FREQUENCY-DOMAIN BASED WAVEFORM DESIGN FOR BINARY EXTENDED-TARGET CLASSIFICATION

Sultan Z. Alshirah [◇•*], *Shahzad Gishkori* [◇] *and Bernard Mulgrew* [◇]

[◇] The School of Engineering, Institute for Digital Communications, The University of Edinburgh, UK
{s.alshirah, s.gishkori, bernie.mulgrew}@ed.ac.uk
[•] King Abdulaziz City for Science and Technology (KACST), Riyadh, Saudi Arabia
salshirah@kacst.edu.sa

ABSTRACT

In this paper, an optimal radar waveform-design scheme is proposed, using frequency snapshots (i.e., frequency-domain processing), based on an objective function derived directly from a binary classification/identification criterion constrained by waveform-energy. We consider an extended-target model where the targets frequency response is assumed to be complex Gaussian. High- and low-energy solutions for the optimal waveform are explored and closed-form expressions in both the scenarios are derived.

***Index Terms*—** Binary target classification, radar waveform design, extended-target, complex signals

1. INTRODUCTION

In recent decades, a lot of research has been done on designing optimal radar waveforms. It has been shown that radar waveforms affect the performance of radar systems in many ways for different kind of scenarios. For example, linearly-frequency-modulated (LFM) waveforms are known to improve detection in many scenarios by allowing high energy of long pulses while maintaining the radar range resolution of a short pulse for targets that can be modelled as point-targets [1]. The point-target model assumes that the radar waveform reflected off a remote target is only delayed and attenuated by a certain scalar factor which depends on the physical and electromagnetic properties of the target (e.g. radar cross-section (RCS)). In scenarios where the radar is wideband or the target has large physical extent or both, target models such as point-target model become less accurate in capturing target behaviour [2]. A more accurate model is to assume that a target has an extended impulse response (IR) that convolves with the transmitted radar waveform once it hits the target. This is called an extended-target model. The target IR is a feature that can be exploited to improve detection and classification performance as each target class can have its distinct IR.

Limited work has been done on designing optimal waveform that maximises target classification performance. Generally, researchers have followed two main approaches: i) To design the waveform pre-employment based on sufficient prior knowledge available to maximise target classification [3, 4, 5] and ii) to design the radar waveform on-the-fly using a mix of special criterion (e.g., maximising mutual information) and hypothesis testing, until a certain target classification maximisation is achieved [6, 7, 8]. Most of the research adopting the first approach derives the optimal waveform for target classification in the time domain from objective functions that are not directly classification-based, such as, signal-to-interference-plus-noise ratio (SINR), mutual information, the Euclidean or Mahalanobis distance between classes [3, 5]. In this paper, we derive the optimal complex-valued waveform to maximise binary target classification in the frequency domain between classes with random target responses under either coloured or white noise. We formulate the optimisation problem to find the optimal waveform under constant energy constraint. We derive and explore the high and low energy solutions for this design problem which, to the authors' knowledge, has not been studied in the literature.

Notations: x is a scalar, $\mathbf{x}$ (small case bold) is a column vector, X is the Fourier transform of $\mathbf{x}$, $\mathbf{X}$ (capital case bold) is a matrix, $(\cdot)^H$ is Hermitian transpose, $\circledast$ describes continuous-time domain convolution, $\mathrm{diag}(X)$ is a diagonal matrix with X constituting its diagonal elements, $\Re x$ and $\Im x$ are the real and imaginary parts of x, respectively.

2. MODELLING

2.1. Signal Model

In the time domain, the IR of the target is convolved with the waveform. The received signal $y(t)$ in the time domain is given by, $y(t) = x(t) \circledast h(t) + n(t)$, where $x(t)$ is the radar waveform, $h(t)$ is the target IR and $n(t)$ is the noise at the receiver. In the frequency domain, the target frequency response is multiplied with the frequency transform of the radar waveform. Then, the signal model employing the frequency

*The first author is partially supported by King Abdulaziz City for Science and Technology (KACST), Riyadh, Saudi Arabia.

snapshot model [9] with vector notation, would be as follows:

$$Y = \boldsymbol{\Omega}_X H + N \quad (1)$$

where $\boldsymbol{\Omega}_X = \mathrm{diag}(X)$, X is the frequency transform of the radar waveform $\mathbf{x}$, H is the frequency response of the extended target and N is the frequency transformed noise vector $\mathbf{n}$ at the receiver. All vectors are of size $m \times 1$. Note, H can represent a sum of all the signal dependent variation or interferences as long as it satisfies the assumptions about its distribution. For example, in [3] (where the derivation is done in the time domain), the authors assumed IR to be a sum of a deterministic part and a zero-mean random (i.e., clutter) part, resulting in the distribution of IR having the mean equal to the deterministic part and the covariance determined by the covariance of the clutter. This, similarly, can be assumed in the frequency domain where we assume that H is a sum of a (deterministic) target frequency response M and the clutter frequency vector C, resulting in $H \sim \mathbb{C}\mathcal{N}(M, \boldsymbol{\Sigma}_C)$, where $\boldsymbol{\Sigma}_C$ is the covariance of the clutter. Whether the signal-independent variation is due to time or frequency response of the target or due to clutter, the signal model is applicable and identical results are to be expected.

2.2. Probability of misclassification and optimal waveform design

The binary classification problem is defined as follows: we assume a target can either belong to class, $i = 1$ or $i = 2$ (i.e. ω_i, for $i = 1, 2$, where ω is *the state of nature* as defined in [10]) where the statistical properties of each class's frequency response are known a priori. If it is assumed that frequency responses of both the classes have complex normal distribution with different means but same covariance matrix, i.e. $H_i \sim \mathbb{C}\mathcal{N}(M_i, \boldsymbol{\Sigma}_H)$, and the noise to be a complex normal random process with zero mean vector and covariance matrix $\boldsymbol{\Sigma}_N$, i.e., $N \sim \mathbb{C}\mathcal{N}(0, \boldsymbol{\Sigma}_N)$. Then, the distribution of the frequency response of the received data vector Y, depending on the class i, would be: $\mathbb{C}\mathcal{N}_{m\times 1}(\mathbf{m}_i, \boldsymbol{\Sigma}_Y)$ for $i = 1, 2$ where $\mathbf{m}_i = \boldsymbol{\Omega}_X M_i$ and $\boldsymbol{\Sigma}_Y = \boldsymbol{\Omega}_X \boldsymbol{\Sigma}_H \boldsymbol{\Omega}_X^H + \boldsymbol{\Sigma}_N$. To derive the probability of misclassification, we start with a minimum-error-rate discriminant functions presented in [10]:

$$\begin{aligned} g_i(Y) =& \ln p(Y|\omega_i) + \ln P(\omega_i) \\ =& -(Y - \mathbf{m}_i)^H \boldsymbol{\Sigma}_Y^{-1} (Y - \mathbf{m}_i) \\ & - m \ln\pi - \ln|\boldsymbol{\Sigma}_Y| + \ln P(\omega_i) \end{aligned} \quad (2)$$

where $p(Y|\omega_i)$ is the likelihood function of Y given ω_i and $P(\omega_i)$ is the prior probability. If the prior probabilities for all classes are identical (i.e. $P(\omega_1) = P(\omega_2) = 0.5$), it is straightforward to derive the minimum-error-rate classifier and the best hyperplane for classification by rearranging the following equation that defines the separating hyperplane into the form of a linear function:

$$g_1(Y) = g_2(Y) \quad (3)$$

which results in the following expression:

$$\Re\{(\mathbf{m}_1 - \mathbf{m}_2)^H \boldsymbol{\Sigma}_Y^{-1} Y + \frac{\mathbf{m}_2^H \boldsymbol{\Sigma}_Y^{-1} \mathbf{m}_2 - \mathbf{m}_1^H \boldsymbol{\Sigma}_Y^{-1} \mathbf{m}_1}{2}\} = 0 \quad (4)$$

making the weight vector $W = \boldsymbol{\Sigma}_Y^{-1}(\mathbf{m}_1 - \mathbf{m}_2)$ and the scalar bias $w_0 = (\mathbf{m}_2^H \boldsymbol{\Sigma}_Y^{-1} \mathbf{m}_2 - \mathbf{m}_1^H \boldsymbol{\Sigma}_Y^{-1} \mathbf{m}_1)/2$ as the standard linear classifier equation is $f(Y) = W^H Y + w_0$.

2.2.1. Deriving the probability of misclassification

The probability of misclassification can be calculated given the distribution of the classifier function in (4) which is defined as:

$$f(Y) = (\mathbf{m}_1 - \mathbf{m}_2)^H \boldsymbol{\Sigma}_Y^{-1} Y + \frac{\mathbf{m}_2^H \boldsymbol{\Sigma}_Y^{-1} \mathbf{m}_2 - \mathbf{m}_1^H \boldsymbol{\Sigma}_Y^{-1} \mathbf{m}_1}{2} \quad (5)$$

using the equation:

$$P_{mc} = p(\Re f(Y) \leq 0|\omega_1)P(\omega_1) + p(\Re f(Y) > 0|\omega_2)P(\omega_2) \quad (6)$$

where: $f(Y) \sim \mathbb{C}\mathcal{N}(\mu_{fi}, \sigma_f^2)$ for $i = 1, 2$. The mean μ_{fi} can be derived as follows:

$$\begin{aligned} \mu_{fi} &= \mathbb{E}\{f(Y)|\omega_i\} \\ &= (\mathbf{m}_1 - \mathbf{m}_2)^H \boldsymbol{\Sigma}_Y^{-1} \mathbf{m}_i + \frac{\mathbf{m}_2^H \boldsymbol{\Sigma}_Y^{-1} \mathbf{m}_2 - \mathbf{m}_1^H \boldsymbol{\Sigma}_Y^{-1} \mathbf{m}_1}{2} \end{aligned} \quad (7)$$

and similarly, the variance σ_f^2:

$$\begin{aligned} \sigma_f^2 &= \mathbb{E}\{(f(Y) - \mathbb{E}\{f(Y)\})(f(Y) - \mathbb{E}\{f(Y)\})^H\} \\ &= (\mathbf{m}_1 - \mathbf{m}_2)^H \boldsymbol{\Sigma}_Y^{-1} (\mathbf{m}_1 - \mathbf{m}_2) \end{aligned} \quad (8)$$

Also, it can be shown that $\Re\mu_{f1} = +\sigma_f^2/2$ and $\Re\mu_{f2} = -\sigma_f^2/2$. Then,

$$\begin{aligned} & p(\Re f(Y) \leq 0|\omega_1) = \\ & \int_{-\infty}^{0} \int_{-\infty}^{\infty} \xi(\Re f(Y)) \xi(\Im f(Y)) \cdot \mathrm{d}\Im f(Y) \cdot \mathrm{d}\Re f(Y) \end{aligned} \quad (9)$$

and,

$$\begin{aligned} & p(\Re f(Y) > 0|\omega_2) = \\ & \int_{0}^{\infty} \int_{-\infty}^{\infty} \xi(\Re f(Y)) \xi(\Im f(Y)) \cdot \mathrm{d}\Im f(Y) \cdot \mathrm{d}\Re f(Y) \end{aligned} \quad (10)$$

where: $\xi(x) = \frac{1}{\sqrt{\pi\sigma_x^2}} \exp\{-\frac{(x - m_x)^H (x - m_x)}{\sigma_x^2}\}$.

By solving the integration by substitution in addition to employing the definition of the Q-function,

$$p(\Re f(Y) \leq 0\}|\omega_1) = \mathrm{Q}(+\sqrt{2}\Re\mu_{f1}/\sigma_f) \quad (11)$$

$$p(\Re f(Y) > 0\}|\omega_2) = \mathrm{Q}(-\sqrt{2}\Re\mu_{f2}/\sigma_f) \quad (12)$$

Finally,

$$P_{mc} = \mathrm{Q}\Big(\sqrt{(\mathbf{m}_1 - \mathbf{m}_2)^H \boldsymbol{\Sigma}_Y^{-1} (\mathbf{m}_1 - \mathbf{m}_2)/2}\,\Big). \quad (13)$$

2.2.2. *Waveform design*

The derivation of the probability of misclassification leads to the expression in (13). In order to improve target classification, the optimal waveform should be designed to maximise:

$$\begin{aligned}\Phi(\mathbf{\Omega}_X) &= (\mathbf{m}_1 - \mathbf{m}_2)^H \mathbf{\Sigma}_Y^{-1} (\mathbf{m}_1 - \mathbf{m}_2) \\ &= (M_1 - M_2)^H \mathbf{\Omega}_X^H (\mathbf{\Omega}_X \mathbf{\Sigma}_H \mathbf{\Omega}_X^H + \mathbf{\Sigma}_N)^{-1} \mathbf{\Omega}_X (M_1 - M_2)\end{aligned} \tag{14}$$

which is the Mahalanobis distance between the means. Now, after constraining the energy of the waveform in the time domain, the optimisation problem to find the optimal waveform to maximise classification can be formulated as follows:

$$\begin{aligned}\underset{\mathbf{\Omega}_X}{\arg\max} \quad & \Phi(\mathbf{\Omega}_X) \\ \text{s.t.} \quad & \mathrm{tr}(\mathbf{\Omega}_X \mathbf{\Omega}_X^H) = m\epsilon_x\end{aligned} \tag{15}$$

where $\mathrm{tr}(\cdot)$ is the trace function, ϵ_x is waveform energy in time domain and m is the number of frequency bins.

We now explore the optimisation problem under two extreme conditions: First, for a system allowing very high energy waveform such that $\mathbf{\Omega}_X \mathbf{\Sigma}_Y \mathbf{\Omega}_X^H + \mathbf{\Sigma}_N \approx \mathbf{\Omega}_X \mathbf{\Sigma}_Y \mathbf{\Omega}_X^H$, while $\mathbf{\Omega}_X$ is invertible, $\Phi(\mathbf{\Omega}_X)$ becomes approximately independent of the waveform design and will be given by,

$$\Phi(\mathbf{\Omega}_X) = (M_1 - M_2)^H \mathbf{\Sigma}_H^{-1} (M_1 - M_2) \tag{16}$$

This also applies if the clutter-to-noise ratio (CNR)/signal-to-noise ratio (SNR) (depending on what source of signal dependent variation is assumed) is significantly large while the waveform has modest or high energy. This conclusion cannot be obtained if the derivation is time-domain based as the convolution matrix that is neither square nor invertible. Second, if the waveform energy is low such that $\mathbf{\Omega}_X \mathbf{\Sigma}_Y \mathbf{\Omega}_X^H + \mathbf{\Sigma}_N \approx \mathbf{\Sigma}_N$, then:

$$\Phi(\mathbf{\Omega}_X) = (M_1 - M_2)^H \mathbf{\Omega}_X^H \mathbf{\Sigma}_N^{-1} \mathbf{\Omega}_X (M_1 - M_2) \tag{17}$$

which clearly leads to the optimal waveform being the eigenvector of the matrix $\mathbf{\Omega}_M^H \mathbf{\Sigma}_N^{-1} \mathbf{\Omega}_M$ corresponding to its largest eigenvalue of where $\mathbf{\Omega}_M = \mathrm{diag}(M_1 - M_2)$. Similarly, this is also applicable if SNR/CNR are significantly low.

If the above conditions do not hold and the energy of the waveform is constrained to ϵ_x, there is no closed form solution to this optimisation problem. This is because of the presence of $\mathbf{\Omega}_X$ inside $\mathbf{\Sigma}_Y$. In this case, finding the optimal waveform can be achieved with an off-the-shelf optimisation solver software.

3. RESULTS AND DISCUSSION

In this section, we present three simulations in terms of the probability of correct classification vs energy per transmission ϵ_x for three different purposes: i) The first simulation shows classification performance maximisation achieved using the optimised waveform over a waveform with flat spectrum like LFM, ii) The second simulation shows that the optimised waveform performs the same if not better than that defined in [3] (in the time domain) and iii) The third simulation shows the high energy limit that no waveform with high energy can outperform. The measurement performance is generated with 100,000 runs while the theoretical performance is calculated from (13). The complex means of the classes and the covariance are generated randomly at the start of each simulation. To design the radar waveform for each simulation, Matlab optimisation toolbox is used where "interior-point" algorithm is employed to solve the constrained complex optimisation problem.

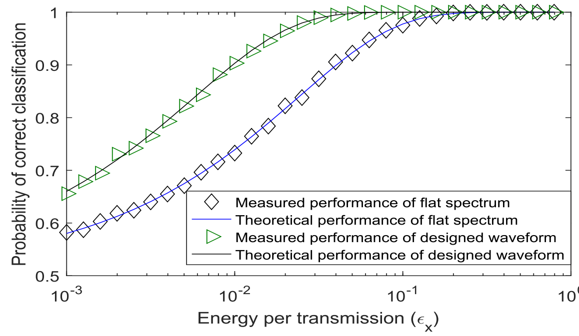

Fig. 1. Measured and theoretical probability of correct classification vs energy per transmission for an optimised waveform and a flat spectrum waveform.

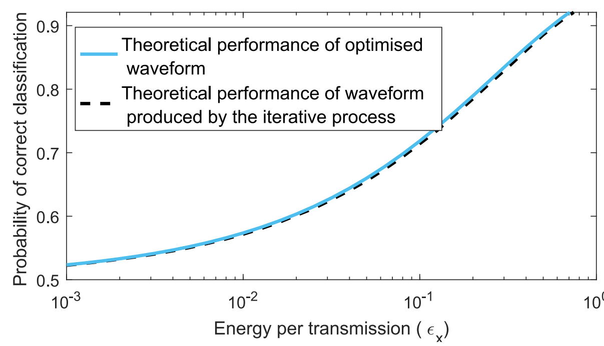

Fig. 2. Theoretical probability of correct classification vs energy per transmission for an optimised waveform and a waveform generated by iterative process explained in [3] from multiple under same conditions.

Fig. 1 shows that as the energy per transmission increases, the two waveforms eventually achieve perfect classification. However, for less energy, the optimised waveform brings better target identification performance than the classical flat spectrum waveform. For example, at $\epsilon_x = 0.02$, a probability of correct classification of approximately 0.97 can be

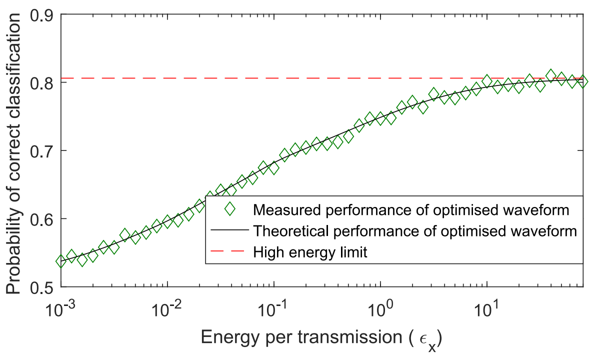

Fig. 3. The theoretical and measured probability of correct classification against energy per transmission for optimised waveform at high energy levels (solid) and the high energy limit (dashed) calculated from (13) and (16).

achieved employing the designed waveform while the other can achieve 0.82. Fig. 2 shows the theoretical performance for two time-domain based waveforms: one designed by the iterative process presented in [3] for the same objective function while the other is designed by the "interior-point" algorithm by Matlab using a time-domain version of the optimisation problem presented above. The figure shows that the optimised waveform achieves similar performance with slight improvement over the one designed by the iterative process. The measured performance does not add much information to the figure as it will follow the corresponding theoretical one as shown previously in Fig. 1. Fig. 3 shows the performance of the optimised waveform for high energy values and the maximum performance that can be achieved according to (13) and (16). The result shows the probability of correct classification is bounded by the high energy approximation. Also, low energy limit can be calculated using (13) and (17) which equals 0.5 at $\epsilon_x = 10^{-9}$ in this scenario.

4. CONCLUSION

In this paper, we have derived the optimal waveform design maximising binary target classification directly from the probability of misclassification assuming the target frequency response to be extended complex Gaussian random response. We have derived an explicit expression for the probability of misclassification based on the waveform and prior knowledge about target classes and the noise at the receiver. The design problem is set up based on the expression derived in addition to constant energy constraint. The optimised waveform is shown to maximise classification performance and slightly outperforms that in [3]. We have derived the high energy and low energy solutions for waveform design which have not been studied in the literature. We showed that, even the optimised waveform with sufficiently high energy, cannot achieve better performance than that defined by the high energy limit while also concluding that the waveform design becomes irrelevant at such high energy. We have derived the low energy waveform and concluded that it would be the eigenvector of a matrix containing the difference between classes' means and noise covariance matrix as given in (17).

5. REFERENCES

[1] S. D. Blunt and E. L. Mokole, "Overview of radar waveform diversity," *IEEE Aerospace and Electronic Systems Magazine*, vol. 31, no. 11, pp. 2–42, November 2016.

[2] M. R. Bell, "Information theory and radar waveform design," *IEEE Transactions on Information Theory*, vol. 39, no. 5, pp. 1578–1597, Sep 1993.

[3] D. A. Garren, M. K. Osborn, A. C. Odom, J. S. Goldstein, S. U. Pillai, and J. R. Guerci, "Enhanced target detection and identification via optimised radar transmission pulse shape," *IEE Proceedings - Radar, Sonar and Navigation*, vol. 148, no. 3, pp. 130–138, June 2001.

[4] D. A. Garren, A. C. Odom, M. K. Osborn, J. S. Goldstein, S. U. Pillai, and J. R. Guerci, "Full-polarization matched-illumination for target detection and identification," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 38, no. 3, pp. 824–837, July 2002.

[5] S. Z. Alshirah and B. Mulgrew, "Improved 2-class target classification performance using radar waveform design," in *IEEE Radar Conference*, April 2018, pp. 0458–0461.

[6] R. Romero and N. A. Goodman, "Improved waveform design for target recognition with multiple transmissions," in *2009 International Waveform Diversity and Design Conference*, Feb 2009, pp. 26–30.

[7] Q. J. O. Tan, R. A. Romero, and D. C. Jenn, "Target recognition with adaptive waveforms in cognitive radar using practical target rcs responses," in *IEEE Radar Conference*, April 2018, pp. 0606–0611.

[8] J. Bae and N. A. Goodman, "Target recognition with high-fidelity target signatures and adaptive waveforms in mimo radar," in *IEEE CAMSAP*, Dec 2015, pp. 285–288.

[9] S. Kay, "Optimal signal design for detection of Gaussian point targets in stationary Gaussian clutter/reverberation," *IEEE Journal of Selected Topics in Signal Processing*, vol. 1, no. 1, pp. 31–41, June 2007.

[10] Richard O Duda, *Pattern Classification*, Wiley, New York, second edition. edition, 2000.