TOEPLITZ MATRIX COMPLETION FOR DIRECTION FINDING USING A MODIFIED NESTED LINEAR ARRAY

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ABSTRACT

A modified nested linear array (MNLA) has been reported recently for a greater potential in increasing the degree-of-freedom. However, there exist some "holes" in the difference co-array, which results in missing "lags" and limited performance of direction-of-arrival (DOA) estimation. In order to tackle this problem, this paper applies a Toeplitz matrix completion technique to MNLA, and investigates the performance of DOA estimation on this basis. Particularly, a semidefinite program with trace minimization is derived to obtain the covariance matrix with Hermitian and Toeplitz structure. The recovered Toeplitz covariance matrix is then utilized to perform DOA estimation. Various numerical examples are provided to verify the effectiveness and superiority of the proposed method.

Index Terms— DOA estimation, direction finding, Toeplitz matrix completion, nested array

1. INTRODUCTION

The design of sparse sensor arrays can date back to 1900s [1, 2], when configurations such as minimum-redundancy linear arrays, fully augmentable arrays and partially augmentable arrays were proposed. In array signal processing community, sparse sensor arrays have received particular attention, mainly due to their capability in increasing the degree of freedom (DOF) compared to the classical arrays, like uniform linear array (ULA). However, the above-mentioned sparse arrays do not have closed-form expressions to indicate where to put the sensors when arbitrary number of sensors is deployed.

In recent years, two kinds of sparse array, named coprime and nested arrays, have been proposed by P. P. Vaidyanathan *et al.* in [3–7]. These arrays have closed-form expressions to demonstrate how to place the sensors. Thereafter, modified arrays based on the coprime and the nested arrays have been proposed [8–12], among which the modified nested linear array (MNLA) has been proposed recently in [8]. The MNLA belongs to partially augmentable arrays, and it has larger DOF and better performance in direction-of-arrival (DOA) estimation when compared with other existing sparse arrays.

As there are probably lags missing in the difference co-array of some of the aforementioned array configurations and the missing lags cause performance degradation in DOA estimation, several techniques [2, 13–16] have thus been proposed to deal with the problem. For instance, the maximum entropy Toeplitz completion and the minimum nuclear norm Toeplitz completion were proposed in [2] and [13], respectively, in which the optimization problems were convex and hence were easy to be solved. A minimum-size virtual array which has the equal aperture with original array and has filled co-array is introduced in [14]. However, the minimum-size virtual array cannot be guaranteed to have the same number of sensors as the original one, and it may need more sensors. More recently, a virtual array interpolation method which applies atomic norm minimization to coprime array has been proposed in [15, 16], where better performance of DOA estimation especially for spatially close sources is disclosed.

The interest of this paper is to solve the problem caused by missing lags in MNLA. The missing lags in the difference co-array in MNLA may result in performance degradation of DOA estimation. To tackle this problem, a new method based on Toeplitz covariance matrix completion is devised. Overall, the completion process includes the following two steps: 1) construct covariance matrix with Toeplitz structure using the existing lags while the missing lags were replaced by zeros; 2) derive and solve a semidefinite program with trace minimization to obtain the final Toeplitz covariance matrix. With the so-obtained Toeplitz covariance matrix, direction finding is performed with the aid of traditional approaches.

The rest of this paper is organized as follows. Section 2 describes the signal model. Section 3 presents the proposed method. Section 4 verifies the proposal with numerical examples, and Section 5 concludes this paper.

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2. SIGNAL MODEL

To begin with, the original two-level nested linear array (ONLA) [6] as well as the MNLA [8] are briefly reviewed, and the signal model for DOA estimation using MNLA is introduced.

2.1. Original Two-Level Nested Linear Array

The ONLA is a co-array consisting of two ULAs, whose sensor positions are given by two sets $S_M = \{md_1 \mid m = 0, 1, \cdots, M-1\}$ and $S_N = \{Md_1 + nd_2 \mid n = 0, 1, \cdots, N-1\}$, where d_1 and d_2 are the interspaces, M and N are the sensor numbers of the first and the second ULA, respectively. d_1 and d_2 satisfy: $d_2 = (M+1)d_1$. This co-array can generate a virtual array with virtual sensor locations marked by the difference set between S_M and S_N.

It should be pointed out that the difference set of S_M and S_N is a continuous number set, and its DOF is $2N(M+1)-1$. The DOF is maximized given fixed total number of physical sensors $L = M + N$. The solution is as follows:

1) $M = N = \frac{L}{2}$, if L is even, and the resulting DOF is $\frac{L^2-2}{2} + L$;

2) $M = \frac{L-1}{2}$, $N = \frac{L+1}{2}$, if L is odd, and the resulting DOF is $\frac{L^2-1}{2} + L$.

2.2. Modified Nested Linear Array

The MNLA extends the structure of ONLA with enlarged DOF. As illustrated in Fig. 1, an MNLA includes two ULAs and an additional sensor. We choose M and $N-1$ sensors to form the first and the second ULAs with inter-spacings d_1 and d_2, respectively, and a separate sensor is with $d_1 + d_2$ apart from the last sensor of the second ULA. The gap between the first and the second ULAs is $\lfloor \frac{M+N-1}{2} \rfloor d_1$, and d_1 and d_2 satisfy:

$$d_2 = \left(M + \left\lfloor \frac{M+N-1}{2} \right\rfloor \right) d_1 \tag{1}$$

where $\lfloor \cdot \rfloor$ is the floor function. That is to say, the sensor locations of the first and the second ULAs are given by S_M and S_{N-1}, respectively, which are represented as follows

$$S_M = \{md_1 \mid m = 0, 1, \cdots, M-1\} \tag{2a}$$

$$S_{N-1} = \{d_1 d_2 + nd_2 - d_1 \mid n = 0, 1, \cdots, N-2\}. \tag{2b}$$

The additional sensor is located at

$$S_1 = \{d_1 d_2 + (N-1)d_2\}. \tag{3}$$

2.3. Signal Model for MNLA

Consider an MNLA with total sensor number being $L = M + N$, whose sensor locations are depicted as (2a)-(3), and K narrow-band far-field signals impinging onto the MNLA from directions $\{\theta_1, \theta_2, \cdots, \theta_K\}$. Denote the sensor locations as $P \triangleq S_M \cup S_{N-1} \cup S_1 = [p_1, p_2, \cdots, p_L]$. The observation data of MNLA can be formed as

$$\mathbf{x}(t) = \mathbf{B}\mathbf{s}(t) + \mathbf{n}(t) \tag{4}$$

where t is the time index, and $\mathbf{B} = [\mathbf{b}(\theta_1), \cdots, \mathbf{b}(\theta_K)]$ is the steering matrix with $\mathbf{b}(\theta) = [\beta^{p_1}(\theta), \cdots, \beta^{p_L}(\theta)]^T$ and $\beta(\theta) = e^{j2\pi \frac{\sin(\theta)}{\lambda}}$, λ denotes the wavelength of the incident signals and $(\cdot)^T$ stands for transpose operator.

Assuming that the signals are mutually uncorrelated, and the noises follow independent and identically Gaussian distribution, the data covariance matrix is defined as follows

$$\mathbf{R}_x \triangleq E\{\mathbf{x}(t)\mathbf{x}^H(t)\} = \mathbf{B}\mathbf{R}_s\mathbf{B}^H + \sigma_n \mathbf{I} \tag{5}$$

where $\mathbf{R}_s \triangleq E\{\mathbf{s}(t)\mathbf{s}^H(t)\}$, $E\{\cdot\}$ and $(\cdot)^H$ represent the mathematical expectation and conjugate transpose operators, respectively, σ_n denotes the noise power, and $\mathbf{I}$ stands for identity matrix. It is worth noting that when in the case of ULAs, the resulting covariance matrix is a Toeplitz Hermitian matrix. However, here, the array configuration under consideration is MNLA rather than ULA. Thus, $\mathbf{R}_x$ is not a Toeplitz matrix, and the DOA estimation performance would be limited whenever it is directly used base on traditional direction finding techniques. To this end, a Toeplitz matrix completion procedure is applied to transform the MNLA covariance matrix to a ULA counterpart before performing DOA estimation. A better performance is expected by recovering those missing lags, as shall be demonstrated by the simulation results.

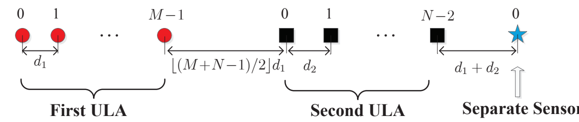

Fig. 1. Illustration of configuration of MNLA.

3. PROPOSED METHOD

In this section, a progress of Toeplitz matrix completion is firstly proposed to be applied to MNLA, and then, the obtained covariance matrix with Toeplitz structure is utilized to perform DOA estimation.

3.1. Progress of Toeplitz Matrix Completion

As discussed in [8], $\mathbf{R}_x$ obtained from an MNLA has some lags being missed. As a numerical demonstration to better illustrate the problem, we set $L = 5$, $M = 2$, $N = 3$, $K = 1$ and $d_1 = 1$. According to (1)–(3), we have $d_2 = 4$, $S_M \cup S_{N-1} \cup S_1 = \{0, 1, 3, 7, 12\}$ (i.e., $p_1 = 0, p_2 = 1, p_3 = 3, p_4 = 7, p_5 = 12$). In this example, the observation data is as follows

$$\mathbf{x}^{(\mathrm{Ex})}(t) = \mathbf{b}(\theta_1)s_1(t) + \mathbf{n}(t) \tag{6}$$

where the superscript $(\cdot)^{(\mathrm{Ex})}$ is used to indicate that the related term is for this example only. The corresponding data covariance matrix is given by

$$\begin{aligned}\mathbf{R}_x^{(\mathrm{Ex})} =& \mathbf{b}(\theta_1)\sigma_s\mathbf{b}^H(\theta_1)+\sigma_n\mathbf{I}\\ =&\sigma_s\begin{bmatrix} \beta^0(\theta_1) & \beta^{-1}(\theta_1) & \beta^{-3}(\theta_1) & \beta^{-7}(\theta_1) & \beta^{-12}(\theta_1)\\ \beta^1(\theta_1) & \beta^0(\theta_1) & \beta^{-2}(\theta_1) & \beta^{-6}(\theta_1) & \beta^{-11}(\theta_1)\\ \beta^3(\theta_1) & \beta^2(\theta_1) & \beta^0(\theta_1) & \beta^{-4}(\theta_1) & \beta^{-9}(\theta_1)\\ \beta^7(\theta_1) & \beta^6(\theta_1) & \beta^4(\theta_1) & \beta^0(\theta_1) & \beta^{-5}(\theta_1)\\ \beta^{12}(\theta_1) & \beta^{11}(\theta_1) & \beta^9(\theta_1) & \beta^5(\theta_1) & \beta^0(\theta_1)\end{bmatrix}\\ &+\sigma_n\mathbf{I}. \end{aligned} \tag{7}$$

It can be seen from (7) that $\mathbf{R}_x^{(\mathrm{Ex})}$ is not a Toeplitz matrix and some lags, including $\beta^8(\theta_1)$, $\beta^{10}(\theta_1)$, $\beta^{-8}(\theta_1)$, and $\beta^{-10}(\theta_1)$, are missing.

In order to obtain the Toeplitz covariance matrix counterpart from $\mathbf{R}_x^{(\mathrm{Ex})}$ and to recover the missing lags, we propose to apply the following two-step scheme to the MNLA covariance matrix.

Step 1: Construct a Toeplitz matrix $\mathbf{T}_0$ with the existing lags in $\mathbf{R}_x$, in which the missing lags are replaced by zeros and in the next step, we intend to recover these missing lags based on trace minimization. For instance, in the above example, the Toeplitz covariance matrix is constructed as

$$\mathbf{T}_0^{(\mathrm{Ex})} = \mathrm{toep}(\mathbf{c},\mathbf{r}) \tag{8}$$

where $\mathrm{toep}(\mathbf{c},\mathbf{r})$ denotes the Toeplitz matrix with $\mathbf{c}$ as the first column and $\mathbf{r}$ as the first row. Moreover, we have

$$\begin{aligned}\mathbf{c} =&[\beta^0(\theta_1),\beta^1(\theta_1),\beta^2(\theta_1),\beta^3(\theta_1),\beta^4(\theta_1),\beta^5(\theta_1),\\ &\beta^6(\theta_1),\beta^7(\theta_1),0,\beta^9(\theta_1),0,\beta^{11}(\theta_1),\beta^{12}(\theta_1)]^T \end{aligned}\tag{9}$$

$$\begin{aligned}\mathbf{r} =&[\beta^0(\theta_1),\beta^{-1}(\theta_1),\beta^{-2}(\theta_1),\beta^{-3}(\theta_1),\beta^{-4}(\theta_1),\beta^{-5}(\theta_1),\\ &\beta^{-6}(\theta_1),\beta^{-7}(\theta_1),0,\beta^{-9}(\theta_1),0,\beta^{-11}(\theta_1),\beta^{-12}(\theta_1)]. \end{aligned}\tag{10}$$

Step 2: Note that the constructed Toeplitz matrix $\mathbf{T}_0$ in Step 1 can be viewed as the covariance matrix of the observation data using the sensors located in the difference set (e.g., 0, 1, 2, 3, 4, 5, 6, 7, NaN, 9, NaN, 11, 12). The missing lags are equivalent to the broken-down sensors (e.g., 8 and 10 in the example). Therefore, we can formulate the following low-rank matrix recovery problem

$$\min_{\mathbf{w}} \ \mathrm{rank}(\mathbf{T}) \quad \text{s.t. } \mathbf{T} = \mathbf{T}_0+\sum_{l\in\mathcal{H}}\left(w_l\mathbf{I}_{(+)}^l + w_l^*\mathbf{I}_{(-)}^l\right) \tag{11}$$

where $\mathrm{rank}(\cdot)$ represents the rank of a matrix, l is the position of the lth hole, $\mathcal{H}$ is the set of hole positions (in the above example, we have $\mathcal{H}=\{8,10\}$), $\mathbf{w}=\{w_l\}_{l\in\mathcal{H}}$, $(\cdot)^*$ denotes the complex conjugate operator, $\mathbf{I}_{(+)}$ and $\mathbf{I}_{(-)}$ are defined as

$$\mathbf{I}_{(+)}=\begin{bmatrix}0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1\\ & & & & 0\end{bmatrix}, \ \mathbf{I}_{(-)}=\begin{bmatrix}0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0\end{bmatrix}.$$

It is known that the nuclear norm, which is defined as the sum of all the singular values of a matrix, can be chosen as an alternative of the rank [13, 17–19]. Therefore, (11) is recast to the following problem:

$$\min_{\mathbf{w}} \ \|\mathbf{T}\|_* \quad \text{s.t. } \mathbf{T} = \mathbf{T}_0+\sum_{l\in\mathcal{H}}\left(w_l\mathbf{I}_{(+)}^l + w_l^*\mathbf{I}_{(-)}^l\right) \tag{12}$$

where $\|\cdot\|_*$ denotes the nuclear norm of a matrix.

Moreover, as $\mathbf{T}$ is Hermitian and positive semidefinite, its nuclear norm is equal to the sum of its nonnegative eigenvalues, or saying, its trace [18]. Hence, the aforementioned problem can be further rewritten as

$$\min_{\mathbf{w}} \ \mathrm{trace}(\mathbf{T}) \quad \text{s.t. } \begin{cases}\mathbf{T}=\mathbf{T}_0+\sum_{l\in\mathcal{H}}\left(w_l\mathbf{I}_{(+)}^l + w_l^*\mathbf{I}_{(-)}^l\right),\\ \mathbf{T}\succeq\mathbf{0}\end{cases} \tag{13}$$

where $\mathrm{trace}(\cdot)$ is the trace of a matrix, and $\mathbf{T}\succeq\mathbf{0}$ indicates that $\mathbf{T}$ is a positive semidefinite matrix. This is a semidefinite program, and there exist many efficient algorithms and high-quality softwares to solve these types of problems.

3.2. DOA Estimation with Toeplitz Covariance Matrix

Let $\mathbf{T}^\star$ be the solution to the problem (13), and then classical approaches like multiple signal classification (MUSIC) algorithm [20] can be adopted for DOA estimation. More specially, the eigenvalue decomposition of $\mathbf{T}^\star$ is first conducted as follows

$$\mathbf{T}^\star = \mathbf{U}_s\mathbf{\Sigma}_s\mathbf{U}_s^H + \mathbf{U}_n\mathbf{\Sigma}_n\mathbf{U}_n^H \tag{14}$$

where $\mathbf{\Sigma}_s$ is a diagonal matrix with its diagonal containing the largest K eigenvalues, $\mathbf{U}_s$ consists of the eigenvectors corresponding to the largest K eigenvalues, while $\mathbf{\Sigma}_n$ is a diagonal matrix with its diagonal containing the remaining eigenvalues , $\mathbf{U}_n$ consists of the eigenvectors corresponding to the remaining eigenvalues. Next, the spatial spectrum is constructed as follows

$$P(\theta) = \mathbf{a}^H(\theta)\mathbf{U}_n\mathbf{U}_n^H\mathbf{a}(\theta) \tag{15}$$

where $\mathbf{a}(\theta) = [\beta^0(\theta),\beta^1(\theta),\cdots,\beta^{D_m}(\theta)]$ with D_m being the maximal number in the difference set among the physical sensor positions. Finally, the directions of the K incoming signals can be determined through searching for the minima of $P(\theta)$ in the spatial region of interest.

4. SIMULATION RESULTS

In this section, several simulation experiments are carried out to verify the effectiveness of the proposed method. For comparison purpose, the DOA estimation using Toeplitz covariance matrix completion method with ONLA [6], MUSIC method [20] with MNLA, and MUSIC method with ULA are simultaneously experimented. Note that MNLA, ONLA, and ULA share the same number of sensors in all the experiments.

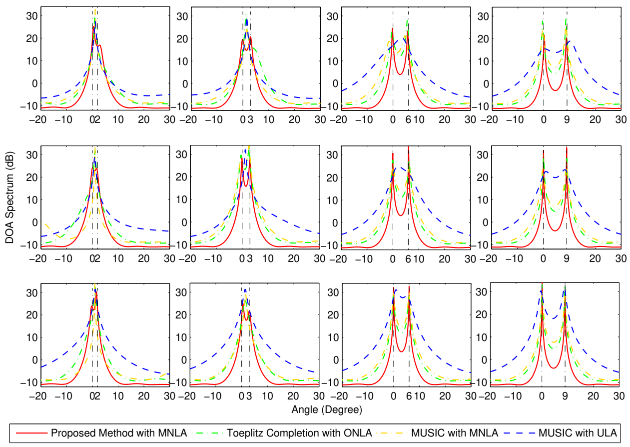

Fig. 2. Angle resolution comparison for different SNRs and different angle separations between sources Δ_θ. SNR is set as -5 dB, 0 dB, and 5 dB (from row 1 to row 3), and the angle separation Δ_θ between two sources is set as $2°$, $3°$, $6°$, and $9°$ (from column 1 to column 4).

4.1. Angle Resolution Comparison

In this example, we compare the DOA estimation resolution in different signal-to-noise ratio (SNR) and different angle separations. Two ($K = 2$) signals are impinging upon a linear array with $L = 5$ sensors. The number of snapshots is 1000. The spatial spectrum among these methods are plotted in Fig. 2. Obviously, it is seen that, in the harsh environment (very low SNR and very small Δ_θ), all the methods performs poorly; while as the SNR or Δ_θ increases, they all can estimate two directions more and more accurately. Additionally, the proposed method with MNLA firstly achieves satisfactory performance when the SNR or Δ_θ increases. Moreover, the proposed method has sharpest peaks in or nearby the real impinging directions in any experimental situations.

4.2. RMSE Comparison

We now compare the root-mean-squared error (RMSE) of the DOA estimates. The RMSE is calculated as

$$\mathrm{RMSE} = \sqrt{\frac{1}{QK}\sum_{q=1}^{Q}\sum_{k=1}^{K}(\hat{\theta}_{kq} - \theta_k)^2} \tag{16}$$

where $\hat{\theta}_{kq}$ is the DOA estimate of the k-th signal in the q-th Monte Carlo run, and Q is the total number of runs. MUSIC method is applied on a filled-sensor ULA which has the same physical aperture with MNLA to set a lower bound for RMSE.

First, two signals from $\{0°, 10°\}$ impinge onto a linear array with $L = 5$ sensors. SNR ranges from -4 dB to 12 dB. The number of snapshots is 500, and the total number of Monte Carlo runs is $Q = 1000$. The curves of RMSE versus SNR are drawn in Fig. 3. The gap between the proposed method and the lower RMSE bound is smaller than any other gaps between other testing methods and the lower bound.

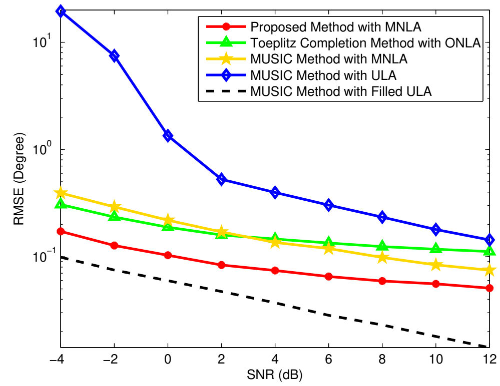

Fig. 3. RMSE versus SNR with 500 snapshots.

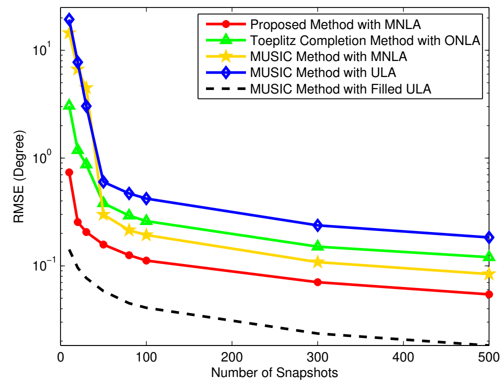

Fig. 4. RMSE versus snapshot number with $\mathrm{SNR} = 10$ dB.

Then, the SNR is fixed to be 10 dB, while the number of snapshots varies from 10 to 500, and other parameters are kept unchanged. The curves of RMSE versus number of snapshots are showed in Fig. 4. Again, it is observed that the proposed method has the closest RMSE to the lower bound when compared to other methods for different snapshots.

5. CONCLUSION

In this paper, DOA estimation using a modified nested linear array has been investigated. A Toeplitz matrix completion scheme has been devised for MNLA which involves missing lags. The recovery process included two steps: 1) construct a covariance matrix with Toeplitz structure using the existing lags while the missing lags are replaced by zeros; 2) formulate and solve a semidefinite program with trace minimization to obtain the recovered Toeplitz covariance matrix. The Toeplitz covariance matrix is subsequently used for direction finding. Simulation results have validated the superior performance of the proposed method compared with other methods.

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