A NEW QUADRATIC MATRIX INEQUALITY APPROACH TO ROBUST ADAPTIVE BEAMFORMING FOR GENERAL-RANK SIGNAL MODEL

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ABSTRACT

The worst-case robust adaptive beamforming problem for general-rank signal model is considered. This is a nonconvex problem, and an approximate version of it (by introducing a matrix decomposition on the presumed covariance matrix of the desired signal) has been studied in the literature. Herein the original robust adaptive beamforming problem is tackled. Resorting to the strong duality of a linear conic program, the robust beamforming problem is reformulated into a quadratic matrix inequality (QMI) problem. There is no general method for solving a QMI problem in the literature. Herein, employing a linear matrix inequality (LMI) relaxation technique, the QMI problem is turned into a convex semidefinite programming problem. Due to the fact that there often is a positive gap between the QMI problem and its LMI relaxation, a deterministic approximate algorithm is proposed to solve the robust adaptive beamforming in the QMI form. Last but not the least, a sufficient optimality condition for the existence of an optimal solution for the QMI problem is derived. To validate our theoretical results, simulation examples are presented, which also demonstrate the improved performance of the new robust beamformer in terms of the output signal-to-interference-plus-noise ratio.

Index Terms— Robust adaptive beamforming, general-rank signal model, quadratic matrix inequality problem, linear matrix inequality relaxation, deterministic approximate algorithm

1. INTRODUCTION

Robust adaptive beamforming techniques provide a powerful approach to significantly improve the array output signal-to-interference-plus-noise ratio (SINR) and other performance metrics such as for example mainlobe width and sidelobe levels. Here the robustness typically means the ability of a method to perform well under any imperfect or incomplete knowledge about the source, propagation and sensor array geometry. In particular, when it is difficult to obtain the knowledge of the desired signal covariance matrix, a mismatch between the presumed and actual source covariance matrices causes dramatic performance degradation, and the robust adaptive beamforming techniques are very efficient against the mismatch [1, 2].

Many robust adaptive beamforming approaches have been proposed for the scenario of a rank-one signal model (see [3] and reference therein). However, it is of practical interest to consider a general-rank signal model, as the signal source often can be incoherently scattered, then robust adaptive beamforming for general-rank signal models becomes necessary. In [4], an efficient robust adaptive beamforming technique for general-rank signal model has been proposed, and a beamformer in closed-form has been computed in terms of a principal eigenvector of the product between inverse of the worst-case of the sample data covariance matrix and the worst-case of the presumed covariance for the signal. The authors of [5] have presented a new method to the robust adaptive beamforming with general-rank signal model, taking into account a positive semidefinite (PSD) constraint over the actual signal covariance matrix (the presumed covariance plus errors). The resultant robust adaptive beamforming problem has been formulated by introducing a matrix decomposition (e.g. spectral or Cholesky type) of the presumed signal covariance and putting the error term into both of the matrices obtained from the decomposition (rather than the worst-case of the actual signal covariance matrix). It turns out that the robust problem is a nonconvex quadratic program, and an iterative algorithm using semidefinite program (SDP) has been proposed to find a suboptimal solution. In [6], two beamformers have been derived in closed-form for the robust adaptive beamforming problem established in [5], and this gave the low complexity robust beamformers. Under the assumption that the interference is well separated from the signal, the authors of [7] have proposed a method using SDP relaxation and bisection search, to solve the robust beamforming problem formulated in [5]. In [8, 9], the aforementioned beamforming problem is termed as a difference-of-convex (DC) optimization problem, and a polynomial time DC (POTDC) algorithm has been proposed; the authors show that the POTDC converges to a local optimal solution, and under the condition that the error norm bound is sufficiently small, the local solution is indeed a globally optimal solution. Rather employing the SDP relaxation technique (applied in [5, 6, 7, 8, 9]), we in [10] have proposed an approximate algorithm for the robust beamforming problem using second-order cone programs, which makes the computation complexity of the algorithm significantly lower.

In this paper, we study the original robust adaptive beamforming problem, i.e., without performing a matrix decomposition over the presumed signal covariance matrix (unlike what has been done in [5, 6, 7, 8, 9, 10]). Resorting to the strong duality theorem of a linear conic program (e.g., see [11]), the robust adaptive beamforming problem is reformulated into a nonconvex quadratic matrix inequality (QMI) problem. Unfortunately, there is no general method to solve a QMI problem in the open literature. In order to tackle the problem, we employ a linear matrix inequality (LMI) relaxation and turn the QMI problem into a convex LMI problem. Based on an optimal solution of the LMI problem, we propose a deterministic approximate algorithm to find a solution for the robust adaptive beamforming problem (i.e. the QMI problem). Last but not the least, we present a sufficient condition for the existence of an optimal solution for the robust adaptive beamforming problem.

2. SIGNAL MODEL AND PROBLEM FORMULATION

The output signal of a narrowband receive beamformer can be written as

$$x(t) = \boldsymbol{w}^H \boldsymbol{y}(t)$$

where $\boldsymbol{w}$ is the $N \times 1$ vector of beamformer complex weight coefficients, $\boldsymbol{y}(t)$ is the $N \times 1$ complex snapshot vector of array observations, and N is the number of antenna elements of the array. The observation vector is given by

$$\boldsymbol{y}(t) = \boldsymbol{s}(t) + \boldsymbol{i}(t) + \boldsymbol{n}(t) \tag{1}$$

where $\boldsymbol{s}(t)$, $\boldsymbol{i}(t)$, and $\boldsymbol{n}(t)$ are the statistically independent components of the desired signal, interference, and noise, respectively. The output SINR of the beamformer is given by

$$\text{SINR} = \frac{\boldsymbol{w}^H \boldsymbol{R}_s \boldsymbol{w}}{\boldsymbol{w}^H \boldsymbol{R}_{i+n} \boldsymbol{w}} \tag{2}$$

where the desired signal covariance matrix is $\boldsymbol{R}_s \triangleq \mathsf{E}[\boldsymbol{s}(t)\boldsymbol{s}^H(t)]$ and the interference and noise covariance matrix is $\boldsymbol{R}_{i+n} \triangleq \mathsf{E}[(\boldsymbol{i}(t) + \boldsymbol{n}(t))(\boldsymbol{i}(t) + \boldsymbol{n}(t))^H]$. Note that the SINR value (2) is unaltered when the norm of beamvector $\boldsymbol{w}$ changes. Matrix $\boldsymbol{R}_s$ herein can be of rank one or higher, i.e., Rank $(\boldsymbol{R}_s) \in \{1, \ldots, N\}$. Both rank-one (corresponding to the case of the point source) and higher-rank $\boldsymbol{R}_s$ are common in many practical situations occurring in wireless communications, radar and sonar (see [1, 3, 4, 5]).

Suppose that $\boldsymbol{R}_s$ and $\boldsymbol{R}_{i+n}$ are known perfectly in some ways, then an optimal beamforming problem of maximizing the SINR can be cast into:

$$\underset{\boldsymbol{w} \neq \boldsymbol{0}}{\text{maximize}} \quad \frac{\boldsymbol{w}^H \boldsymbol{R}_s \boldsymbol{w}}{\boldsymbol{w}^H \boldsymbol{R}_{i+n} \boldsymbol{w}} \quad . \tag{3}$$

It is evident that the optimal value for (3) is $\lambda_{\max}(\boldsymbol{R}_{i+n}^{-1/2} \boldsymbol{R}_s \boldsymbol{R}_{i+n}^{-1/2})$ (assuming that $\boldsymbol{R}_{i+n}$ is of full rank), and the optimal solution is a principal eigenvector of $\boldsymbol{R}_{i+n}^{-1/2} \boldsymbol{R}_s \boldsymbol{R}_{i+n}^{-1/2}$ (an eigenvector corresponding to the largest eigenvalue).

In practical applications, however, the interference-plus-noise covariance matrix $\boldsymbol{R}_{i+n}$ is not available. Thus, the sample covariance matrix for $\boldsymbol{R} = \mathsf{E}[\boldsymbol{y}(t)\boldsymbol{y}^H(t)]$:

$$\hat{\boldsymbol{R}} = \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{y}(t)\boldsymbol{y}^H(t) \tag{4}$$

is used to replace $\boldsymbol{R}_{i+n}$ in the optimal beamforming design (3), as a compromise. In (4), T stands for the number of training snapshots. On the other hand, often the signal covariance matrix $\boldsymbol{R}_s$ is only known imperfectly; in other words, there is always a certain mismatch between the presumed signal covariance matrix $\hat{\boldsymbol{R}}_s$ and the actual signal covariance matrix $\boldsymbol{R}_s$. The beamvector obtained by maximizing the SINR defined by $\hat{\boldsymbol{R}}_s$ and $\hat{\boldsymbol{R}}$ (without taking into account the error terms), however, gives rise to bad performance of the array. Therefore, in order to improve the performance, robust adaptive beamforming has been considered, and there are a number of papers working on this subject (for example, see [2, 3] for an overview and references therein) in the last two decades.

Herein, let us consider the robust adaptive beamforming problem with general-rank $\boldsymbol{R}_s$, aiming to propose a new efficient method to find a robust beamformer with improved performance. Toward the end, the following robust adaptive beamforming problem maximizing the worst-case SINR (cf. [12]) is studied:

$$\underset{\boldsymbol{w} \neq \boldsymbol{0}}{\text{maximize}} \quad \underset{\boldsymbol{\Delta}_1 \in \mathcal{B}_1, \boldsymbol{\Delta}_2 \in \mathcal{B}_2}{\text{minimize}} \quad \frac{\boldsymbol{w}^H (\hat{\boldsymbol{R}}_s + \boldsymbol{\Delta}_2) \boldsymbol{w}}{\boldsymbol{w}^H (\hat{\boldsymbol{R}} + \boldsymbol{\Delta}_1) \boldsymbol{w}} \tag{5}$$

where the uncertainty sets $\mathcal{B}_1$ and $\mathcal{B}_2$ are given by

$$\mathcal{B}_1 = \{\boldsymbol{\Delta}_1 \in \mathbb{C}^{N \times N} \mid \|\boldsymbol{\Delta}_1\| \leq \gamma, \ \hat{\boldsymbol{R}} + \boldsymbol{\Delta}_1 \succeq \boldsymbol{0}\}, \tag{6}$$

and

$$\mathcal{B}_2 = \{\boldsymbol{\Delta}_2 \in \mathbb{C}^{N \times N} \mid \|\boldsymbol{\Delta}_2\| \leq \epsilon, \ \hat{\boldsymbol{R}}_s + \boldsymbol{\Delta}_2 \succeq \boldsymbol{0}\}, \tag{7}$$

respectively. In (6) and (7), the matrix norms are a Frobenius norm (which is effective throughout the paper).

Since $\boldsymbol{\Delta}_1$ and $\boldsymbol{\Delta}_2$ are separable, (5) can be recast into:

$$\underset{\boldsymbol{w} \neq \boldsymbol{0}}{\text{maximize}} \quad \frac{\underset{\boldsymbol{\Delta}_2 \in \mathcal{B}_2}{\min} \ \boldsymbol{w}^H (\hat{\boldsymbol{R}}_s + \boldsymbol{\Delta}_2) \boldsymbol{w}}{\underset{\boldsymbol{\Delta}_1 \in \mathcal{B}_1}{\max} \ \boldsymbol{w}^H (\hat{\boldsymbol{R}} + \boldsymbol{\Delta}_1) \boldsymbol{w}}. \tag{8}$$

It is straightforward to show that the denominator of the objective function of (8) equals $\boldsymbol{w}^H (\hat{\boldsymbol{R}} + \gamma \boldsymbol{I}) \boldsymbol{w}$, where γ represents the diagonal loading factor [4]. Thus, (8) can be reexpressed as:

$$\underset{\boldsymbol{w} \neq \boldsymbol{0}}{\text{maximize}} \quad \frac{\underset{\boldsymbol{\Delta}_2 \in \mathcal{B}_2}{\min} \ \boldsymbol{w}^H (\hat{\boldsymbol{R}}_s + \boldsymbol{\Delta}_2) \boldsymbol{w}}{\boldsymbol{w}^H (\hat{\boldsymbol{R}} + \gamma \boldsymbol{I}) \boldsymbol{w}}, \tag{9}$$

which is equivalent to the following problem:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \underset{\boldsymbol{\Delta}_2 \in \mathcal{B}_2}{\min} \ \boldsymbol{w}^H (\hat{\boldsymbol{R}}_s + \boldsymbol{\Delta}_2) \boldsymbol{w} \\ \text{subject to} & \boldsymbol{w}^H (\hat{\boldsymbol{R}} + \gamma \boldsymbol{I}) \boldsymbol{w} = 1. \end{array} \tag{10}$$

Problems (9) and (10) are equivalent in the sense that they share the same optimal value and if $\boldsymbol{w}^\star$ solves (10), then it is optimal for (9) too. Therefore, we only need to focus on how to solve the latter maximin problem (10) in order to solve robust adaptive beamforming problem (5).

In existing works (e.g. [5, 6, 7, 8, 9, 10]), incorporating the PSD constraint $\hat{\boldsymbol{R}}_s + \boldsymbol{\Delta}_2 \succeq \boldsymbol{0}$, the objective $\boldsymbol{w}^H (\hat{\boldsymbol{R}}_s + \boldsymbol{\Delta}_2) \boldsymbol{w}$ of (10) is replaced with

$$\boldsymbol{w}^H (\boldsymbol{Q} + \boldsymbol{\Delta}_3)^H (\boldsymbol{Q} + \boldsymbol{\Delta}_3) \boldsymbol{w}, \tag{11}$$

where $\hat{\boldsymbol{R}}_s = \boldsymbol{Q}^H \boldsymbol{Q}$, $\boldsymbol{Q} \in \mathbb{C}^{M \times N}$, $N \geq M =$ Rank $(\boldsymbol{R}_s)$, and the norm of distortion $\boldsymbol{\Delta}_3$ is simply bounded by η: $\mathcal{B}_3 = \{\boldsymbol{\Delta}_3 \in \mathbb{C}^{M \times N} \mid \|\boldsymbol{\Delta}_3\| \leq \eta\}$. With the new objective, problem (10) is reformulated into (see e.g. [10]):

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \|\boldsymbol{Q}\boldsymbol{w}\| - \eta \|\boldsymbol{w}\| \\ \text{subject to} & \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} + \gamma \|\boldsymbol{w}\|^2 \leq 1, \end{array} \tag{12}$$

which further is tantamount to the following quadratic program:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{minimize}} & \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} + \gamma \|\boldsymbol{w}\|^2 \\ \text{subject to} & \|\boldsymbol{Q}\boldsymbol{w}\| - \eta \|\boldsymbol{w}\| \geq 1. \end{array} \tag{13}$$

Problem (13) has been studied well in the literature through convex approximation. However, herein we deal with the original robust beamforming problem (10) via QMI approaches, without utilizing the objective (11).

3. A QUADRATIC MATRIX INEQUALITY APPROACH FOR THE ROBUST ADAPTIVE BEAMFORMING PROBLEM

In this section, we propose a QMI method to solve robust adaptive beamforming problem (5) for general-rank signal model. In addition, sufficient conditions for a globally optimal solution for the robust beamforming problem are derived.

3.1. A QMI approach to solve (5)

Observe that

$$\begin{array}{ll} \min_{\boldsymbol{\Delta}} & \operatorname{tr}(\boldsymbol{A}\boldsymbol{\Delta}) \\ \text{s.t.} & \boldsymbol{R}+\boldsymbol{\Delta} \succeq \boldsymbol{0} \\ & \|\boldsymbol{\Delta}\| \leq \epsilon, \end{array}$$

has the dual

$$\begin{array}{ll} \max_{\boldsymbol{X}} & -\epsilon\|\boldsymbol{X}-\boldsymbol{A}\| - \operatorname{tr}(\boldsymbol{R}\boldsymbol{X}) \\ \text{s.t.} & \boldsymbol{X} \succeq \boldsymbol{0}. \end{array}$$

See [13] for the proof on how to get the dual problem. Clearly, they are strictly feasible (as long as $\boldsymbol{R}$ is PSD, which is a mild assumption) and the strong duality holds between them.

Therefore, it follows that (10) can be reformulated into:

$$\begin{array}{ll} \underset{\boldsymbol{w},\boldsymbol{X}}{\text{maximize}} & \operatorname{tr}(\hat{\boldsymbol{R}}_s(\boldsymbol{w}\boldsymbol{w}^H-\boldsymbol{X})) - \epsilon\|\boldsymbol{w}\boldsymbol{w}^H-\boldsymbol{X}\| \\ \text{subject to} & \boldsymbol{w}^H(\hat{\boldsymbol{R}}+\gamma\boldsymbol{I})\boldsymbol{w}=1 \\ & \boldsymbol{X}\succeq\boldsymbol{0}, \end{array} \tag{14}$$

and setting $\boldsymbol{Y}=\boldsymbol{w}\boldsymbol{w}^H-\boldsymbol{X}$, one obtains the following problem:

$$\begin{array}{ll} \underset{\boldsymbol{w},\boldsymbol{Y}}{\text{maximize}} & \operatorname{tr}(\hat{\boldsymbol{R}}_s\boldsymbol{Y}) - \epsilon\|\boldsymbol{Y}\| \\ \text{subject to} & \boldsymbol{w}^H(\hat{\boldsymbol{R}}+\gamma\boldsymbol{I})\boldsymbol{w}=1 \\ & \boldsymbol{w}\boldsymbol{w}^H-\boldsymbol{Y}\succeq\boldsymbol{0}. \end{array} \tag{15}$$

Note that QMI problem[1] (15) amounts to the original robust adaptive beamforming problem (5). The second constraint in (15) is a typically nonconvex QMI constraint. In the literature, there is no general method to solve a QMI problem. Herein in order to tackle the problem, let us first take a look into the LMI relaxation problem:

$$\begin{array}{ll} \underset{\boldsymbol{W},\boldsymbol{Y}}{\text{maximize}} & \operatorname{tr}(\hat{\boldsymbol{R}}_s\boldsymbol{Y}) - \epsilon\|\boldsymbol{Y}\| \\ \text{subject to} & \operatorname{tr}((\hat{\boldsymbol{R}}+\gamma\boldsymbol{I})\boldsymbol{W})=1 \\ & \boldsymbol{W}-\boldsymbol{Y}\succeq\boldsymbol{0},\ \boldsymbol{W}\succeq\boldsymbol{0}. \end{array} \tag{16}$$

Unfortunately, there often is a nonzero gap between the QMI problem and its LMI relaxation. In that case, we wish to either establish a sufficient global optimality condition for the QMI problem, or find a suboptimal/approximate solution for the QMI problem within polynomial time complexity. Let us first propose an approximate algorithm for (15) and then present the sufficient global optimality conditions.

Toward the end, (16) is transformed equivalently into:

$$\begin{array}{ll} \underset{\boldsymbol{W},\boldsymbol{Y},t}{\text{maximize}} & \operatorname{tr}(\hat{\boldsymbol{R}}_s\boldsymbol{Y}) - \epsilon t \\ \text{subject to} & \operatorname{tr}((\hat{\boldsymbol{R}}+\gamma\boldsymbol{I})\boldsymbol{W})=1 \\ & \boldsymbol{W}-\boldsymbol{Y}\succeq\boldsymbol{0} \\ & \|\boldsymbol{Y}\|\leq t,\ \boldsymbol{W}\succeq\boldsymbol{0}, \end{array} \tag{17}$$

the dual of which is claimed in the following lemma:

Lemma 3.1 *The dual of* (16) *is the following linear conic program:*

$$\begin{array}{ll} \underset{z,\boldsymbol{Z}}{\text{minimize}} & z \\ \text{subject to} & \epsilon \geq \|\boldsymbol{Z}-\hat{\boldsymbol{R}}_s\|, \\ & z(\hat{\boldsymbol{R}}+\gamma\boldsymbol{I})-\boldsymbol{Z}\succeq\boldsymbol{0}, \\ & \boldsymbol{Z}\succeq\boldsymbol{0}. \end{array} \tag{18}$$

[1] If the terms of optimization variables in the matrix inequality are quadratic or linear, then we call it QMI problem, which is in line with LMI problem (all terms of optimization variables in the matrix inequality must be linear).

Due to the limited space, we omit the proof.

It is evident that problems (17) and (18) are strictly feasible Thus, the strong duality holds between them. It is easily verified (see [11, Theorem 1.4.1] or [14, Equation (5.48)]) that the complementary conditions for primal problem (17) and dual problem (18) are

$$\epsilon t + \operatorname{tr}(\boldsymbol{Y}(\boldsymbol{Z}-\hat{\boldsymbol{R}}_s)) = 0, \tag{19}$$

$$\operatorname{tr}((z(\hat{\boldsymbol{R}}+\gamma\boldsymbol{I})-\boldsymbol{Z})\boldsymbol{W}) = 0, \tag{20}$$

$$\operatorname{tr}((\boldsymbol{W}-\boldsymbol{Y})\boldsymbol{Z}) = 0. \tag{21}$$

Observing that $\operatorname{tr}(\boldsymbol{W}(\hat{\boldsymbol{R}}+\gamma\boldsymbol{I}))=1$, we further have a compact form of the complementary conditions:

$$z=\operatorname{tr}(\boldsymbol{W}\boldsymbol{Z})=\operatorname{tr}(\boldsymbol{Y}\boldsymbol{Z})=\operatorname{tr}(\boldsymbol{Y}\hat{\boldsymbol{R}}_s)-\epsilon t. \tag{22}$$

Suppose that $\{\boldsymbol{W}^\star,\boldsymbol{Y}^\star;z^\star,\boldsymbol{Z}^\star\}$ is an optimal primal-dual pair for (16) and (18), through a primal-dual interior point method. If the rank of $\boldsymbol{W}^\star$ is one (i.e. $\boldsymbol{W}^\star=\boldsymbol{w}\boldsymbol{w}^H$), then $\boldsymbol{w}$, together with $\boldsymbol{Y}^\star$, is optimal for (15). Suppose that the rank of $\boldsymbol{W}^\star$ is greater than one. We attempt to look for a rank-one approximate solution for (16). For that purpose, we resort to the rank-one matrix decomposition lemma in [15].

Lemma 3.2 (Theorem 2.1 in [15]) *Suppose that $\boldsymbol{X}$ is an $N\times N$ complex Hermitian PSD matrix of rank R, and $\boldsymbol{A}$, $\boldsymbol{B}$ are two $N\times N$ given Hermitian matrices. Then, there is a rank-one decomposition $\boldsymbol{X}=\sum_{r=1}^R \boldsymbol{x}_r\boldsymbol{x}_r^H$ such that*

$$\boldsymbol{x}_r^H\boldsymbol{A}\boldsymbol{x}_r=\frac{\operatorname{tr}(\boldsymbol{A}\boldsymbol{X})}{R} \quad \textit{and} \quad \boldsymbol{x}_r^H\boldsymbol{B}\boldsymbol{x}_r=\frac{\operatorname{tr}(\boldsymbol{B}\boldsymbol{X})}{R},\ r=1,\ldots,R$$

(*synthetically denoted as* $\mathcal{D}(\boldsymbol{X},\boldsymbol{A},\boldsymbol{B})$).

Accordingly, we apply the lemma and obtain $\boldsymbol{W}^\star=\sum_{i=1}^R \boldsymbol{w}_i\boldsymbol{w}_i^H$ (where R is the rank of $\boldsymbol{W}^\star$) such that

$$\operatorname{tr}((\hat{\boldsymbol{R}}+\gamma\boldsymbol{I})(R\boldsymbol{w}_i\boldsymbol{w}_i^H))=\operatorname{tr}((\hat{\boldsymbol{R}}+\gamma\boldsymbol{I})\boldsymbol{W}^\star)=1, \tag{23}$$

and

$$\operatorname{tr}((R\boldsymbol{w}_i\boldsymbol{w}_i^H)\boldsymbol{Z}^\star)=\operatorname{tr}(\boldsymbol{W}^\star\boldsymbol{Z}^\star)=z^\star,\ i=1,\ldots,R. \tag{24}$$

Here (24) means that each $R\boldsymbol{w}_i\boldsymbol{w}_i^H$ (together with $\boldsymbol{Y}^\star$, $\|\boldsymbol{Y}^\star\|$, $z^\star$, and $\boldsymbol{Z}^\star$) fulfills the optimality conditions stated in (22), while (23) implies that $\sqrt{R}\boldsymbol{w}_i$ complies only with the first constraint in (15). If some $\sqrt{R}\boldsymbol{w}_{i_0}$ (together with $\boldsymbol{Y}^\star$) satisfies the second constraint in (15), then one can conclude that $\sqrt{R}\boldsymbol{w}_{i_0}$ is optimal.

In order to obtain the objective function value of (15) at $\sqrt{R}\boldsymbol{w}_i$, we substitute $R\boldsymbol{w}_i\boldsymbol{w}_i^H$ into (16), getting:

$$\begin{array}{ll} \underset{\boldsymbol{Y}}{\text{maximize}} & \operatorname{tr}(\hat{\boldsymbol{R}}_s\boldsymbol{Y}) - \epsilon\|\boldsymbol{Y}\| \\ \text{subject to} & R\boldsymbol{w}_i\boldsymbol{w}_i^H-\boldsymbol{Y}\succeq\boldsymbol{0} \end{array} \tag{25}$$

(recalling the feasible condition of $R\boldsymbol{w}_i\boldsymbol{w}_i^H$ in (23)). If the optimal value of (25) is equal to that of (16) at some $R\boldsymbol{w}_{i_0}\boldsymbol{w}_{i_0}^H$, then $\sqrt{R}\boldsymbol{w}_{i_0}$ is optimal for (15).

Suppose that $\bar{\boldsymbol{Y}}_i$ is an optimal solution for (25), $i=1,\ldots,R$. Select this $\boldsymbol{Y}_1:=\arg\max\{\operatorname{tr}(\hat{\boldsymbol{R}}_s\bar{\boldsymbol{Y}}_i)-\epsilon\|\bar{\boldsymbol{Y}}_i\| \mid i=1,\ldots,R\}$. Thus, $(R\boldsymbol{w}_1\boldsymbol{w}_1^H,\boldsymbol{Y}_1)$ is returned as a suboptimal solution for (16), since it is feasible, and consequently $\sqrt{R}\boldsymbol{w}_1$ is an approximate solution for (10) (i.e. problem (5)). Algorithm 1 summarizes the procedure of finding a solution for original problem (5).

Algorithm 1 Solution procedure for robust adaptive beamforming problem (5)

Input: $\hat{\boldsymbol{R}}$, $\hat{\boldsymbol{R}}_s$, γ, ϵ;
Output: A solution $\boldsymbol{w}$ for (5);

1: solve the SDP (16), and find optimal solution $(\boldsymbol{W}^\star, \boldsymbol{Y}^\star)$ together with the dual optimal one $(\boldsymbol{Z}^\star, z^\star)$ for (18);
2: if $\boldsymbol{W}^\star$ is of rank one (i.e. $\boldsymbol{W}^\star = \boldsymbol{w}\boldsymbol{w}^H$), then output $\boldsymbol{w}$ as the optimal solution and terminate;
3: implement the rank-one decomposition $\mathcal{D}(\boldsymbol{W}^\star, \hat{\boldsymbol{R}} + \gamma \boldsymbol{I}, \boldsymbol{Z}^\star)$, getting $\boldsymbol{W}^\star = \sum_{i=1}^{R} \boldsymbol{w}_i \boldsymbol{w}_i^H$;
4: solve SDPs (25) and obtain $\{\boldsymbol{Y}_i^\star\}$;
5: compute $\boldsymbol{Y}_{i_0} := \arg\max\{\mathrm{tr}\,(\hat{\boldsymbol{R}}_s \boldsymbol{Y}_i^\star) - \epsilon \|\boldsymbol{Y}_i^\star\| \mid i = 1, \ldots, R\}$;
6: output $\sqrt{R}\boldsymbol{w}_{i_0}$.

The computational cost of the algorithm for QMI problem (15) is dominated by solving $(R+1)$ SDPs, which are the relaxation problems for QMI problems. In [5, 7, 9], quadratic problem (13) is approximated by solving a sequence of SDPs, which are the relaxation problems for QCQPs. In [4, 6], closed-form solutions are discussed, where in [4] the matrix in the solution includes an indefinite covariance while in [6] the matrix contains coarse approximation.

3.2. Sufficient Optimality Conditions for QMI Problem (15)

A sufficient condition is built herein for the existence of rank-one solutions for (16), i.e. a sufficient optimality condition for the QMI problem (15) is obtained in the following theorem.

Theorem 3.3 *Suppose that* $(\boldsymbol{W}^\star, \boldsymbol{Y}^\star)$ *is optimal for* (16). *If*

$$\mathrm{tr}\,(\boldsymbol{W}^\star - \boldsymbol{Y}^\star) \geq \mathrm{tr}\,\boldsymbol{W}^\star \sqrt{N-1}\left(1 + \frac{\lambda_{\max}(\hat{\boldsymbol{R}}_s)}{\epsilon}\right),$$

then a rank-one solution for LMI relaxation problem (16) *can be constructed within polynomial time complexity.*

Due to the limited space, we omit the proof.

4. SIMULATION RESULTS

We consider the simulated scenario with a uniform linear array of $N = 10$ omnidirectional sensors spaced half a wavelength apart. The additive noise variance in each sensor is set to 0 dB. Suppose that an interference source with the interference-to-noise ratio (INR) 30 dB impinges on the sensor array. Both the desired signal and the interference are locally incoherently scattered sources. The signal of interest and the interference have Gaussian and uniform angular power densities with the central angles 30° and 10°, respectively, and the angular spreads 4° and 10°, respectively. The presumed desired signal is assumed to have Gaussian angular power density with central angle and angular spread 34° and 6°, respectively. The sample data covariance matrix is estimated with $T = 50$ snapshots. The diagonal loading parameter $\gamma = 0.1\|\hat{\boldsymbol{R}}\|$ is set and the norm bound $\epsilon = 0.3\|\hat{\boldsymbol{R}}_s\|$ is chosen herein. The norm bound $\eta = 0.9\sqrt{\mathrm{tr}\,(\hat{\boldsymbol{R}}_s)}$ is selected for the methods in [4, 5, 6, 9]. All results are averaged over 100 simulation runs.

Example 1: This example examines the array output SINR versus SNR among the new proposed beamformer herein and the beamformers proposed in [4, 5, 6, 9], termed "New Beamformer", "SGLW Beamfomer", "CG Beamformer", "XMW Beamformer", and "KV Beamformer", respectively. Fig. 1 displays the output SINRs versus SNR. As can be seen, the new QMI beamforming method leads to significant performance improvement comparing with the ones proposed in [4, 5, 6, 9].

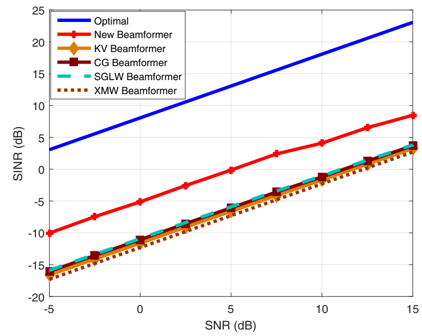

Fig. 1. The beamformer output SINR versus SNR, with INR=30 dB and $T = 50$

Example 2: It is known that when the angle spread of the desired source varies, the rank of the actual covariance $\boldsymbol{R}_s$ of the desired source changes, which can affect the performance of the beamformer. This example tests how much the output SINR can fluctuate if the angle spread is set to $1^\circ, 2^\circ, 5^\circ, 9^\circ, 14^\circ$. Suppose that SNR=10 dB. All other simulation settings are the same as those in Example 1. Fig. 2 plots the output SINRs versus the rank of the actual correlation matrix $\boldsymbol{R}_s$. As observed, our robust beamformer outperforms the previous ones.

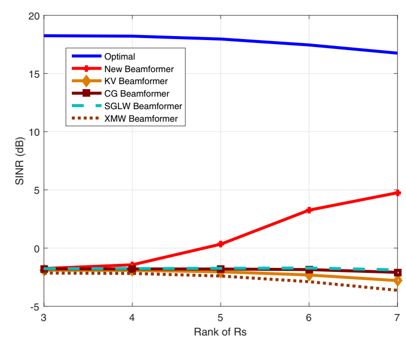

Fig. 2. The array output SINR versus the rank of $\boldsymbol{R}_s$, with INR=30 dB, SNR=10 dB, and $T = 50$

5. CONCLUSION

We have considered the robust adaptive beamforming problem for general-rank signal model. Unlike solving the problem by introducing a matrix decomposition on the presumed matrix of the signal of interest, we have studied the original robust adaptive beamforming problem. Resorting to the strong duality theorem for a linear conic program, we have reformulated the beamforming problem into a nonconvex QMI problem, and relaxed it into a convex LMI problem. Due to the nonzero gap between the QMI and LMI problems, we have proposed a deterministic approximate algorithm with polynomial-time computational complexity, in order to solve the robust beamforming problem. Besides, a sufficient optimality condition is derived for the robust adaptive beamforming problem. The improved performance of the proposed robust beamformer has been demonstrated by simulations in terms of the array output SINR.

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