MVDR ROBUST ADAPTIVE BEAMFORMING DESIGN WITH DIRECTION OF ARRIVAL AND GENERALIZED SIMILARITY CONSTRAINTS

*Yongwei Huang** *Mingkang Zhou** *Sergiy A. Vorobyov*†

*Guangdong University of Technology
Sch. of Informat. Engn.
University Town, Guangzhou, China
ywhuang@gdut.edu.cn, 2111603045@mail2.gdut.edu.cn

†Aalto University
Dept. of Signal Process. & Acoust.
Konemiehentie, Finland
svor@ieee.org

ABSTRACT

The MVDR robust adaptive beamforming design problem based on estimation of the signal-of-interest (SOI) steering vector is considered. In this case, the optimal beamformer is obtained by computing the sample matrix inverse and an optimal estimate of the SOI steering vector. In order to find the optimal steering vector estimate of the SOI, a new beamformer output power maximization problem is formulated subject to a double-sided norm perturbation constraint, a generalized similarity constraint, and a direction-of-arrival (DOA) constraint that guarantees that the DOA of the SOI is away from the DOA region of all linear combinations of the interference steering vectors. It turns out that the power maximization problem is a nonconvex quadratically constrained quadratic program (QCQP) with two homogenous and one inhomogeneous constraints. In general, a globally optimal solution for the QCQP is not guaranteed; however, we herein derive sufficient optimality conditions to ensure the existence of an optimal solution, and develop an efficient algorithm to find the solution. To validate our results, simulation examples are presented, and they demonstrate the improved performance of the new robust adaptive beamformer in terms of the output SINR.

Index Terms— Robust adaptive beamforming, steering vector estimation, double-sided norm constraint, DOA constraint, ellipsoidal constraint

1. INTRODUCTION

In array processing, robust adaptive beamforming has particularly been recognized as a fundamental problem and drawn much research interest, due to its wide applications to radar, sonar, communications, microphone array speech/audio processing, etc. [1]. In the last two decades, a number of robust adaptive beamforming techniques have been established, and substantial progress has been made, partially supported by significant developments in convex and robust optimization [2, 3, 4].

In [5], doubly constrained robust Capon beamformer has been proposed as a natural extension of the standard Capon beamformer. The constraints there include a norm constraint and a similarity constraint over the signal-of-interest (SOI) steering vector. The new beamformer can be computed at a manageable cost. The work has been extended by the authors of [6], changing the similarity constraint to an ellipsoidal constraint. The formulated problem is a nonconvex quadratically constrained quadratic program (QCQP) with one homogenous and one inhomogeneous constraints. An efficient algorithm therein for the QCQP has been developed. In [7, 8], a beamformer output power maximization problem is considered, with a norm constraint on steering vector and a direction-of-arrival (DOA) constraint preventing the DOA of the desired signal from converging to the DOA set of all linear combinations of interference steering vectors. It is highlighted that therein the only prior information about the angular sector of the desired signal and antenna array geometry is required. The resulting power maximization problem is a form of QCQP problem, with the objective and the two constraint functions all in homogenous quadratic form. It has been known that the QCQP problem can be solved up to global optimality using a semidefinite program (SDP) relaxation technique, followed by a purification procedure for getting a rank-one solution [9, 10, 11, 12] if necessary. However, thanks to the structured matrices in the QCQP problem, the procedure of determining a rank-one solution can be simplified significantly, as stated in [7, Theorem 1].

In this paper, we propose a new minimum variance distortionless response (MVDR) robust adaptive beamforming design with improved performance by introducing more practical constraints to power maximization problem in [7]. First, we propose a new DOA (quadratic) constraint which is different from the one in [7], aiming to ameliorate the array output performance, in terms of the output signal-to-interference-plus-noise ratio (SINR) as well as the output power. Second, we extend the steering vector norm equality constraint to a double-sided constraint, allowing a certain range of the norm error perturbations. The new constraint is to account for the steering vector gain perturbations caused, e.g., by the sensor amplitude errors, phase errors, the sensor position errors, etc. [5]. Third, we consider a generalized similarity constraint which includes the ellipsoidal constraint as a special case. The formulated problem is a nonconvex QCQP with a double-sided constraint, a homogenous and an inhomogeneous constraints. It is known that a globally optimal solution for such problem is not secured in general. Thus, we derive sufficient optimality conditions for the problem and design efficient algorithms to solve it. Our simulation results show the new beamformer outperforms (demonstrating higher output SINR) the state-of-the-art beamforming in this class proposed in [7].

2. SIGNAL MODEL AND PROBLEM FORMULATION

A receive narrowband beamformer is applied to an output of a linear array of N antenna elements. Then the output signal of the beamformer at the time instant k can be written as

$$\boldsymbol{y}(k) = \boldsymbol{w}^H \boldsymbol{x}(k), \tag{1}$$

where $\boldsymbol{w}$ is the $N \times 1$ vector of complex weight coefficients, i.e., the beamvector, and $\boldsymbol{x}(k)$ is the complex vector of the antenna array measurements. The array observation vector in (1) is given by $\boldsymbol{x}(k) = \boldsymbol{s}(k) + \boldsymbol{i}(k) + \boldsymbol{n}(k)$, where $\boldsymbol{s}(k)$, $\boldsymbol{i}(k)$, and $\boldsymbol{n}(k)$ are statistically independent vectors corresponding to the SOI, interference, and sensor noise, respectively. The SOI can be written under the point source assumption as $\boldsymbol{s}(k) = s(k)\boldsymbol{a}$, where $s(k)$ is the signal waveform and $\boldsymbol{a}$ is the steering vector.

The optimal weight vector $\boldsymbol{w}^\star$ can be found from the optimal solution of the following SINR maximization problem

$$\underset{\boldsymbol{w}}{\text{maximize}}\ \text{SINR} = \underset{\boldsymbol{w}}{\text{maximize}}\ \frac{\sigma_s^2|\boldsymbol{w}^H\boldsymbol{a}|^2}{\boldsymbol{w}^H\boldsymbol{R}_{i+n}\boldsymbol{w}} \tag{2}$$

where σ_s^2 is the SOI power and $\boldsymbol{R}_{i+n} = \mathsf{E}[(\boldsymbol{i}(k)+\boldsymbol{n}(k))(\boldsymbol{i}(k)+\boldsymbol{n}(k))^H]$ is the interference-plus-noise covariance matrix. Since the exact covariance matrix $\boldsymbol{R}_{i+n}$ is unknown in practice, the following sample data covariance matrix computed based on T available snapshots $\hat{\boldsymbol{R}} = \frac{1}{T}\sum_{k=1}^{T}\boldsymbol{x}(k)\boldsymbol{x}^H(k)$ often is employed instead of $\boldsymbol{R}_{i+n}$ in the SINR maximization problem (2). It is evident that the SINR maximization problem is tantamount to the following optimization problem

$$\underset{\boldsymbol{w}}{\text{minimize}}\ \boldsymbol{w}^H\hat{\boldsymbol{R}}\boldsymbol{w}\ \ \text{subject to}\ |\boldsymbol{w}^H\boldsymbol{a}| = 1 \tag{3}$$

with the optimal solution

$$\boldsymbol{w}^\star = \frac{1}{\boldsymbol{a}^H\hat{\boldsymbol{R}}^{-1}\boldsymbol{a}}\hat{\boldsymbol{R}}^{-1}\boldsymbol{a}, \tag{4}$$

referred to as MVDR sampling matrix invertion (SMI) beamformer [1] or Capon beamformer [2]. Let us also note here that the corresponding array output power $\mathsf{E}[|\boldsymbol{y}(k)|^2]$ is given by

$$\mathsf{E}[|\boldsymbol{y}(k)|^2] = \mathsf{E}[|\boldsymbol{w}^{\star H}\boldsymbol{x}(k)|^2] \approx \boldsymbol{w}^{\star H}\hat{\boldsymbol{R}}\boldsymbol{w}^\star = \frac{1}{\boldsymbol{a}^H\hat{\boldsymbol{R}}^{-1}\boldsymbol{a}}. \tag{5}$$

In practice, the desired signal steering vector $\boldsymbol{a}$ is usually known imprecisely, while only a presumed steering vector $\hat{\boldsymbol{a}}$ can be estimated based on the knowledge of antenna array geometry, parameters of the SOI, and also some additional assumptions about propagation media and antenna array calibration. As a result, in many practical scenarios, the performance of the beamformer (4) degrades dramatically because of the mismatch between the actual steering vector $\boldsymbol{a}$ and the presumed steering vector $\hat{\boldsymbol{a}}$, as well as the inaccurate estimate $\hat{\boldsymbol{R}}$. To mitigate the degradation, a number of robust adaptive beamforming techniques based on optimal estimation of the desired steering vector have been proposed. Among them, the MVDR robust adaptive beamformer adopts the beamvector (4) with $\boldsymbol{a}$ therein replaced by an estimate $\hat{\boldsymbol{a}}$ that is optimized via a certain method, while assuming that $\hat{\boldsymbol{R}}$ is a sufficiently good estimate of $\boldsymbol{R}$ (see e.g. [2, 4, 5, 6, 7, 8, 13, 14, 15]).

Particularly, in [7], the optimal steering vector $\hat{\boldsymbol{a}}$ is obtained by maximizing the beamformer output power (5) subject to a DOA constraint needed to separate the DOA of the SOI from the directions given by linear combinations of the interference steering vectors, as well as a norm constraint on the steering vector. It turns out remarkably that the prior information required there includes only the approximate knowledge of the antenna array geometry and the angular sector where the desired source lies. In this paper, we extend the work in [7] by considering a new DOA constraint, a double-sided norm constraint, and additionally a generalized similarity constraint, aiming to improve the beamformer performance.

Mathematically, using (5) as an objective, we consider the following output power maximization problem for finding $\hat{\boldsymbol{a}}$:

$$\begin{aligned} \underset{\boldsymbol{a}}{\text{minimize}}\quad & \boldsymbol{a}^H\hat{\boldsymbol{R}}^{-1}\boldsymbol{a} \\ \text{subject to}\quad & \boldsymbol{a}^H\boldsymbol{C}\boldsymbol{a} \geq \Delta_1 \\ & N(1-\eta_1) \leq \|\boldsymbol{a}\|^2 \leq N(1+\eta_2) \\ & \|\boldsymbol{Q}^H(\boldsymbol{a}-\boldsymbol{a}_0)\|^2 \leq \epsilon, \end{aligned} \tag{6}$$

where in the first quadratic constraint $\boldsymbol{C} = \int_\Theta \boldsymbol{d}(\theta)\boldsymbol{d}^H(\theta)d\theta$ with $\boldsymbol{d}(\theta)$ being the steering vector associated with direction θ that has the structure defined by the antenna array geometry, and with the angular sector $\Theta = [\theta_{\min}, \theta_{\max}]$ being the direction set of SOI. In (6), the parameter Δ_1 is obtained by $\Delta_1 = \min_{\theta\in\Theta}\boldsymbol{d}^H(\theta)\boldsymbol{C}\boldsymbol{d}(\theta)$. Moreover, $\boldsymbol{a}_0$ is the presumed steering vector of SOI and $\boldsymbol{Q}$ together with ϵ and $\boldsymbol{a}_0$ defines a convex set.

In the first constraint of problem (6), Δ_1 is a boundary line to distinguish approximately whether or not the direction of $\boldsymbol{a}$ is in the SOI angular sector Θ. Specifically, if

$$\boldsymbol{a}^H\boldsymbol{C}\boldsymbol{a} \geq \Delta_1, \tag{7}$$

then the direction of $\boldsymbol{a}$ is treated as being inside Θ, which means that $\boldsymbol{a}$ never converges to any steering vector of a linear combination of the interferers, with the steering vector's DOA inside the complement $\tilde{\Theta}$ of Θ. We call (7) a DOA constraint. In contrast, another quadratic constraint has been proposed in [7], and it takes the form

$$\boldsymbol{a}^H\tilde{\boldsymbol{C}}\boldsymbol{a} \leq \Delta_0, \tag{8}$$

where $\tilde{\boldsymbol{C}} = \int_{\tilde{\Theta}} \boldsymbol{d}(\theta)\boldsymbol{d}^H(\theta)d\theta$ and $\Delta_0 = \max_{\theta\in\Theta}\boldsymbol{d}^H(\theta)\tilde{\boldsymbol{C}}\boldsymbol{d}(\theta)$. Like Δ_1, Δ_0 is another benchmark line. In other words, if $\boldsymbol{a}$ complies with (8), then $\boldsymbol{a}$ is counted as a possible steering vector of SOI. In order to illustrate the effectiveness of Δ_1 and Δ_0, we draw in Fig. 1 two subfigures for the desired sector $\Theta = [0°, 10°]$: one for $\boldsymbol{d}^H(\theta)\boldsymbol{C}\boldsymbol{d}(\theta)$ with Δ_1 and the other for $\boldsymbol{d}^H(\theta)\tilde{\boldsymbol{C}}\boldsymbol{d}(\theta)$ with Δ_0.

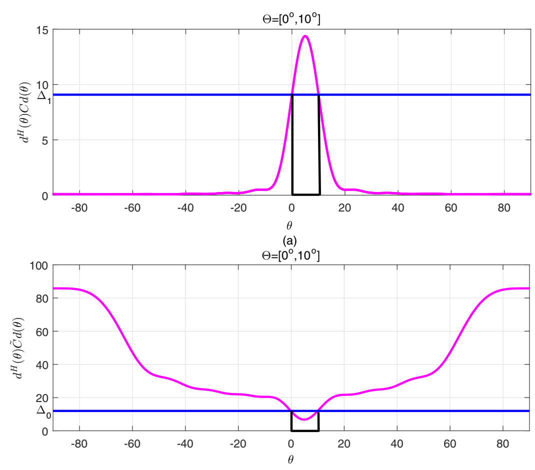

Fig. 1. Two benchmark lines Δ_1 and Δ_0 with the angular sector $\Theta = [0°, 10°]$; (a) for $\theta \in \Theta$, $\boldsymbol{d}^H(\theta)\boldsymbol{C}\boldsymbol{d}(\theta) \geq \Delta_1$ (b) for $\theta \in \Theta$, $\boldsymbol{d}^H(\theta)\tilde{\boldsymbol{C}}\boldsymbol{d}(\theta) \leq \Delta_0$

Note that the definition of the matrix $\tilde{\boldsymbol{C}}$ ($\boldsymbol{C}$) requires only the knowledge about the antenna array geometry and the angular sector, and Δ_0 (Δ_1) can be compuated efficiently, and no other prior knowledge is required.

The second constraint of (6) is the double-sided constraint that allows a certain range of the norm error perturbations, and accounts for the steering vector gain perturbations caused, e.g., by the sensor amplitude errors, phase errors, the sensor position errors, etc. (cf. [5, pp. 2408 and 2414]). The third constraint is the generalized similarity constraint. If $\boldsymbol{Q}$ is of full row rank, then it is an ellipsoidal constraint; in particular, when $\boldsymbol{Q}$ is the identity matrix, it is the traditional similarity constraint. The constraint ensures that the optimal estimate, in terms of the norm of difference, is sufficiently close to a given steering vector.

Problem (6) is a typical QCQP with a double-sided constraint, homogeneous, and inhomogeneous constraints. It is known that in general, a globally optimal solution for it cannot be secured, and deriving sufficient optimality conditions is necessary. Herein, we focus on how to get a globally optimal solution for (6) under some sufficient optimality conditions.

3. SOLVING THE OPTIMAL STEERING VECTOR ESTIMATION PROBLEM

The SDP relaxation problem for the optimal steering vector estimation problem (6) is cast as

$$\begin{array}{ll} \underset{\boldsymbol{Y}}{\text{minimize}} & \text{tr}\,(\boldsymbol{A}_0\boldsymbol{Y}) \\ \text{subject to} & \text{tr}\,(\boldsymbol{A}_1\boldsymbol{Y}) \geq \Delta_1 \\ & N(1-\eta_1) \leq \text{tr}\,(\boldsymbol{A}_2\boldsymbol{Y}) \leq N(1+\eta_2) \\ & \text{tr}\,(\boldsymbol{A}_3\boldsymbol{Y}) \leq \epsilon \\ & \text{tr}\,(\boldsymbol{A}_4\boldsymbol{Y}) = 1 \\ & \boldsymbol{Y} \succeq \boldsymbol{0}, \end{array} \tag{9}$$

with

$$\boldsymbol{A}_0 = \begin{bmatrix} \hat{\boldsymbol{R}}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix},\ \boldsymbol{A}_1 = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix},\ \boldsymbol{A}_2 = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix}, \tag{10}$$

and

$$\boldsymbol{A}_3 = \begin{bmatrix} \boldsymbol{Q}\boldsymbol{Q}^H & -\boldsymbol{Q}\boldsymbol{Q}^H\boldsymbol{a}_0 \\ -\boldsymbol{a}_0^H\boldsymbol{Q}\boldsymbol{Q}^H & \boldsymbol{a}_0^H\boldsymbol{Q}\boldsymbol{Q}^H\boldsymbol{a}_0 \end{bmatrix},\ \boldsymbol{A}_4 = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{bmatrix}. \tag{11}$$

Suppose that $\boldsymbol{Y}^\star = \boldsymbol{y}^\star\boldsymbol{y}^{\star H}$, with $\boldsymbol{y}^\star = [\boldsymbol{a}^\star; t^\star] \in \mathbb{C}^{N+1}$, is an optimal solution for (9). Then it is known that $\boldsymbol{a}^\star/t^\star$ is an optimal solution for (6) (e.g., see [10, 16]).

Assume that

$$\boldsymbol{Y}^\star = \begin{bmatrix} \boldsymbol{X}^\star & \boldsymbol{x}^\star \\ \boldsymbol{x}^{\star H} & 1 \end{bmatrix}. \tag{12}$$

is a general-rank solution for (9). Having it, let us establish several sufficient conditions under which SDP problem (9) possesses a rank-one solution obtained from $\boldsymbol{Y}^\star$.

Theorem 3.1 *Suppose that $\boldsymbol{Y}^\star$ (as defined in (12)) is the optimal solution for SDP problem (9), and one of the following two inequality conditions*

$$\text{tr}\,(\boldsymbol{A}_1\boldsymbol{Y}^\star) = \text{tr}\,(\boldsymbol{C}\boldsymbol{X}^\star) > \Delta_1 \tag{13}$$

and

$$\text{tr}\,(\boldsymbol{A}_3\boldsymbol{Y}^\star) = \text{tr}\,(\boldsymbol{Q}\boldsymbol{Q}^H\boldsymbol{X}^\star) - 2\Re(\boldsymbol{a}_0^H\boldsymbol{Q}\boldsymbol{Q}^H\boldsymbol{x}^\star) + \boldsymbol{a}_0^H\boldsymbol{Q}\boldsymbol{Q}^H\boldsymbol{a}_0 < \epsilon \tag{14}$$

is satisfied. Then SDP problem (9) has rank-one solution that can be found in polynomial time.

The proof is similar to the proof of Theorem III.3 in [17] (which deals with (8)), and we omit it due to the limited space. It however is a construction procedure to generate a rank-one solution for (9), and we state it in a very short way. Essentially, suppose that $\boldsymbol{Y}^\star$ is an optimal solution of rank greater than one for (9), and define

$$\boldsymbol{A}_1' = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0} & -\Delta_1 \end{bmatrix},\ \boldsymbol{A}_2' = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & -b_2 \end{bmatrix}, \tag{15}$$

where $b_2 = \text{tr}\,(\boldsymbol{A}_2\boldsymbol{Y}^\star)$, and

$$\boldsymbol{A}_3' = \begin{bmatrix} \boldsymbol{Q}\boldsymbol{Q}^H & -\boldsymbol{Q}\boldsymbol{Q}^H\boldsymbol{a}_0 \\ -\boldsymbol{a}_0^H\boldsymbol{Q}\boldsymbol{Q}^H & \boldsymbol{a}_0^H\boldsymbol{Q}\boldsymbol{Q}^H\boldsymbol{a}_0 - \epsilon \end{bmatrix}. \tag{16}$$

To get a low-rank (e.g., rank-one) solution, we resort to the following rank-one matrix decomposition lemma.

Lemma 3.2 (Theorem 2.1 in [11]) *Suppose that $\boldsymbol{X}$ is an $N \times N$ complex Hermitian positive semidefinite matrix of rank R, and $\boldsymbol{A}$, $\boldsymbol{B}$ are two given $N \times N$ Hermitian matrices. Then, there exists a rank-one decomposition $\boldsymbol{X} = \sum_{r=1}^{R} \boldsymbol{x}_r\boldsymbol{x}_r^H$ such that*

$$\boldsymbol{x}_r^H\boldsymbol{A}\boldsymbol{x}_r = \frac{\text{tr}\,(\boldsymbol{A}\boldsymbol{X})}{R} \quad \text{and} \quad \boldsymbol{x}_r^H\boldsymbol{B}\boldsymbol{x}_r = \frac{\text{tr}\,(\boldsymbol{B}\boldsymbol{X})}{R},\ r = 1,\ldots,R.$$

The rank-one decomposition synthetically is denoted as $\{\boldsymbol{x}_r\} = \mathcal{D}_1(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{B})$. The solution procedure for finding a rank-one solution for (9) (i.e., a globally optimal solution for (6)) is summarized in Algorithm 1.

Algorithm 1 Procedure for Solving QCQP Problem (6)

Input: $\hat{\boldsymbol{R}}, \boldsymbol{C}, \boldsymbol{Q}, \boldsymbol{a}_0, \Delta_1, \eta_1, \eta_2, \epsilon$;
Output: An optimal solution $\boldsymbol{a}^\star$ of problem (6);

1. define $\boldsymbol{A}_i$ as in (10) and (11), solve SDP (9) finding $\boldsymbol{Y}^\star$ as in (12), and define $\boldsymbol{A}_i'$ as in (15) and (16);
2. if $\text{tr}\,(\boldsymbol{A}_1\boldsymbol{Y}^\star) > \Delta_1$, then implement the matrix decomposition $\{\boldsymbol{y}_i\} = \mathcal{D}_1(\boldsymbol{Y}^\star, \boldsymbol{A}_2', \boldsymbol{A}_3')$ and pick up $\boldsymbol{y}_l = [\boldsymbol{x}_l; t_l]$ with nonzero t_l such that $\text{tr}\,(\boldsymbol{A}_1'\boldsymbol{y}_l\boldsymbol{y}_l^H) > 0$; go to step 4;
3. if $\text{tr}\,(\boldsymbol{A}_3\boldsymbol{Y}^\star) < \epsilon$, then perform the decomposition $\{\boldsymbol{y}_i\} = \mathcal{D}_1(\boldsymbol{Y}^\star, \boldsymbol{A}_1', \boldsymbol{A}_2')$ and select $\boldsymbol{y}_l = [\boldsymbol{x}_l; t_l]$ with nonzero t_l such that $\text{tr}\,(\boldsymbol{A}_3'\boldsymbol{y}_l\boldsymbol{y}_l^H) < 0$;
4. output $\boldsymbol{a}^\star = \boldsymbol{x}_l/t_l$.

The computational cost in Algorithm 1 is dominated by the cost of solving SDP problem (9), which is of the order $O(N^{4.5})$ [18].

In the following analysis, we focus on the scenario when $\boldsymbol{Y}^\star$ (as defined in (12)) is such that:

$$\text{tr}\,(\boldsymbol{A}_3\boldsymbol{Y}^\star) = \text{tr}\,(\boldsymbol{Q}\boldsymbol{Q}^H\boldsymbol{X}^\star) - 2\Re(\boldsymbol{a}_0^H\boldsymbol{Q}\boldsymbol{Q}^H\boldsymbol{x}^\star) + \boldsymbol{a}_0^H\boldsymbol{Q}\boldsymbol{Q}^H\boldsymbol{a}_0 = \epsilon. \tag{17}$$

Under condition (17), it is still possible to find a mild sufficient condition under which SDP problem (9) has a rank-one solution. The following theorem establishes such condition.

Theorem 3.3 *Suppose that $\boldsymbol{Y}^\star$ (as defined in (12)) is optimal for SDP problem (9). Suppose also that condition (17) holds true. If the number $\boldsymbol{a}_0^H\boldsymbol{Q}\boldsymbol{Q}^H\boldsymbol{x}^\star$, which is in general a complex number, is neither a positive number nor zero (i.e., $\boldsymbol{a}_0^H\boldsymbol{Q}\boldsymbol{Q}^H\boldsymbol{x}^\star \ngeq 0$), then SDP problem (9) has a rank-one solution.*

The proof is similar to that of Theorem 4 in [17], and we omit it due to the limited space.

One more sufficient condition ensuring that the solution of SDP problem (9) is of rank one can be derived based on Theorem 2.3 in [19].

Theorem 3.4 *Suppose that $\boldsymbol{Y}^\star$ (as defined in* (12)*) is optimal for SDP problem* (9)*. If the rank of $\boldsymbol{Y}^\star$ is three or above, then there is a rank-one solution for SDP problem* (9)*.*

The proof is similar to Theorem III.6 in [17], and we omit it due to the limit space.

Summarizing, if one of the following conditions is satisfied, then a rank-one optimal solution for SDP problem (9) can be found in polynomial time:

1. Rank $(\boldsymbol{Y}^\star)$=1 or Rank $(\boldsymbol{Y}^\star)$ $\geq$3;
2. Rank $(\boldsymbol{Y}^\star) = (\geq)2$ and tr $(\boldsymbol{A}_1\boldsymbol{Y}^\star) > \Delta_1$;
3. Rank $(\boldsymbol{Y}^\star) = (\geq)2$ and tr $(\boldsymbol{A}_3\boldsymbol{Y}^\star) < \epsilon$;
4. Rank $(\boldsymbol{Y}^\star) = (\geq)2$, tr $(\boldsymbol{A}_3\boldsymbol{Y}^\star) = \epsilon$, and $\boldsymbol{a}_0^H\boldsymbol{Q}\boldsymbol{Q}^H\boldsymbol{x}^\star \not\geq 0$,

where $\boldsymbol{Y}^\star$ (as in (12)) is a general-rank solution for (9).

4. SIMULATION RESULTS

Consider a uniform linear array with $N = 12$ omni-directional antenna elements spaced half a wavelength apart of each other. The array noise is a Gaussian vector with zero mean and covariance $\boldsymbol{I}$. Two interferers with the same interference-to-noise ratio (INR) of 30 dB are assumed to impinge upon the array from the angles $\theta_1 = 25^\circ$ and $\theta_2 = 85^\circ$ with respect to the array broadside, and the desired signal is always present in the training data cell. The training sample size T is preset to 100. The angular sector Θ of interest is $[50^\circ, 60^\circ]$, and the presumed direction is assumed to be $\theta_0 = 55^\circ$, while the actual signal impinges upon the array from direction $\theta = 55^\circ$ (i.e., there is no look direction mismatch). The norm perturbation parameters η_1 and η_2 for the proposed beamformers are both set to 0.5. All results are averaged over 200 simulation runs. We take into account mismatch caused also by wavefront distortion in an inhomogeneous medium [7]. Specifically, we assume that the signal steering vector is distorted by wave propagation effects in the way that independent-increment phase distortions are accumulated by the components of the steering vector, and assume that the phase increments are independent Gaussian variables each with zero mean and standard deviation 0.02, and they are randomly generated and remain unaltered in each simulation run.

We adopt the method in [13] to generate an ellipsoid

$$\mathcal{E} = \{\boldsymbol{a} \mid (\boldsymbol{a}-\boldsymbol{a}_0)^H\boldsymbol{P}^{-1}(\boldsymbol{a}-\boldsymbol{a}_0) \leq \epsilon\}, \tag{18}$$

where the actual steering vector is located. In other words, $\boldsymbol{Q}\boldsymbol{Q}^H = \boldsymbol{P}^{-1}$. We collect $L = 64$ equally spaced samples at the angle sector Θ. Then the center $\boldsymbol{a}_0$ and the matrix $\boldsymbol{P}$ are, respectively, the sample mean and the sample covariance matrix of different steering vectors with angles in the sector. In other words, $\boldsymbol{a}_0 := \bar{\boldsymbol{a}} = \frac{1}{L}\sum_{l=1}^{L}\boldsymbol{a}(\theta_l)$ and $\boldsymbol{P} = \frac{1}{L}\sum_{l=1}^{L}(\boldsymbol{a}(\theta_l)-\bar{\boldsymbol{a}})(\boldsymbol{a}(\theta_l)-\bar{\boldsymbol{a}})^H$, where

$$\theta_l = \frac{\theta_{\min}+\theta_{\max}}{2} + \left(-\frac{1}{2}+\frac{l-1}{L-1}\right)(\theta_{\max}-\theta_{\min}),\ l = 1,\dots,L.$$

In order to guarantee that $\boldsymbol{P}$ is positive definite in our case, let $\boldsymbol{P} := \boldsymbol{P} + 0.1\boldsymbol{I}$. The parameter ϵ of the ellipsoid takes value of $0.45N$.

We will compare the performance of the beamformer computed by (6) with that of the beamformer computed by the following optimal estimation problem of steering vector:

$$\begin{array}{ll} \underset{\boldsymbol{a}}{\text{minimize}} & \boldsymbol{a}^H\hat{\boldsymbol{R}}^{-1}\boldsymbol{a} \\ \text{subject to} & \boldsymbol{a}^H\tilde{\boldsymbol{C}}\boldsymbol{a} \leq \Delta_0 \\ & N(1-\eta_1) \leq \|\boldsymbol{a}\|^2 \leq N(1+\eta_2) \\ & \|\boldsymbol{Q}^H(\boldsymbol{a}-\boldsymbol{a}_0)\|^2 \leq \epsilon \end{array} \tag{19}$$

(which is problem (15) in [17], and the difference between (6) and (19) is in the DOA constraint), and with the beamformer provided by [7, Eqns. (23)-(25)], which are termed as "New Beamformer 1", "New Beamformer 2" and "KVH Beamformer", respectively.

It can be seen from Fig. 2 that the two proposed beamformers show significant performance superiority comparing with the KVH beamformer in the SNR region of $[-10, 30]$ dB. Thus, it can be concluded that if the sector-of-interest is far enough from the antenna array broadside and is rather close to the antenna end-fire, the DOA (quadratic) constraint in the KVH beamformer is insufficient to prevent the SOI cancellation for insufficiently high SNR. For the two new proposed beamformers, however, the ellipsoidal constraint ensures that the optimal estimate of the steering vector is sufficiently close to the center (i.e., the average of the steering vectors with their DOAs inside the angular sector-of-interest). We observe also that the average SINR of New Beamformer 1 is higher than that of New Beamformer 2, especially in moderate and high SNR region. This implies that changing the DOA constraint from (8) (in problem (19)) to (7) (in problem (6)) does lead to performance improvement. In words, the new DOA constraint (7) is more effective than (8). We report that there are a few problem instances of both (6) and (19) having a rank-two solution and Algorithm 1 must be applied to find a rank-one optimal solution.

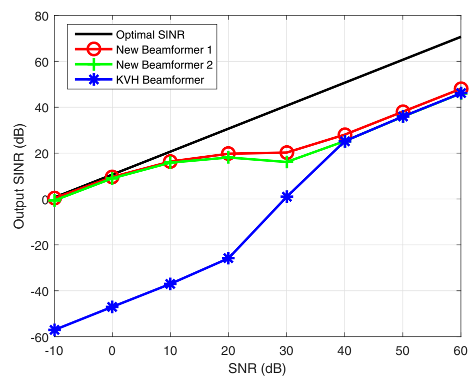

Fig. 2. Average beamformer output SINR versus SNRs, with $\Theta = [50^\circ, 60^\circ]$ and $T = 100$.

5. CONCLUSION

We have considered the MVDR robust adaptive beamforming problem based on optimal steering vector estimation with limited prior knowledge. A new beamformer design problem has been studied, by considering the beamformer output power maximization problem, subject to a DOA constraint enforcing the desired signal DOA to be far away from the DOA interval(s) of all linear combinations of the interference steering vectors, a relaxed double-sided norm constraint, and a generalized similarity constraint. It has turned out that the maximization problem is a non-convex QCQP with three constraints, including a homogeneous, a double-sided, and an inhomogeneous constraints. Therefore, there is no guarantee for a globally optimal solution for the QCQP within polynomial time complexity. Thus, we have established several sufficient optimality conditions, based on which an algorithm is designed to find a globally optimal solution. The performance improvement of the proposed robust beamformer has been demonstrated by simulations in terms of the array output SINR.

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