Robust Capon Beamforming via ADMM

Wen Fan[1], *Junli Liang*[1], *Guoyang Yu*[1], *H. C. So*[2], *and Jian Li*[3]

1. School of Electronics and Information,
Northwestern Polytechnical University, China.
(e-mail: liangjunli@nwpu.edu.cn, fanwen@mail.nwpu.edu.cn).
2. Department of Electronic Engineering, City University of Hong Kong, China.
3. Department of Electronic Engineering and Information Science, University of Science and Technology of China, Hefei 230000, China.

***Abstract*—This paper proposes two methods for robust Capon beamforming. One is for the doubly constrained robust Capon beamforming problem, where the unit modular constraints on the elements of the steering vector of interest are enforced to circumvent the look direction error or phase perturbations of the signal-of-interest; and another addresses robust beamforming in impulsive noise environment, where we consider the l_p-norm minimization ($0 < p < 2$) of the output while constraining the mainlobe response ripple term. We apply the splitting technique to simplify the resultant nonconvex optimization problem and solve it using alternating direction method of multipliers. The performance of the proposed methods is demonstrated via numerical examples.**

***Index Terms*—Robust Capon beamformer (RCB), Alternating direction method of multipliers (ADMM), Unit modular constraints, Impulsive noise.**

I. Introduction

Adaptive beamforming plays an important role in the fields of array signal processing such as sonar, radar, and wireless communications, and thus has received much attention [1]-[11]. The task of adaptive beamforming is to suppress interferences and noise and at the same time enhance the signal-of-interest (SOI) at the output of a sensor array by spatial filtering.

The classical Capon beamformer designs the weight vector to minimize the array output power with a unity constraint on the gain in the incoming angle of SOI. However, it will exhibit poor performance when there exists a mismatch between the presumed and actual array responses such as look direction error, imperfect array calibration, unknown wavefront distortions, phase perturbation and impulsive noise environment [1], [10], [12].

To improve the robustness, [9] proposed a robust beamforming method via optimizing the performance in the worst case, where the unknown mismatch vector between the presumed and actual steering vectors of SOI is assumed to be norm-bounded by some known constant. Then, the resulting second-order cone programming formulation is efficiently solved using the standard interior point method. Besides, [10] proposed a robust minimum variance beamformer to guarantee to satisfy the minimum gain constraint for all values in the ellipsoid, which covers the possible range of values of the interested steering vector due to imprecise knowledge of the array manifold. Then, the Lagrange multiplier method is applied to find the true steering vector. In [6], Li *et al.* first proposed a robust Capon beamforming method with an uncertainty ellipsoid constraint on the mismatch vector, and then add a norm constraint on the steering vector to constrain the white noise gain at the output, i.e., the so-called doubly constrained robust Capon beamformer [6]. Other robust beamforming techniques can be found in [1], and references cited therein.

In this paper, two robust beamforming methods are proposed. The first one addresses the problem of doubly constrained robust Capon beamformer, where the unit modular constraints on the elements of the steering vector of interest are enforced. The main reason is that when there exists the look direction error or phase perturbation, the unit modulus constraints on the elements of the steering vector of SOI are more precise than the constant norm constraint. The second one is devised for impulsive noise environment, i.e., there are some random errors with extreme values, the minimum variance (MV) (i.e., Capon [2]) based methods are sensitive to impulsive noise and thus its performance degrades severely due to the idealized Gaussian distribution assumption. Unlike the l_2-norm, the l_p-norm, where $0 < p < 2$, may achieve outlier-resistant purpose [20], [12], thus we consider the l_p-norm minimization of the output while constraining the mainlobe response ripple term. To handle the resulting nonconvex optimization problem, we decouple the convex inequality constraint with the nonconvex unit-modulus constraint, and solve it via the alternating direction method of multipliers (ADMM) [14]–[19].

Throughout the paper, vectors and matrices are denoted by boldface lowercase and uppercase letters, respectively. The $(\cdot)^T$, $(\cdot)^H$, and $(\cdot)^{-1}$ are the transpose, conjugate transpose, and matrix inverse operators, respectively. The $\mathbf{I}_m$ represents the $m \times m$ identity matrix. A square diagonal matrix with elements $\{a_1, \cdots, a_n\}$, are denoted by $\mathrm{diag}\{[a_1, \cdots, a_n]\}$. The $|\cdot|$ and $\angle(\cdot)$ are the magnitude and phase of a complex-valued scalar, respectively.

II. Problem Formulation

Consider an array composed of M elements. Let $\mathbf{R}$ denote the positive definite covariance matrix of the array output vector [1], [5], [6], i.e.,

$$\mathbf{R} = \delta^2 \mathbf{a}\mathbf{a}^H + \mathbf{Q}, \tag{1}$$

where δ^2 and $\mathbf{a}$ are the power of the SOI and its steering vector, respectively, and $\mathbf{Q}$ contains the interference and noise components.

The standard Capon beamformer minimizes the output power [2], i.e.,

$$\min_{\mathbf{w}} \ \mathbf{w}^H \mathbf{R}\mathbf{w}, \ \text{ s.t. } \ \mathbf{w}^H \mathbf{a} = 1, \tag{2}$$

where $\mathbf{R} = \mathrm{E}\{\mathbf{x}_n \mathbf{x}_n^H\}$ is not available in reality and thus is replaced by the sample covariance matrix $\hat{\mathbf{R}} = \frac{1}{N}\sum_{n=1}^{N} \mathbf{x}_n \mathbf{x}_n^H$, where N denotes the number of snapshots. According to the Capon objective function

$$\mathbf{w}^H \mathbf{R}\mathbf{w} = \frac{1}{N}\sum_{n=1}^{N} |\mathbf{w}^H \mathbf{x}_n|^2, \tag{3}$$

it is easily found that the Capon beamformer actually minimizes the l_2-norm.

A. Doubly Constrained Robust Capon Beamformer

In practical applications, when there exists look direction error or phase perturbations in the provided steering vector $\bar{\mathbf{a}}$, [6] formulates the so-called doubly constrained robust Capon beamformer model to determine the true steering vector $\mathbf{a}$:

$$\min_{\mathbf{a}} \ \mathbf{a}^H \mathbf{R}^{-1}\mathbf{a} \ \text{ s.t. } \|\mathbf{a} - \bar{\mathbf{a}}\|^2 \leq \varepsilon, \ \|\mathbf{a}\|^2 = M, \tag{4}$$

where $\varepsilon > 0$ is a user-defined parameter, and $\mathbf{a} = [a(1) \ \cdots \ a(M)]^T$. In fact, the element of the steering vector of SOI is unimodular in the presence of look direction error or phase perturbations. Therefore, we consider the following optimization model:

$$\begin{aligned} &\min_{\mathbf{a}} \ \mathbf{a}^H \mathbf{R}^{-1}\mathbf{a} \\ &\text{s.t.} \quad \|\mathbf{a} - \bar{\mathbf{a}}\|^2 \leq \varepsilon, \ |a(m)| = 1, \quad m = 1, \ldots, M. \end{aligned} \tag{5}$$

In certain scenarios, the modulus of individual elements may not be a constant. Thus, we can also extend the above model by replacing the unimodulus constraint by the double constraints $1 - \epsilon \leq |a(m)| \leq 1 + \epsilon, m = 1, \ldots, M$.

The task of this work is to solve (5) to determine the true steering vector of SOI $\mathbf{a}$ and then obtain the weight vector $\mathbf{w}$ (i.e., $\mathbf{w} = \frac{\mathbf{R}^{-1}\mathbf{a}}{\mathbf{a}^H \mathbf{R}^{-1}\mathbf{a}}$) by solving the standard Capon beamformer problem (2) [1], [2].

B. Robust Adaptive Beamforming in Impulsive Noise Environment

Most existing data-dependent beamforming algorithms are based on the MV criterion (3) [20]. However, when there exists impulsive noise or outliers in actual applications, i.e., there are some random errors with extreme values, the MV based approach is sensitive to impulsive noise and thus its performance degrades severely due to the idealized Gaussian distribution assumption. Unlike the l_2-norm, the l_p-norm, where $0 < p < 2$, may achieve outlier-resistant purpose, and thus has been applied into the signal processing problems in impulsive noise environment. Therefore, we consider replacing the objective function (3) with the following objective function [20]:

$$\begin{aligned} &\min_{\mathbf{w}} \ \sum_{n=1}^{N} |\mathbf{w}^H \mathbf{x}_n|^p \\ &\text{s.t.} \ \ 1 - \epsilon \leq |\mathbf{w}^H \mathbf{a}(\theta_m)|^2 \leq 1 + \varepsilon, m = 1, \cdots, M, \end{aligned} \tag{6}$$

where $\{\theta_m\}_m^M = 1$ denotes the M grid angles in the main beam.

III. Proposed Algorithms

The ADMM based algorithms are derived in this section to solve the problems (5) and (6).

A. Solution to (5)

The main difficulty of (5) lies in the nonconvex constraints $|a(m)| = 1$ for $m = 1, \cdots, M$. To separate it from other convex constraints, we introduce auxiliary variables $\mathbf{a} = \mathbf{b}$, and rewrite (5) in the following equivalent form:

$$\begin{aligned} &\min_{\mathbf{a},\mathbf{b}} \ \mathbf{a}^H \mathbf{R}^{-1}\mathbf{a} \\ &\text{s.t.} \quad \|\mathbf{a} - \bar{\mathbf{a}}\|^2 \leq \varepsilon, \\ &\qquad \mathbf{a} = \mathbf{b}, \ |b(m)| = 1, \quad m = 1, \ldots, M, \end{aligned} \tag{7}$$

where $\mathbf{b} = [b(1) \ \cdots \ b(M)]^T$. Based on (7), we construct the augmented Lagrangian:

$$\begin{aligned} &\mathcal{L}(\mathbf{a}, \mathbf{b}, \boldsymbol{\lambda}) = \mathbf{a}^H \mathbf{R}^{-1}\mathbf{a} + \Re\{\boldsymbol{\lambda}^H(\mathbf{a} - \mathbf{b})\} + \frac{\rho}{2}\|\mathbf{a} - \mathbf{b}\|^2, \\ &\text{s.t.} \quad \|\mathbf{a} - \bar{\mathbf{a}}\|^2 \leq, \ |b(m)| = 1, \quad m = 1, \ldots, M, \end{aligned} \tag{8}$$

where $\rho > 0$ is a user-defined parameter, and $\boldsymbol{\lambda} = [\lambda_1, \cdots, \lambda_M]^T$ contains the Lagrange multipliers corresponding to the constraints $a(m) = b(m)$ for $m = 1, \cdots, M$.

Then, based on the ADMM [14], we determine $\{\mathbf{a}, \mathbf{b}, \boldsymbol{\lambda}\}$ via the following iterative steps:

Step 1: Determine $\mathbf{a}$ with given $\{\mathbf{b}(t), \boldsymbol{\lambda}(t)\}$ from:

$$\begin{aligned} \mathbf{a}(t+1) = &\arg \min_{\mathbf{a}} \ \mathcal{L}(\mathbf{a}, \mathbf{b}(t), \boldsymbol{\lambda}(t))) \\ &\text{s.t. } \|\mathbf{a} - \bar{\mathbf{a}}\|^2 \leq \varepsilon. \end{aligned} \tag{9}$$

Ignoring the irrelevant terms to $\mathbf{a}$, (9) can be simplified as :

$$\begin{aligned} &\min_{\mathbf{a}} \ \mathbf{a}^H \mathbf{R}^{-1}\mathbf{a} - \frac{\rho}{2}\left(\hat{\mathbf{b}}^H(t)\mathbf{a} + \mathbf{a}^H \hat{\mathbf{b}}(t)\right) \\ &\text{s.t.} \ \|\mathbf{a} - \bar{\mathbf{a}}\|^2 \leq \varepsilon, \end{aligned} \tag{10}$$

where $\mathbf{a}^H\mathbf{a} = M$ is applied and $\hat{\mathbf{b}}(t) = \mathbf{b}(t) - \frac{\boldsymbol{\lambda}(t)}{\rho}$. To simplify the problem in (10), we define:

$$\breve{\mathbf{a}} = \mathbf{a} - \bar{\mathbf{a}}, \tag{11}$$

and replace $\mathbf{a}$ by $\breve{\mathbf{a}} + \bar{\mathbf{a}}$ in (10), yielding:

$$\min_{\breve{\mathbf{a}}} \ \breve{\mathbf{a}}^H \mathbf{R}^{-1}\breve{\mathbf{a}} + \mathbf{c}^H \breve{\mathbf{a}} + \breve{\mathbf{a}}^H \mathbf{c} \ \text{ s.t. } \ \|\breve{\mathbf{a}}\|^2 \leq \varepsilon, \tag{12}$$

where $c = \mathbf{R}^{-1}\bar{\mathbf{a}} - \frac{\rho}{2}\hat{\mathbf{b}}$. Then, based on (12), we define the Lagrangian as [13], [23]:

$$\mathcal{F}(\breve{\mathbf{a}}, \gamma) = \breve{\mathbf{a}}^H \mathbf{R}^{-1}\breve{\mathbf{a}} + \mathbf{c}^H \breve{\mathbf{a}} + \breve{\mathbf{a}}^H \mathbf{c} + \gamma(\|\breve{\mathbf{a}}\|^2 - \varepsilon). \tag{13}$$

Partial differentiating $\mathcal{F}(\breve{\mathbf{a}}, \gamma)$ w.r.t. $\mathbf{a}$ and γ, respectively, and setting the results to be zeros yield the so-called two Lagrange equations [23], i.e.,

$$2\mathbf{R}^{-1}\breve{\mathbf{a}} + 2\mathbf{c} + 2\gamma\breve{\mathbf{a}} = 0 \tag{14}$$

and $\|\breve{\mathbf{a}}\|^2 - \varepsilon = 0$. From (14) we derive the analytical solution:

$$\breve{\mathbf{a}} = -(\mathbf{R}^{-1} + \gamma\mathbf{I}_M)^{-1}\mathbf{c}. \tag{15}$$

Inserting (15) into $\|\breve{\mathbf{a}}\|^2 - \varepsilon = 0$ results in the function of the Lagrange multiplier γ as:

$$g(\gamma) = \mathbf{c}^H(\mathbf{R}^{-1} + \gamma\mathbf{I}_M)^{-2}\mathbf{c} - \varepsilon = 0, \tag{16}$$

which shows that the optimal value $\breve{\gamma}$ is one of the roots of $g(\gamma) = 0$ [23]. Denote the eigenvalue decomposition of $\mathbf{R}$ as:

$$\mathbf{R} = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^H, \tag{17}$$

where $\mathbf{\Sigma} = \mathrm{diag}\{[\sigma_1, \sigma_2, \cdots, \sigma_M]\}$ is the diagonal eigenvalue matrix and the eigenvalues are arranged in the descending order, i.e., $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_M > 0$. In addition, the corresponding eigenvectors $\{\mathbf{u}_m\}$ form the eigenvector matrix $\mathbf{U} = [\mathbf{u}_1\ \mathbf{u}_2 \cdots \mathbf{u}_M]$. Thus,

$$\begin{aligned} g(\gamma) &= \mathbf{c}^H\mathbf{U}(\Sigma^{-1} + \gamma\mathbf{I}_M)^{-2}\mathbf{U}^H\mathbf{c} - \varepsilon \\ &= \sum_{m=1}^{M} \frac{|\mathbf{c}^H\mathbf{u}_m|^2}{(\frac{1}{\sigma_m} + \gamma)^2} - \varepsilon, \end{aligned} \tag{18}$$

where $0 < \frac{1}{\sigma_1} \leq \frac{1}{\sigma_2} \leq \cdots \leq \frac{1}{\sigma_M}$. Since $\lim_{\gamma \to -\frac{1}{\sigma_1}} g(\gamma) = +\infty$ and $\lim_{\gamma \to +\infty} g(\gamma) = -\varepsilon$, and

$$\frac{\partial g(\gamma)}{\partial \gamma} = \sum_{m=1}^{M} \frac{-2|\mathbf{c}^H\mathbf{u}_m|^2}{(\frac{1}{\sigma_m} + \gamma)^3} < 0,\ \gamma \in (-\frac{1}{\sigma_1}, +\infty), \tag{19}$$

$g(\gamma)$ is a monotonically decreasing function [23] and thus there is a unique solution $\breve{\gamma}$ to $g(\gamma) = 0$ in the region $\gamma \in (-\frac{1}{\sigma_1}, +\infty)$. Therefore, we can determine the Lagrange multiplier $\breve{\gamma}$ by a simple bisection method, and insert it into (15) to yield $\breve{\mathbf{a}}$. Furthermore, with the definition in (11), $\mathbf{a}(t+1)$ is easily obtained as:

$$\mathbf{a}(t+1) = \bar{\mathbf{a}} + \breve{\mathbf{a}}. \tag{20}$$

Step 2: Update $\mathbf{b}$ with given $\{\mathbf{a}(t+1), \boldsymbol{\lambda}(t)\}$ from:

$$\begin{aligned} \mathbf{b}(t+1) = &\arg\min_{\mathbf{b}}\ \mathcal{L}(\mathbf{a}(t+1), \mathbf{b}, \boldsymbol{\lambda}(t))) \\ &\text{s.t. } |b(m)| = 1, \quad m = 1, \ldots, M. \end{aligned} \tag{21}$$

Ignoring the irrelevant terms to $\mathbf{b}$, (21) can be simplified as

$$\min_{\mathbf{b}} \frac{\rho}{2}\|\hat{\mathbf{a}}(t+1) - \mathbf{b}\|^2 \text{ s.t. } |b(m)| = 1, m = 1, \ldots, M. \tag{22}$$

where $\hat{\mathbf{a}}(t+1) = \mathbf{a}(t+1) + \frac{\boldsymbol{\lambda}(t)}{\rho}$.

Thus, the solution to (21) is given by:

$$b_m(t+1) = e^{j\angle(\hat{a}_m(t+1))}, \text{for } m = 1, \cdots, M. \tag{23}$$

Step 3: Update Lagrange multiplier vector $\boldsymbol{\lambda}$ as:

$$\boldsymbol{\lambda}(t+1) = \boldsymbol{\lambda}(t) + \rho(\mathbf{a}(t+1) - \mathbf{b}(t+1)), \tag{24}$$

Steps 1-3 are repeated until a predefined maximum iteration number T is reached. The proposed method is summarized in Algorithm 1.

Algorithm 1: Doubly constrained robust Capon beamformer via ADMM

Initialization: $\boldsymbol{\lambda}(0)$, $\mathbf{b}(0)$, ρ, ε, and T;
for $t = 0, \ldots, T$
  Obtain $\mathbf{a}(t+1)$ using (11)-(20);
  Determine $\mathbf{b}(t+1)$ using (23);
  Update $\{\boldsymbol{\lambda}(t+1)\}$ using (24);
end for $t = T$.
Output $\mathbf{w} = \frac{\mathbf{R}^{-1}\mathbf{a}(T)}{\mathbf{a}^H(T)\mathbf{R}^{-1}\mathbf{a}(T)}$.

B. Solution to (6)

With the introduction of auxiliary variables $y_n = \mathbf{w}^H\mathbf{x}_n$, (6) becomes

$$\begin{aligned} \min_{\mathbf{w}, \{y_n\}} \quad & \sum_{n=1}^{N} |y_n|^p \\ \text{s.t.} \quad & y_n = \mathbf{w}^H\mathbf{x}_n, \\ & 1 - \epsilon \leq |\mathbf{w}^H\mathbf{a}(\theta_m)|^2 \leq 1 + \epsilon, m = 1, \cdots, M, \end{aligned} \tag{25}$$

The augmented Lagrangian of (25) is

$$\begin{aligned} \mathcal{L}(\mathbf{w}, \{y_n, \lambda_n\}) = \sum_{n=1}^{N} \Big(& |y_n|^p + \Re\{\lambda_n^*(y_n - \mathbf{w}^H\mathbf{x}_n)\} \\ & + \frac{\rho}{2}|y_n - \mathbf{w}^H\mathbf{x}_n|^2 \Big) \\ \text{s.t. } 1 - \epsilon \leq |\mathbf{w}^H\mathbf{a}(\theta_m)|^2 &\leq 1 + \epsilon, m = 1, \cdots, M, \end{aligned} \tag{26}$$

where $\rho > 0$ is a user-defined parameter, and $\{\lambda_n\}$ are the Lagrange multipliers corresponding to the constraints $y_n = \mathbf{w}^H\mathbf{x}_n$. Then, based on the ADMM [13], we determine $(\mathbf{w}, \{y_n, \lambda_n\})$ via the following iterative steps:

$$y_n(t+1) := \arg\min_{y_n} \mathcal{L}(\mathbf{w}(t), \{y_n, \lambda_n(t)\}), \tag{27a}$$

$$\mathbf{w}(t+1) := \arg\min_{\mathbf{w}} \mathcal{L}(\mathbf{w}, \{y_n(t+1), \lambda_n(t)\}), \tag{27b}$$

$$\lambda_n(t+1) := \lambda_n(t) + y_n(t+1) - \mathbf{w}(t+1)^H\mathbf{x}_n. \tag{27c}$$

The solutions to the subproblems in (27a) and (27b) are discussed as follows:

1) Solution to (27a): By ignoring the constant term, the optimization problem (27a) can be simplified as:

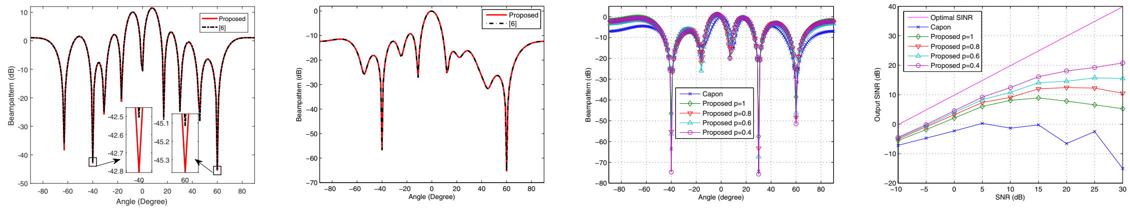
$$\min_{y_n} \quad |y_n|^p + \frac{\rho}{2}|y_n - \tilde{y}_n|^2, n = 1, \cdots, N, \tag{28}$$

where $\tilde{y}_n(t) = \mathbf{w}(t)^H\mathbf{x}_n - \frac{\lambda_n(t)}{\rho}$. Obviously, the optimal phase of y_n is equivalent to that of $\tilde{y}_n(t)$. Therefore, (28) reduces to the amplitude (real-valued) optimization problem:

$$\min_{y_n^a} \quad (y_n^a)^p + \frac{\rho}{2}(y_n^a - \tilde{y}_n^a(t))^2 \quad, n = 1, \cdots, N, \tag{29}$$

where y_n^a and $\tilde{y}_n^a(t)$ are the magnitudes of y_n and $\tilde{y}_n(t)$, respectively. Similar to [21], we compute the first-, second-, and third-order derivatives of (29), analyze the convexity or concavity property of the corresponding piecewise functions of a single nonnegative variable y_n^a, and determine the optimal y_n^a via selecting the global optimum from the local optima of the piecewise functions (or see (41)–(47) of [21] for details). Once the optimal y_n^a is obtained, the optimal $y_n(t)$ is given by:

$$y_n(t+1) = y_n^a e^{j\angle\tilde{y}_n(t)}, \text{for}, n = 1, \cdots, N, \tag{30}$$



(a) (b) (c) (d)

Fig. 1: (a) Generated beampatterns in Exp. 1; (b) Generated beampatterns in Exp. 2; (c) Generated beampatterns in Exp. 3; (d) Output SINR versus SNR in Exp. 3.

2) Solution to (27b): With the given $\{y_n(t+1), \lambda_n(t)\}$, and by ignoring irrelevant terms in (27b), we have

$$\min_{\mathbf{w}} \sum_{n=1}^{N} \left(-\Re\{\lambda_n(t)^* \mathbf{w}^H \mathbf{x}_n\} + \frac{\rho}{2} |y_n - \mathbf{w}^H \mathbf{x}_n|^2 \right)$$
$$\text{s.t. } 1-\epsilon \le |\mathbf{w}^H \mathbf{a}(\theta_m)|^2 \le 1+\epsilon, m = 1, \cdots, M. \quad (31)$$

To simplify (31), we define $\boldsymbol{\lambda}(t) = [\lambda_1(t), \cdots, \lambda_N(t)]^T$, $\mathbf{X} = [\mathbf{x}_n, \cdots, \mathbf{x}_N]$ and $\mathbf{y}(t+1) = [y_1(t+1), \cdots, y_N(t+1)]^T$, and rewrite (31) in a compact form as

$$\min_{\mathbf{w}} \ \frac{\rho}{2} \mathbf{w}^H \mathbf{X}\mathbf{X}^H \mathbf{w} - \Re\{\mathbf{w}^H \mathbf{b}\}$$
$$\text{s.t. } 1-\epsilon \le |\mathbf{w}^H \mathbf{a}(\theta_m)|^2 \le 1+\epsilon, m = 1, \cdots, M. \quad (32)$$

where $\mathbf{b} = \mathbf{X}\left(\boldsymbol{\lambda}(t) + \rho \mathbf{y}(t+1)\right)$. This optimization problem can be solved by using the technique discussed in [22] (see Eqs. (63)–(76) of [22] for details).

IV. Numerical Examples

In this section, numerical examples are presented to evaluate the performance of the proposed methods. Here we consider robust beamforming with a 10-element uniform linear array with half wavelength inter-element spacing. The first experiment focuses on the look direction error, the second pays attention to the phase perturbation case whereas the third addresses the impulsive noise environment.

A. Experiment 1: Look direction error

In the first experiment, we address the robust beamforming problem with look direction error. The actual DOA of the desired signal is $0°$ while those of two interferences are $-40°$ and $60°$. Both the signal-to-noise ratio (SNR) and interference-to-noise ratio are 10 dB. When the provided DOA of the signal is $1°$, there exists a $1°$ look direction error between the provided and actual DOAs. In this experiment, ε, ρ, and T are set as 0.845, 0.1, and 10000, respectively. We also implement the method in [6] for comparison purpose, and plot the generated beampatterns in Fig. 1(a), which show that the proposed method can generate more deeper notch nulls than [6]. The corresponding signal-to-interference-pulse-noise ratio (SINR) of the proposed method and the algorithm in [6] are 7.75 dB and 7.71 dB, respectively.

B. Experiment 2: Phase perturbations

In the second experiment, we evaluate robust beamforming problem with phase perturbations, where the phase errors are generated via the uniform distribution $[0, 0.3]$. Except $\varepsilon = 0.00038343$ we adopt the same parameters as the first experiment. Similarly, we also implement the method in [6]. We plot the generated beampatterns in Fig. 1(b), which again show more deep notch nulls for the proposed method. The resulting SINR of the proposed method and the algorithm in [6] are 7.07 dB and 7.00 dB, respectively.

C. Experiment 3: Impulsive noise environment

In the third experiment, we examine the robust beamforming problem in impulsive noise environment, where the impulsive noise is generated via the α-stable process with $\alpha = 0.8$. In the experiment we set the number of snapshots $N = 300, \epsilon = 0.001$ and $\rho = 0.01$, The desired signal direction is $0°$ while those of three interferences are $-40°, 30°$ and $60°$. The SNR is -10 dB. We implement the standard Capon beamformer for comparison. The generated beampatterns with different values of p are shown in Fig. 1(c). We see that more deep notch nulls in the proposed method at the interference directions, which implies that it has better performance for impulsive noise environment. In addition, Fig. 1(d) plots the output SINR versus SNR for $p = 1, 0.8, 0.6, 0.4$. Figs. 1(c)–(d) show that the use of proposed method with smaller p is helpful for performance improvement.

V. Conclusion

This paper proposed two new algorithms to tackle the robust Capon beamforming problem, one is for look direction error or phase perturbations; another is for impulsive noise environment. To improve the approach efficiently, we have decoupled the convex quadratic inequality constraint and the nonconvex unit modular constraint.

Acknowledgement

This work was supported in part by Natural Science Foundation of China (NSFC 61471295), Aeronautical Science Foundation of China (G20172053017), and Central University funds (G2016KY0308, G2016KY0002, 17GH030144).

References

[1] J. Li, and P. Stoica. *Robust Adaptive Beamforming.* Hoboken, NJ: Wiley, 2006.

[2] J. Capon, "High-resolution frequency wavenumber spectrum analysis," *Proceedings of the IEEE*, vol. 57, no.8, pp. 1408-1418, Aug. 1969.

[3] B. D. Van Veen, and K. M. Buckley, "Beamforming: A versatile approach to spatial filtering," *IEEE ASSP Magazine*, vol.5, no.2, pp. 4-24, 1988.

[4] A. B. Gershman, N. D. Sidiropoulos, S. Shahabazpanahi, M. Bengtsson, and B. Ottersten, "Convex optimization-based beamforming," *IEEE Signal Processing Magazine*, vol. 27, no.3, pp.62-75, May 2010.

[5] J. Li, P. Stoica, and Z. Wang, "On Robust Capon beamforming and diagonal loading," *IEEE Transactions on Signal Processing*, Vol. 51, No. 7, pp. 1702-1715, Jul. 2003.

[6] J. Li, P. Stoica, and Z. Wang, "Doubly constrained robust Capon beamforming," *IEEE Transactions on Signal Processing*, Vol. 52, No. 9, pp. 2407-2423, Sep. 2004.

[7] H. Cox, R. M. Zeskind, and M. M. Owen, "Robust adatpvie beamforming," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 35, no.10, pp. 1365-1376, Oct. 1987.

[8] A. B. Gershman, "Robust adaptive beamforming in sensor arrays," *International Journal of Electronics and Communications*, vol. 53, pp. 305-314, Dec. 1999.

[9] S. A. Vorobyov, A. B. Gershman, and Z. Q. Luo, "Robust adaptive beamforming using worst-case performance optimization: a solution to the signal mismatch problem," *IEEE Transactions Signal Processing*, vol. 51, no.2, pp.313-324, Feb. 2003.

[10] R. G. Lorenz, and S. P. Boyd,"Robust minimum variance beamforming," *IEEE Transactions Signal Processing*, vol. 53, no.5, pp.1684-1696, May 2005.

[11] P. Stoica, Z. Wang, and J. Li, "Robust Capon beamforming," *IEEE Signal Processing Letters*, vol. 10, no.6, pp. 172-175, 2003.

[12] X.Jiang, W.-J.Zeng, H.C.So, A.M.Zoubir and T.Kirubarajan, "Beamforming via nonconvex linear regression," *IEEE Transactions on Signal Processing*, vol.64, no.7, pp.1714-1728, Apr. 2016.

[13] S. P. Boyd, and L. Vandenberghe, *Convex Optimization.* Cambridge, U.K.:Cambridge Univ. Press, 2004.

[14] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, "Distributed optimization and statistical learning via the alternating direction method of multipliers," *Foundations and Trends in Machine Learning*, vol. 3, no. 1, pp. 1-122, 2010.

[15] J. Liang, H. C. So, J. Li, and A. Farina, "Unimodular sequence design based on alternating direction method of multipliers," *IEEE Transactions Signal Processing*, vol. 64, no. 20, pp. 5367-5381,2016.

[16] J. Liang, H. C. So, J. Li, A. Farina, and D. Zhou,"On optimizations with magnitude constraints on frequency or angular responses," *Signal Processing*, vol.145, pp.214-224, 2018.

[17] K. Huang, N. D. Sidiropoulos, "Consensus-ADMM for general quadratically constrained quadratic programming," *IEEE Transactions Signal Processing*, vol. 26, no. 20, pp. 5297-5310, 2016.

[18] J. Liang, G. Yu, B. Chen, and M. Zhao, "Decentralized dimensionality reduction for distributed tensor data across sensor networks," *IEEE Trans. Neural Networks and Learning Systems*, vol. 27, no.11, pp. 2174-286, 2016.

[19] W. Fan, J. Liang and J. Li, "Constant Modulus MIMO Radar Waveform Design With Minimum Peak Sidelobe Transmit Beampattern," *IEEE Transactions on Signal Processing*, vol. 66, no. 16, pp. 4207-4222, Aug. 2018.

[20] X. Jiang, W. Zeng, A. Yasotharan, H. C. So and T. Kirubarajan, "Minimum dispersion beamforming for non-gaussian signals," *IEEE Transactions on Signal Processing*, vol. 62, no. 7, pp. 1879-1893, Apr. 2014.

[21] J. Liang, X. Zhang, H. C. So and D. Zhou, "Sparse array beampattern synthesis via alternating direction method of multipliers," *IEEE Transactions Antennas Propagation*, vol. 66, no. 5, pp. 2333-2345, May 2018.

[22] J. Liang, H. C. So, J. Li, A. Farina, D. Zhou, "On optimizations with magnitude constraints on frequency or angular responses," *Signal Processing*, Vol. 145, pp. 214-224, 2018,

[23] J. Liang, M. Zhang, X. Zeng and G. Yu, "Distributed dictionary learning for sparse representation in sensor networks," in *IEEE Transactions on Image Processing*, vol. 23, no. 6, pp. 2528-2541, Jun. 2014.