Specifically, we concluded that unique registrability is equivalent to global rigidity of the body graph. This opens up the possibility of using non-trivial results from rigidity theory to efficiently test unique registrability. For instance, in two dimensions, we only need to test 3-connectivity of the body graph to establish unique registrability. Moreover, unique registrability can be verified in three or more dimensions simply by testing generic global rigidity of the body graph, for which there exists a polynomial-time randomized algorithm [17].

6. APPENDIX: PROOF OF THEOREM 8

We will show that unique registrability is equivalent to global rigidity of the body graph framework corresponding to the ground-truth. Then, Assumption (A2) along with Proposition 6 allows us to remove the dependence on the framework, thus proving the theorem.

We first make some definitions specialized to the registration problem which allow us to express the question of uniqueness registrability in a form amenable to a rigidity theoretic analysis.

Definition 10 (Node-patch framework)**.** *Given a correspondence graph $\Gamma_C = (\mathcal{S}, \mathcal{P}, \mathcal{E})$, and a map $\mathbf{x} : \mathcal{S} \to \mathbb{R}^d$ that assigns coordinates to the nodes, the pair $(\Gamma_C, \mathbf{x})$ is called a node-patch framework.*

Definition 11 (Equivalence of node-patch frameworks)**.** *Two node-patch frameworks $(\Gamma_C, \mathbf{x})$ and $(\Gamma_C, \mathbf{y})$ are said to be equivalent, denoted by $(\Gamma_C, \mathbf{x}) \sim (\Gamma_C, \mathbf{y})$, if $\mathbf{x}(k) = \mathcal{Q}_i \mathbf{y}(k), (k, i) \in \mathcal{E}$, where $\mathcal{Q}_i$ is a rigid transform.*

Definition 12 (Congruence of node-patch frameworks)**.** *Two node-patch frameworks $(\Gamma_C, \mathbf{x})$ and $(\Gamma_C, \mathbf{y})$ are said to be congruent, denoted by $(\Gamma_C, \mathbf{x}) \equiv (\Gamma_C, \mathbf{y})$, if $\mathbf{x}(k) = \mathcal{Q}\mathbf{y}(k),\ k \in \mathcal{S}$, where $\mathcal{Q}$ is a rigid transform.*

Given a solution $(\mathbf{X}, \boldsymbol{\mathcal{R}})$ to REG, where $\mathbf{X} = (x_k)_{k=1}^N$, $\boldsymbol{\mathcal{R}} = (\mathcal{R}_i)_{i=1}^M$, we will denote by $\mathbf{x}$ the map that assigns to node k the coordinates x_k, and say that $(\Gamma_C, \mathbf{x})$ is the node-patch framework corresponding to the solution $(\mathbf{X}, \boldsymbol{\mathcal{R}})$.

Proposition 13. *Let $(\mathbf{X}, \boldsymbol{\mathcal{R}})$ and $(\mathbf{Y}, \boldsymbol{\mathcal{T}})$ be two solutions to REG. Then the corresponding node-patch frameworks $(\Gamma_C, \mathbf{x})$ and $(\Gamma_C, \mathbf{y})$ are equivalent.*

Proof. Since $(\mathbf{X}, \boldsymbol{\mathcal{R}})$ and $(\mathbf{Y}, \boldsymbol{\mathcal{T}})$ are solutions to REG, we have that $\mathbf{x}(k) = \mathcal{R}_i(x_{k,i})$ and $\mathbf{y}(k) = \mathcal{T}_i(x_{k,i}), k \in P_i, i \in [1 : M]$. Thus $\mathbf{x}(k) = \mathcal{Q}_i \mathbf{y}(k)$, where $\mathcal{Q}_i = \mathcal{R}_i \circ \mathcal{T}_i^{-1}$. □

Proposition 14. *Let $(\mathbf{X}, \boldsymbol{\mathcal{R}})$ be a solution to REG with the corresponding node-patch framework $(\Gamma_C, \mathbf{x})$ and let $\mathbf{y}$ be such that $(\Gamma_C, \mathbf{y}) \sim (\Gamma_C, \mathbf{x})$. Then there exists some $\boldsymbol{\mathcal{T}}$ for which $(\mathbf{Y}, \boldsymbol{\mathcal{T}})$ is a solution of REG.*

Proof. Indeed, $(\Gamma_C, \mathbf{y}) \sim (\Gamma_C, \mathbf{x})$ implies that there exists rigid transforms $(\mathcal{Q}_i)_{i=1}^M$ such that $\mathbf{y}(k) = \mathcal{Q}_i \mathbf{x}(k), (k, i) \in \mathcal{E}$. Since $(\mathbf{X}, \boldsymbol{\mathcal{R}})$ is a solution to REG, we have $\mathbf{x}(k) = \mathcal{R}_i(x_{k,i}), (k, i) \in \mathcal{E}$. Thus, $\mathbf{y}(k) = (\mathcal{Q}_i \circ \mathcal{R}_i)(x_{k,i})$, which shows that $(\mathbf{Y}, \boldsymbol{\mathcal{T}})$ is a solution to REG, where $\mathbf{Y} = (\mathbf{y}(k))_{k=1}^N$ and $\boldsymbol{\mathcal{T}} = (\mathcal{Q}_i \circ \mathcal{R}_i)_{i=1}^M$. □

Foregoing definitions and propositions allow us to express the condition of unique registrability in a compact manner. Namely, let $(\Gamma_C, \bar{\mathbf{x}})$ be the ground-truth node-patch framework. Then, under Assumption (A1), REG has a unique solution if and only if for any node-patch framework $(\Gamma_C, \mathbf{y})$ such that $(\Gamma_C, \mathbf{y}) \sim (\Gamma_C, \bar{\mathbf{x}})$, we have $(\Gamma_C, \mathbf{y}) \equiv (\Gamma_C, \bar{\mathbf{x}})$.

The next two propositions relate node-patch and body graph frameworks.

Proposition 15. *Two node-patch frameworks $(\Gamma_C, \mathbf{x})$ and $(\Gamma_C, \mathbf{y})$ are equivalent (Definition 11) if and only if the body graph frameworks $(\Gamma_B, \mathbf{x})$ and $(\Gamma_B, \mathbf{y})$ are equivalent (Definition 2).*

Proof. Suppose $(\Gamma_C, \mathbf{x}) \sim (\Gamma_C, \mathbf{y})$. Pick an arbitrary edge $(k, l) \in E$. From construction of Γ_B, $(k, l) \in E$ if and only if there is a patch, say P_i, that contains both the nodes k and l. Since $(\Gamma_C, \mathbf{x}) \sim (\Gamma_C, \mathbf{y})$, there exists a rigid transform $\mathcal{Q}_i$ such that $\mathbf{x}(k) = \mathcal{Q}_i \mathbf{y}(k)$ and $\mathbf{x}(l) = \mathcal{Q}_i \mathbf{y}(l)$. This implies that $\mathbf{x}(k) - \mathbf{x}(l) = \mathcal{Q}_i(\mathbf{y}(k) - \mathbf{y}(l))$, from where it follows that $\|\mathbf{x}(k) - \mathbf{x}(l)\| = \|\mathbf{y}(k) - \mathbf{y}(l))\|$. Thus, $(\Gamma_B, \mathbf{x}) \sim (\Gamma_B, \mathbf{y})$.

Conversely, suppose $(\Gamma_B, \mathbf{x}) \sim (\Gamma_B, \mathbf{y})$. Consider an arbitrary patch P_i. Note that any subgraph of Γ_B induced by a patch is a clique. This, along with the assumption that $(\Gamma_B, \mathbf{x}) \sim (\Gamma_B, \mathbf{y})$, implies that $\|\mathbf{x}(k) - \mathbf{x}(l)\| = \|\mathbf{y}(k) - \mathbf{y}(l))\|$ for every $k, l \in P_i$, which, in turn, implies that there exists a rigid transform $\mathcal{Q}_i$ such that $\mathbf{x}(v) = \mathcal{Q}_i \mathbf{y}(v), v \in P_i$. Thus, $(\Gamma_C, \mathbf{x}) \sim (\Gamma_C, \mathbf{y})$. □

Proposition 16. *Two node-patch frameworks $(\Gamma_C, \mathbf{x})$ and $(\Gamma_C, \mathbf{y})$ are congruent (Definition 12) if and only if the body graph frameworks $(\Gamma_B, \mathbf{x})$ and $(\Gamma_B, \mathbf{y})$ are congruent (Definition 3).*

The above result easily follows from Definitions 3 and 12. We are now in a position to complete the proof of Theorem 8. Suppose REG has a unique solution. We will show that the body graph framework $(\Gamma_B, \bar{\mathbf{x}})$ is globally rigid. Consider a framework $(\Gamma_B, \mathbf{y}) \sim (\Gamma_B, \bar{\mathbf{x}})$. Then, by Proposition 15, $(\Gamma_C, \mathbf{y}) \sim (\Gamma_C, \bar{\mathbf{x}})$. By Proposition 14, this implies that $(\Gamma_C, \mathbf{y})$ correponds to a solution of REG. Now, since REG has a unique solution, $(\Gamma_C, \mathbf{y}) \equiv (\Gamma_C, \bar{\mathbf{x}})$. Thus, by Proposition 16, $(\Gamma_B, \mathbf{y}) \equiv (\Gamma_B, \bar{\mathbf{x}})$.

Conversely, suppose $(\Gamma_B, \bar{\mathbf{x}})$ is globally rigid. Let $(\mathbf{Y}, \boldsymbol{\mathcal{T}})$ be a solution to REG. By Proposition 13, $(\Gamma_C, \mathbf{y}) \sim (\Gamma_C, \bar{\mathbf{x}})$. Hence, by Proposition 15, $(\Gamma_B, \mathbf{y}) \sim (\Gamma_B, \bar{\mathbf{x}})$. This, by global rigidity of $(\Gamma_B, \bar{\mathbf{x}})$, implies that $(\Gamma_B, \mathbf{y}) \equiv (\Gamma_B, \bar{\mathbf{x}})$. Finally, by Proposition 16, $(\Gamma_C, \mathbf{y}) \equiv (\Gamma_C, \bar{\mathbf{x}})$.

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