PHASE-ONLY ROBUST MINIMUM DISPERSION BEAMFORMING

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ABSTRACT

A phase-only robust minimum dispersion (PO-RMD) beamformer is devised for non-Gaussian signals. The proposed PO-RMD employs a constant-modulus constraint on the weights, which is equivalent to simply phase shifting at each antenna. It adopts the minimum dispersion criterion to utilize the non-Gaussianity of the signals while employing the worst-case constraint to achieve the robustness against model uncertainty. A gradient projection algorithmic framework is developed to solve the resulting nonconvex optimization problem. In order to find a feasible point in the intersection of the constant-modulus and robustness constraint sets, an alternating projection algorithm is devised. More importantly, the closed-form expressions of the projection onto the two sets are derived, respectively. Simulation results demonstrate the effectiveness, accuracy and robustness of the PO-RMD.

***Index Terms*—** Phase-only beamforming, model mismatch, non-Gaussian signals, gradient projection, alternating projection.

1. INTRODUCTION

As a versatile spatial filtering technique, the beamforming technique relies on determining the complex-valued weight vector to extract the desired signal and suppress the interferences from different directions [1]. That means separate power amplifier and phase shifter have to be used to adjust the amplitude and phase of each array element. However, large-scale systems become increasingly common in both military and civilian applications [2], [3]. As the number of the array elements increases, it is expected that both the cost and energy consumption can still be maintained at a low level. One way to save the cost of beamforming is to use a single power amplifier for all the antennas. That is, the beampattern can be formed by phase shifting only [4]–[6]. This gives rise to renewed attention to phase-only beamforming (POBF) [7]–[9]. The phase-only linearly constrained minimum variance (LCMV) beamforming is proposed in [7], which converts the original nonconvex quadratic optimization problem into a convex optimization problem by the semidefinite relaxation (SDR) technique. However, a major drawback of SDR-based methods is that it is not well-suited for large-scale problems because of dimension lifting of the optimization variable [15]. The POBF can also be formulated as a unit-modulus least squares (ULS) problem and solved using projected gradient descent [8]. A unit-modulus quadratic programs for POBF in wireless sensor networks has been investigated and the alternating direction method of multipliers (ADMM)-based solution is given in [9].

The limits of the existing POBF techniques are two fold. One is that the issue of model mismatch has not been taken into account, which cannot be avoided in practical applications. The other is that the minimum variance (MV)-based methods are only optimal for Gaussian signals and noise. However, many real world signals and noise are non-Gaussian [10]. In this paper, we develop a phase-only robust minimum dispersion (PO-RMD) beamforming technique. The MD criterion [10], [11] with $p \geq 1$ is adopted such that the non-Gaussianity can be utilized to improve the performance of the beamformer. Two constraint sets are incorporated. One is the robustness constraint, i.e., worst-case performance optimization [12], [13]. The other is the constant modulus constraint. Different from the unit-modulus property widely utilized by the existing POBF, we force the weights of the beamformer to have the same modulus $b \in \mathbb{R}^+$. By introducing the parameter b, the feasibility of the constrained optimization problem can be guaranteed. A gradient projection (GP) algorithmic framework is developed to solve the resulting nonconvex optimization problem, which has a very low computational cost.

2. SIGNAL MODEL

Consider narrowband independent signals impinging on an M-element array. The complex baseband received signal vector $\boldsymbol{x}(n) = [x_1(n), \cdots, x_M(n)]^T$ can be written as

$$\boldsymbol{x}(n) = s(n)\boldsymbol{a} + \boldsymbol{r}(n) \tag{1}$$

where the superscript $(\cdot)^T$ represents transpose, n is the discrete time index, $s(n)$ is the SOI, $\boldsymbol{a}$ is the steering vector of

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the SOI, and $\boldsymbol{r}(n)$ represents the sum of interference signals and noise. The form of steering vector depends on the array shape. For uniform linear array (ULA), the steering vector is expressed as $\boldsymbol{a} = \left[1, \mathrm{e}^{-\mathrm{j}2\pi \frac{d}{\eta}\sin\theta}, \cdots, \mathrm{e}^{-\mathrm{j}2\pi(M-1)\frac{d}{\eta}\sin\theta}\right]^T$, where θ is the AOA, η is the wavelength and d is the inter-space of elements of the array. The output of the beamformer is given by

$$y(n) = \boldsymbol{w}^H \boldsymbol{x}(n) = \boldsymbol{w}^H \boldsymbol{a} s(n) + \boldsymbol{w}^H \boldsymbol{r}(n) \tag{2}$$

where the superscript $(\cdot)^H$ denotes Hermitian transpose, $\boldsymbol{w} \in \mathbb{C}^M$ is the complex vector of weights. The aim of beamforming is to extract the SOI $s(n)$ while suppressing the interference and noise by adjusting the amplitude and phase of weight of each array element.

The most well-known data-dependent beamformer, i.e., MVDR [14], uses the MV criterion with a single linear constraint to keep the response distortionless. A major drawback of MVDR beamformer is that it is sensitive to steering vector mismatch. The robust minimum dispersion (RMD) beamformer solves the following optimization problem:

$$\begin{aligned} &\min_{\boldsymbol{w}} \ \mathrm{E}\left[|y(n)|^p\right] = \mathrm{E}\left[|\boldsymbol{w}^H \boldsymbol{x}(n)|^p\right] \\ &\text{s.t.} \ \left|(\boldsymbol{a}+\boldsymbol{e})^H \boldsymbol{w})\right| \geq 1, \ \text{for all } \boldsymbol{e} \in \mathcal{E} \end{aligned} \tag{3}$$

where $\mathrm{E}[\cdot]$ is the expectation operation and $|\cdot|$ is the modulus of a complex number, $p \geq 1$. In (3), the actual steering vector is defined as $\boldsymbol{a}+\boldsymbol{e}$ with $\boldsymbol{e}$ being the steering vector error and the uncertainty region $\mathcal{E}$ is modeled as a ball defined as

$$\mathcal{E} = \{\boldsymbol{e} \mid \|\boldsymbol{e}\| \leq \varepsilon\} \tag{4}$$

with ε being the radius of the ball. The quantity $\mathrm{E}\left[|y(n)|^p\right]$ is called the dispersion of $y(n)$. The dispersion, which is a generalization of variance ($p = 2$), implicitly exploits the higher- or lower-order statistics. When $p = 2$, (3) reduces to the robust minimum variance beamformer (RMVB) [12], [13]. It is pointed out that the MV criterion is only statistically optimal under Gaussian assumption. For the non-Gaussian signals that are frequently encountered in radar, sonar, navigation, and wireless communications systems, the higher- or lower-order statistics contain useful information and can be utilized to improve the performance [10], [11].

3. PHASE-ONLY ROBUST MINIMUM DISPERSION BEAMFORMING

Note that the beamforming weight vector $\boldsymbol{w}$ is complex. It means that not only the phase but also the modulus of each antenna has to be adjusted to extract the SOI, which is costly in large-scale systems. In this section, we develop the PO-RMD beamformer, which only relies on per-antenna phase shifting to enhance the desired signal and mitigate the interference and noise.

3.1. Formulation of PO-RMD

The MD-based phase-only beamformer can be generally formulated as

$$\begin{aligned} &\min_{\boldsymbol{w}} \ \mathrm{E}\left[|\boldsymbol{w}^H \boldsymbol{x}(n)|^p\right] \\ &\text{s.t.} \ \boldsymbol{w} \in \mathcal{B}, \ \boldsymbol{w} \in \mathcal{C} \end{aligned} \tag{5}$$

where

$$\mathcal{B} = \{\boldsymbol{w} \,|\, |\boldsymbol{w}_1| = \cdots = |\boldsymbol{w}_M| = b\} \tag{6}$$

is a constant modulus constraint set with $b \in \mathbb{R}^+$. It can be equivalently written as $\mathcal{B} = \left\{\boldsymbol{w} \,\middle|\, \boldsymbol{w}_m = b\mathrm{e}^{\mathrm{j}\phi_m}, m = 1, \cdots, M\right\}$, where ϕ_m is the phase of the mth antenna. Therefore, it is clear that only the phase of each antenna is utilized to fulfill beamforming. The other constraint set $\mathcal{C}$ in (5) is taken into account to recover the SOI. Different sets $\mathcal{C}$ lead to different beamformers. In order to handle arbitrary model mismatch, we adopt the worst-case constraint as shown in (3) which can be converted into the convex second-order cone (SOC) constraint given by [12]

$$\mathcal{C} = \mathcal{K}, \ \mathcal{K} = \left\{\boldsymbol{w} | \mathrm{Re}(\boldsymbol{a}^H \boldsymbol{w}) \geq \varepsilon \|\boldsymbol{w}\| + 1\right\}. \tag{7}$$

By using the sample mean instead of expectation, (5) can be converted to

$$\begin{aligned} &\min_{\boldsymbol{w}} \ f_p(\boldsymbol{w}) = \|\boldsymbol{X}^H \boldsymbol{w}\|_p^p \\ &\text{s.t.} \ \boldsymbol{w} \in \mathcal{S}, \ \mathcal{S} = \mathcal{B} \cap \mathcal{C} \end{aligned} \tag{8}$$

where $\mathcal{S}$ is the intersection of $\mathcal{B}$ and $\mathcal{C}$, $\boldsymbol{X} = [\boldsymbol{x}(1), \cdots, \boldsymbol{x}(N)]$ is the observed data matrix and $\|\boldsymbol{y}\|_p$ is the ℓ_p-norm of $\boldsymbol{y}$ defined as $\|\boldsymbol{y}\|_p = \left(\sum_{n=1}^N |y(n)|^p\right)^{1/p}$ with $|y(n)| = \sqrt{\mathrm{Re}^2(y(n)) + \mathrm{Im}^2(y(n))}$ being the modulus of $y(n)$. We refer to the solution of (8) as PO-RMD beamformer. For $p = 2$, the PO-RMD reduces to phase-only robust minimum variance beamformer (PO-RMVB), which is optimal for Gaussian signals and noise. For sub-Gaussian signals, larger values of p are suggested [10]. For super-Gaussian case, $p < 2$ is better. The definition of ℓ_p-norm is still of some interest for $0 < p < 1$. However, the resulting function does not define a norm because it violates the triangle inequality [10]. In this work, we consider the case of $p \geq 1$.

3.2. Gradient Projection Method for PO-RMD

Before discussing the algorithm for solving (8), we first need to investigate the feasibility of (8), which is equivalent to determining whether the intersection of the two sets is nonempty. The following proposition provides a guidance for choosing the value of b, which can be proved by Cauchy-Schwartz inequality.

Proposition 1: The intersection of the two sets $\mathcal{B}$ and $\mathcal{C}$ is nonempty, i.e., $\mathcal{S} = \mathcal{B} \cap \mathcal{C} \neq \emptyset$, by choosing

$$b = \frac{1}{\rho M - \varepsilon\sqrt{M}} \tag{9}$$

where $\varepsilon/\sqrt{M} < \rho \leq 1$.

Now we aim at solving (8). Although the robustness constraint (7) constitutes a convex set, the constant modulus constraint (6) is nonconvex. Therefore, the problem of (8) cannot be solved directly by the existing software packages for convex optimization. We develop a GP algorithmic framework, which has a low complexity.

The gradient of the objective $f_p(\boldsymbol{w})$ with respect to (w.r.t.) complex vector $\boldsymbol{w}$ can be calculated as

$$\nabla f_p(\boldsymbol{w}) = \frac{p}{2}\boldsymbol{X}\boldsymbol{D}(\boldsymbol{w})\boldsymbol{X}^H\boldsymbol{w} \tag{10}$$

where $\boldsymbol{D}(\boldsymbol{w}) = \mathrm{diag}\left\{|y(1)|^{p-2}, \cdots, |y(N)|^{p-2}\right\}$.

The GP algorithm generates a sequence $\{\boldsymbol{w}^k\} \in \mathbb{C}^M$ ($k = 1, 2, \cdots$) through the following iterative procedure:

Initialization: Take $\boldsymbol{w}^0 \in \mathcal{S}$.

Iterative step: If the convergence condition is satisfied, then stop. Otherwise, let

$$\boldsymbol{w}^{k+1} = \Pi_{\mathcal{S}}(\boldsymbol{w}^k - \mu_k \nabla f_p(\boldsymbol{w}^k)) \tag{11}$$

where $\mu_k > 0$ is the positive step size in the kth iteration and $\Pi_{\mathcal{S}}(\cdot)$ is the projection operator onto $\mathcal{S}$.

Note that the one-dimensional (1-D) function $g_p(\mu) \triangleq f_p\left(\Pi_{\mathcal{S}}\left(\boldsymbol{w}^k - \mu\nabla f_p(\boldsymbol{w}^k)\right)\right)$ is not convex w.r.t. μ because of the projection operator $\Pi_{\mathcal{S}}(\cdot)$. Therefore, it is difficult to find the global minimum of $g_p(\mu)$ and perform exact line search to determine the optimal step size μ^k. Alternatively, we may choose μ_k to sufficiently decrease the objective function by an inexact line search, e.g., backtracking line search [16].

Now the remaining issue is solving the projection $\Pi_{\mathcal{S}}(\cdot)$ in (11). Note that the set $\mathcal{S}$ denotes the intersection of $\mathcal{B}$ and $\mathcal{C}$. Hence, given $\tilde{\boldsymbol{w}}^k = \boldsymbol{w}^k - \mu_k\nabla f_p(\boldsymbol{w}^k)$, $\Pi_{\mathcal{S}}(\tilde{\boldsymbol{w}}^k)$ can be formulated as finding a common point of the two sets that is closest to $\tilde{\boldsymbol{w}}^k$, i.e.,

$$\min_{\boldsymbol{u}^k} \|\boldsymbol{u}^k - \tilde{\boldsymbol{w}}^k\|^2 \qquad \text{s.t. } \boldsymbol{u}^k \in \mathcal{B}\cap\mathcal{C} \tag{12}$$

which can be solved by alternating projection (AP) as shown in Algorithm 1.

Algorithm 1 Alternating projection

Input: $\tilde{\boldsymbol{w}}^k = \boldsymbol{w}^k - \mu_k\nabla f_p(\boldsymbol{w}^k)$
  Initialize: $\boldsymbol{z} = \tilde{\boldsymbol{w}}^k$
  for $i = 1, 2\cdots$ **do**
    Compute $\boldsymbol{v} = \Pi_{\mathcal{C}}(\boldsymbol{z})$; Compute $\boldsymbol{z} = \Pi_{\mathcal{B}}(\boldsymbol{v})$;
    Stop if termination condition satisfied.
  end for
Output: $\boldsymbol{w}^{k+1} = \boldsymbol{z}$

Then we demonstrate that the projections $\Pi_{\mathcal{C}}(\cdot)$ and $\Pi_{\mathcal{B}}(\cdot)$ have closed-form expressions and can be computed with very low complexity. There is a trick for efficiently computing $\Pi_{\mathcal{C}}(\cdot)$. Since the constant modulus constraint (6) is incorporated into the PO-RMD, the SOC constraint (7) can be simplified as the following half-space constraint:

$$\mathcal{C} = \mathcal{K}_s,\ \mathcal{K}_s = \left\{\boldsymbol{w}|\mathrm{Re}(\boldsymbol{a}^H\boldsymbol{w}) \geq \varepsilon b\sqrt{M}+1\right\} \tag{13}$$

by using $\|\boldsymbol{w}\| = b\sqrt{M}$, which significantly reduces the computational complexity of $\Pi_{\mathcal{C}}(\cdot)$. It is clear that the projection of $\boldsymbol{z}$ onto $\mathcal{C}$ is itself if $\boldsymbol{z} \in \mathcal{C}$. For any $\boldsymbol{z} \notin \mathcal{C}$, the projection onto the half-space set must lie on its boundary. Then $\Pi_{\mathcal{C}}(\boldsymbol{z})$ can be equivalently expressed as the following equality constrained problem:

$$\begin{aligned} &\min_{\boldsymbol{v}}\ \frac{1}{2}\|\boldsymbol{v}-\boldsymbol{z}\|^2 \\ &\text{s.t. } \mathrm{Re}(\boldsymbol{a}^H\boldsymbol{v}) = \varepsilon b\sqrt{M}+1 \end{aligned} \tag{14}$$

which can be converted into the real-valued problem given by

$$\begin{aligned} &\min_{\bar{\boldsymbol{v}}}\ \frac{1}{2}\|\bar{\boldsymbol{v}}-\bar{\boldsymbol{z}}\|^2 \\ &\text{s.t. } \bar{\boldsymbol{a}}^T\bar{\boldsymbol{v}} = \varepsilon b\sqrt{M}+1 \end{aligned} \tag{15}$$

with $\bar{\boldsymbol{v}} = \begin{bmatrix} \boldsymbol{v}_R \\ \boldsymbol{v}_I \end{bmatrix}$, $\bar{\boldsymbol{z}} = \begin{bmatrix} \boldsymbol{z}_R \\ \boldsymbol{z}_I \end{bmatrix}$, $\bar{\boldsymbol{a}} = \begin{bmatrix} \boldsymbol{a}_R \\ \boldsymbol{a}_I \end{bmatrix} \in \mathbb{R}^{2M}$. The closed-form solution of (15) is given by

$$\bar{\boldsymbol{v}} = \bar{\boldsymbol{z}} - \frac{\bar{\boldsymbol{a}}^T\bar{\boldsymbol{z}} - r}{M}\bar{\boldsymbol{a}} \tag{16}$$

where $r \triangleq \varepsilon b\sqrt{M}+1$ and $\|\bar{\boldsymbol{a}}\|^2 = M$ is used. Therefore, considering both cases of $\boldsymbol{z} \in \mathcal{C}$ and $\boldsymbol{z} \notin \mathcal{C}$, the real-valued expanded form of the projection $\boldsymbol{v} = \Pi_{\mathcal{C}}(\boldsymbol{z})$ can be expressed compactly as

$$\bar{\boldsymbol{v}} = \bar{\boldsymbol{z}} - \frac{\min(\bar{\boldsymbol{a}}^T\bar{\boldsymbol{z}} - r, 0)}{M}\bar{\boldsymbol{a}}. \tag{17}$$

On the other hand, it is not difficult to calculate the projection onto $\mathcal{B}$ as

$$\Pi_{\mathcal{B}}(\boldsymbol{v}) = b\left[\mathrm{e}^{\mathrm{j}\angle\boldsymbol{v}_1}, \cdots, \mathrm{e}^{\mathrm{j}\angle\boldsymbol{v}_M}\right]^T \tag{18}$$

where $\angle\boldsymbol{v}_i$ is the phase of $\boldsymbol{v}_i$. It can be seen from (17) and (18) that the two projections can be calculated with a complexity of $\mathcal{O}(M)$. Hence, the dominant cost of the GP algorithm is the calculation the gradient of (10) and evaluation the objective function, which has a complexity of $\mathcal{O}(MN)$ in each iteration.

The convergence of the alternating projection for finding a common point of two sets was previously established for convex sets only [17]. Recently, the convergence of alternating projection for nonconvex sets that satisfies a transversality condition has been investigated [18], [19]. Exploiting the fact that the constant modulus constraint set of (6) satisfies the transversality condition [19], [20], we can establish the convergence of the alternating projection for solving (12), as stated in the following proposition.

Proposition 2: If the initial point is close enough to the intersection of $\mathcal{B}$ and $\mathcal{C}$, then the APA in Algorithm 1 locally converges to a point in $\mathcal{B} \cap \mathcal{C}$ at a linear rate.

4. SIMULATION RESULTS

A ULA of $M = 10$ omnidirectional antennas with a half-wavelength spacing is considered. Three zero-mean sub-Gaussian signals impinge on the array. Unless stated otherwise, the AOA of the SOI is $\theta = 43°$ and the AOAs of the two interferences are $\theta_1 = 30°$ and $\theta_2 = 75°$. The desired signal and interferences adopt quadrature phase shift keying (QPSK) modulation, which corresponds to sub-Gaussian signal, while the noise is Gaussian distributed. The number of samples is $N = 100$. The signal-to-noise ratio (SNR) is defined as $\mathrm{SNR} = 10\log_{10}(\sigma_s^2/\sigma_v^2)$, where σ_s^2 and σ_v^2 are the variances of the SOI and additive noise, respectively. We have $\mathrm{SNR} = 20$ dB. The two interferences are stronger than the SOI with variances being $\sigma_1^2 = \sigma_2^2 = 10\sigma_s^2$. We adopt the output signal-to-interferences-plus noise ratio (SINR) as the performance measure of beamforming, which is defined as

$$\mathrm{SINR} = \frac{\mathrm{E}\left\{\left|s(n)\boldsymbol{w}^H\boldsymbol{a}\right|^2\right\}}{\mathrm{E}\left\{\left|\boldsymbol{w}^H(\boldsymbol{i}(n)+\boldsymbol{v}(n))\right|^2\right\}} = \frac{\sigma_s^2\left|\boldsymbol{w}^H\boldsymbol{a}\right|^2}{\boldsymbol{w}^H\boldsymbol{R}_{i+n}\boldsymbol{w}}. \quad (19)$$

where $\boldsymbol{R}_{i+n}$ is the interferences-plus-noise covariance matrix. The output SINRs of the beamformers, namely, subspace [21], RMVB [12], [13], RMD [11] and the proposed PO-RMD are compared. Different values of p are taken into account. The upper bound of the SINR is the maximum eigenvalue of the matrix $\sigma_s^2\boldsymbol{R}_{i+n}^{-1}\boldsymbol{a}\boldsymbol{a}^H$, which is also provided for comparison. The minimum description length (MDL) principle [22] is adopted to estimate the dimension of the signal-plus-interference subspace of subspace beamformer. When plotting the SINR curves, 200 Monte Carlo trials are performed. The initial value $\boldsymbol{w}_0$ of GP can be obtained as $\boldsymbol{w}^0 = \Pi_{\mathcal{S}}(\boldsymbol{q})$, where $\boldsymbol{q} \in \mathbb{C}^M$ is randomly generalized. The steering vector perturbation defined as $\mathrm{Ptb} = 10\log_{10}\frac{2M\sigma_e^2}{\|\boldsymbol{a}\|^2} = 10\log_{10}(2\sigma_e^2)$ is fixed to $\mathrm{Ptb} = -10$ dB with σ_e^2 being the variance of $\boldsymbol{e}$. We take $\varepsilon = 5.6\sigma_e$ [11]. The value of b is chosen as (9) with $\rho = 0.8 + 0.2\varepsilon/\sqrt{M}$.

We first investigate the convergence behavior of GP for PO-RMD with $p = 2, 4, 8, 20$. Fig. 1 plots the output SINR versus number of iterations with noisy data. We can see from Fig. 1 that PO-RMD with $p > 2$ leads to an improved performance compared with that of $p = 2$ for sub-Gaussian signals. The PO-RMD achieves a satisfactory performance after serval iterations (less than ten). Fig. 2 plots the output SINR versus SNR. It is worthy to note that the performance of PO-RMD is comparable to that of RMD with complex-valued

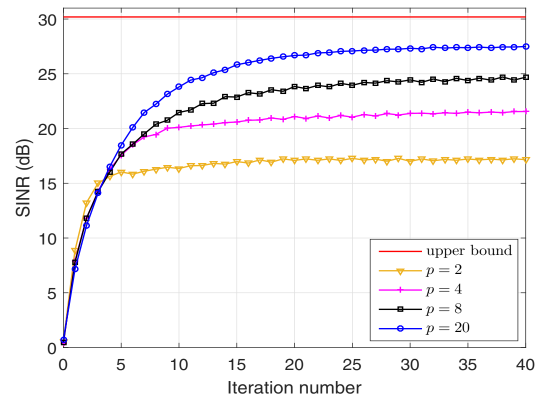

Fig. 1. Output SINR versus iteration number.

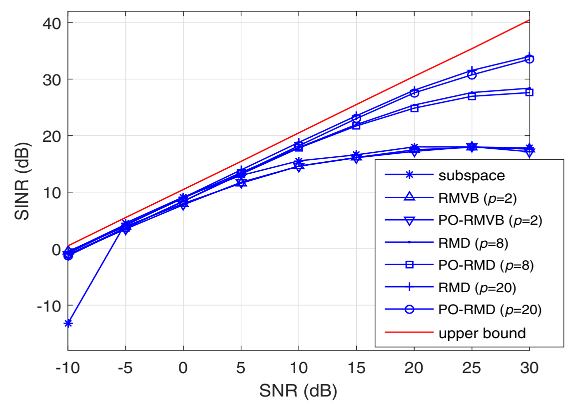

Fig. 2. Output SINR versus SNR.

weights. The performance of PO-RMD is about 10dB higher than that of RMVB when $\mathrm{SNR} > 20$ dB.

5. CONCLUSION

In this paper, a PO-RMD beamformer, whose weights have the same modulus for all the array elements, is proposed. It minimizes the ℓ_p-norm of the array output subject to two constraints, i.e., constant modulus and worst-case constraints. A generic GP framework for efficiently solving the proposed PO-RMD is developed. Note that the closed-form expressions of the projection onto the two constraint sets are derived, respectively, and can be computed efficiently with a very low cost. The PO-RMD substantially improves the SINR performance compared with the MV-based robust beamformers. More importantly, although it only relies on phase shifting, its performance is comparable to that of RMD with complex-valued weights.

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