SPARSE FRACTAL ARRAY DESIGN WITH INCREASED DEGREES OF FREEDOM

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ABSTRACT

Sparse arrays are of great interest since they can identify $\mathcal{O}(N^2)$ uncorrelated sources with N physical sensors. This stems from their large difference coarray, defined as the differences between sensor locations. In a recent study, desired array properties such as closed-form expression for sensor locations, symmetry and large hole-free difference coarray were considered and it was shown that most existing sparse arrays do not exhibit these characteristics simultaneously. Standard Cantor arrays were shown to satisfy all the criteria above, however, their difference coarrays are of size $\mathcal{O}(N^{\log_2 3})$ which is smaller than that obtained with minimum redundancy arrays and nested arrays. In this paper, we introduce a fractal array design where a generator array is extended in a simple recursive fashion. In contrast to previous work, the generator is assumed to be a sparse array with a hole-free difference coarray. We study the resulting arrays and prove they inherit their properties from the generator. Thus, this approach can be used to extend any known sparse configuration to an arbitrarily large array. A small-scale array, which meets all design criteria, can be created and then expanded to generate a symmetric fractal array with a difference coarray of size $\mathcal{O}(N^2)$, unlike Cantor arrays.

***Index Terms*—** Sparse arrays, difference coarray, fractal arrays, increased degrees of freedom

I. INTRODUCTION

Sparse arrays play an important role in many fields such as radar [1–7], radio [8], communications [9] and ultrasound imaging [10–12]. In particular, they are widely used for direction-of-arrival (DOA) estimation. The attractive power of sparse arrays lies in their ability to resolve $\mathcal{O}(N^2)$ uncorrelated sources with N physical sensors. This stems from their difference coarrays, defined as the difference between physical elements, which contain an $\mathcal{O}(N^2)$-long contiguous section.

In a recent study [13], several important design criteria for sparse arrays were discussed. To allow simple and scalable constructions, it is desirable to have closed-form expressions for the sensor locations. Symmetric arrays are favorable in certain applications as they can reduce complexity [14, 15] and improve performance [16–18]. A large difference coarray can increase resolution [19–21] and the number of resolvable sources [20–22]. In addition, having a difference coarray which is a uniform linear array (ULA) facilitates the use of DOA algorithms [20]. It was shown in [13] that most existing sparse arrays such as minimum redundancy arrays (MRA) [19], minimum holes arrays (MHA), nested arrays [20] and coprime arrays [21] do not fulfill these requirements simultaneously. The authors then suggested Cantor arrays [23], which we refer to here as standard Cantor arrays, that meet all the above criteria. However, the proposed arrays restrict the number of sensors N to be a power of two and they lead to a difference coarray of size $N^{\log_2 3} \approx N^{1.58}$, unlike the sparse arrays previously mentioned.

The main contribution of this paper is introducing a fractal array design, based on generalized Cantor arrays [23], in which a generator sparse array is expanded in a simple recursive scheme. This approach differs from previous work [13, 23] by the fact that the array definition directly relates to the generator's difference coarray, assumed to be hole-free. We examine the properties of the proposed fractal arrays and prove that they inherit the characteristics of the generator. In particular, we show that they have a hole-free difference coarray and are symmetric if the generator is symmetric. Moreover, a small-scale symmetric array with large difference coarray can be designed to construct an arbitrarily large fractal array with $\mathcal{O}(N^2)$-long difference coarray where N is the number of physical sensors. Thus, the arrays we present exhibit the same good properties of standard Cantor arrays while providing increased degrees of freedom.

The paper is organized as follows. In Section II we review the signal model and design criteria for sparse arrays. Section III-A briefly describes standard Cantor arrays while Section III-B introduces sparse fractal arrays and investigates their properties. Finally, Section IV concludes the paper.

II. REVIEW OF SPARSE ARRAYS

Consider K narrowband sources with carrier wavelength λ impinging on an N element linear array. The sensor locations are given by $n\lambda/2$ where n belongs to an integer set $\mathbb{G}$ ($|\mathbb{G}|= N$). We denote by $s_k \in \mathbb{C}$ and $\theta_k \in [-\pi/2, \pi/2]$ the complex amplitude and the DOA of the kth source respectively. The received signal $\mathbf{x}$ is modeled as

$$\mathbf{x} = \sum_{k=1}^{K} s_k \mathbf{a}(\theta_k) + \mathbf{n} = \mathbf{A}\mathbf{s} + \mathbf{n} \in \mathbb{C}^N, \tag{1}$$

where $\mathbf{s} = [s_1 \ s_2 \cdots s_k]^T \in \mathbb{C}^K$, $\mathbf{A} = [\mathbf{a}(\theta_1) \ \mathbf{a}(\theta_2) \cdots \mathbf{a}(\theta_K)] \in \mathbb{C}^{N\times K}$ is the array manifold with $\mathbf{a}(\theta) \in \mathbb{C}^{N\times 1}$ being the steering vector at direction θ whose ith entry is $e^{j\pi \sin(\theta) n_i}$ ($n_i \in \mathbb{G}$). The vector $\mathbf{n}$ denotes additive white noise. We assume the sources $\mathbf{s}$ and the noise $\mathbf{n}$ to be zero-mean and uncorrelated, i.e.,

- $\mathbb{E}[\mathbf{s}] = 0, \ \mathbb{E}[\mathbf{n}] = 0,$
- $\mathbb{E}[\mathbf{n}\mathbf{s}^H] = 0,$
- $\mathbb{E}[\mathbf{s}\mathbf{s}^H] = \text{diag}(p_1, p_2, ..., p_K), \ \mathbb{E}[\mathbf{n}\mathbf{n}^H] = p_n\mathbf{I},$

where p_k and p_n are the power of the kth source and the noise respectively.

The covariance matrix of $\mathbf{x}$ can be written as

$$\mathbf{R} = \sum_{k=1}^{K} p_k \mathbf{a}(\theta_k)\mathbf{a}(\theta_k)^H + p_n\mathbf{I}. \tag{2}$$

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The autocorrelation vector $\mathbf{r}$ is given by vectorizing (2) and averaging over duplicated entries:

$$\mathbf{r} = \sum_{k=1}^{K} p_k \mathbf{b}(\theta_k) + p_n \boldsymbol{\delta} = \mathbf{B}\mathbf{p} + p_n \boldsymbol{\delta} \in \mathbb{C}^{|\mathbb{D}|}, \tag{3}$$

where $\boldsymbol{\delta} \in \mathbb{C}^{|\mathbb{D}|}$ is a column vector with 1 in the $\frac{|\mathbb{D}|+1}{2}$-th entry and 0 otherwise. Here $\mathbf{b}(\theta) = \mathbf{a}(\theta) \odot \mathbf{a}(\theta)$ where $\odot$ denotes the Khatri-Rao product and $\mathbf{B} = [\mathbf{b}(\theta_1)\ \mathbf{b}(\theta_2) \cdots \mathbf{b}(\theta_K)] \in \mathbb{C}^{|\mathbb{D}| \times K}$ behaves as an array manifold whose sensor locations are given by the difference coarray $\mathbb{D}$ defined as follows:

Definition 1 (Difference Coarray)**.** Consider a sensor array $\mathbb{G}$. The difference coarray is given by

$$\mathbb{D} \triangleq \{d \mid n_1 - n_2 = d,\ n_1, n_2 \in \mathbb{G}\}.$$

The number of degrees of freedom (DOF) of a sparse array $\mathbb{G}$ is the cardinality of its difference coarray $\mathbb{D}$.

When the size of the difference coarray $\mathbb{D}$ is much larger than that of the physical array $\mathbb{G}$ it allows to recover more uncorrelated sources than sensors by using DOA estimators on the autocorrelation vector or alternatively, to increase the spatial resolution in estimating the DOAs. Hence, it is of great importance to design sparse arrays with nonuniform element spacing which exploit this property of the difference coarray. To that end, we rely on the following definitions:

Definition 2 (Symmetric Array)**.** Consider a sensor array $\mathbb{G}$. The reversed version of an array $\mathbb{G}$ is defined as

$$\hat{\mathbb{G}} \triangleq \{\max(\mathbb{G}) + \min(\mathbb{G}) - n \mid n \in \mathbb{G}\}.$$

An array is symmetric if $\mathbb{G} = \hat{\mathbb{G}}$.

Definition 3 (Hole-Free Difference Coarray)**.** Consider a sensor array $\mathbb{G}$ and the corresponding difference coarray $\mathbb{D}$. We denote by $\mathbb{U}$ the central ULA in $\mathbb{D}$ with unity spacing which includes the element 0. The difference coarray $\mathbb{D}$ is said to be hole-free if $\mathbb{D} = \mathbb{U}$.

Based on the definitions above, we consider the following desired design criteria for sparse arrays [13]:

Criterion 1 (Closed-Form)**.** The sensor locations should be described using a closed-form expression.

Criterion 2 (Symmetric physical array)**.** In various applications symmetric arrays are favorable since they can reduce computational load and improve performance [14–18].

Criterion 3 (Hole-free difference coarray)**.** The performance of any unbiased DOA estimator using sparse arrays is governed by the difference coarray [20]. In particular, the performance of coarray MUSIC relies on $\mathbb{U}$ [21]. Therefore, a hole-free difference array is preferable. Otherwise, a prior interpolation step is required which could increase complexity [24–26].

Criterion 4 (Large difference coarray)**.** It is beneficial that the size of the difference coarray $|\mathbb{D}|$ will increase rapidly with $\mathbb{G}$. In particular, it is desirable that $|\mathbb{D}| = \mathcal{O}(|\mathbb{G}|^2)$.

Known array geometries, which have recently attracted attention in array signal processing, do not satisfy the above four properties simultaneously. ULAs have a small difference coarray of size $\mathcal{O}(N)$. Nested arrays are not symmetric while coprime arrays are not hole-free. MRAs and MHAs cannot be expressed in closed-form. Cantor arrays have large difference coarray but with size $\mathcal{O}(N^{\log_2 3})$. Some of these configurations, such as nested arrays, can be easily modified to satisfy Criteria 1 to 4, however, this often results in increasing the number of physical elements while the size of the difference array remains unchanged. Finally, sparse arrays which meet the criteria above can be found by searching algorithms, however, these techniques are limited to a small number of elements due to complexity.

In the next section, we present a scalable array design which fulfills Criteria 1 to 4 based on generalized Cantor arrays [23].

III. FRACTAL ARRAY DESIGN

In this section, we introduce a fractal array design which has a simple recursive definition and satisfies Criteria 1 to 4. We begin with describing standard Cantor arrays [23] whose difference coarrays were recently investigated [13]. Then, we present an extension of Cantor arrays which is scalable and meets Criteria 1 to 4. In particular, the proposed arrays have a large hole-free difference coarray of size $\mathcal{O}(N^2)$.

A. Standard Cantor Arrays

Standard Cantor arrays are fractal arrays defined recursively as follows:

$$\begin{aligned} &\mathbb{C}_0 \triangleq \{0\}, \\ &\mathbb{C}_{r+1} \triangleq \mathbb{C}_r \cup (\mathbb{C}_r + 3^r), \quad r \in \mathbb{N}, \end{aligned} \tag{4}$$

where $\cup$ denotes the union operation and $\mathbb{A} + \mathbb{B}$ denotes the sum set of two integer sets $\mathbb{A}$ and $\mathbb{B}$ defined as

$$\mathbb{A} + \mathbb{B} \triangleq \{a + b \mid a \in \mathbb{A},\ b \in \mathbb{B}\}.$$

The arrays in (4) are equivalent to the Cantor arrays proposed in [13]. The Cantor arrays are based upon the Cantor sets in fractal theory [27, 28]. An example of such an array is given in Fig. 1(c).

It can be shown that any Cantor array $\mathbb{C}_r$ is symmetric with $N = 2^r$ physical elements. In addition, it was proven in [13] that it has a hole-free difference coarray $\mathbb{D}_r$ of size $|\mathbb{D}_r| = 3^r$. Therefore, Cantor arrays satisfy Criteria 1 to 4 which make them very attractive. On the down side, their difference coarray has size $\mathcal{O}(N^{\log_2 3} \approx \mathcal{O}(N^{1.585})$ which is smaller than $\mathcal{O}(N^2)$, achieved by other sparse arrays such as nested arrays and MRAs. Another restriction of Cantor arrays is that the number of sensors N must be a power of two.

We next present an extension of Cantor arrays which does not exhibit these limitations while fulfilling Criteria 1 to 4.

B. Fractal Arrays with Increased Degrees of Freedom

We introduce a fractal array configuration which has a simple recursive definition as a Cantor array. As we show, its difference coarray can have size $\mathcal{O}(N^2)$ where N is the number of physical elements.

Consider an L element linear array whose sensor locations correspond to an integer set $\mathbb{G}$ ($|\mathbb{G}| = L$). Without loss of generality, we assume that $\min(\mathbb{G}) = 0$, otherwise, it can be translated to satisfy this condition. In addition, the difference coarray $\mathbb{D}$ of $\mathbb{G}$ is assumed to be hole-free ($\mathbb{D} = \mathbb{U}$). We then propose a fractal array

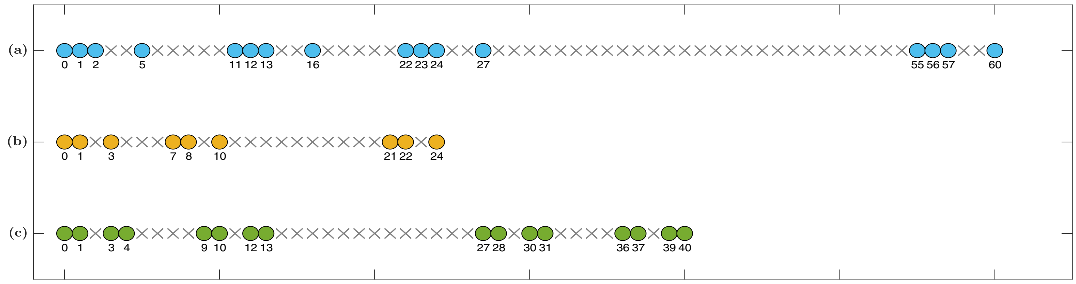

Fig. 1: **Fractal Array Examples.** The fractal arrays for (a) $r = 2$ and a nested array generator $\mathbb{G} = [0\ 1\ 2\ 5]$, (b) $r = 2$ and a MRA $\mathbb{G} = [0\ 1\ 3]$ and (c) $r = 4$ and $\mathbb{G} = [0\ 1]$ (4th order Cantor array).

$\mathbb{F}_r$ defined recursively as

$$\begin{aligned} &\mathbb{F}_0 \triangleq \{0\}, \\ &\mathbb{F}_{r+1} \triangleq \bigcup_{n\in\mathbb{G}} (\mathbb{F}_r + nM^r), \quad r \in \mathbb{N}, \end{aligned} \tag{5}$$

where $M \triangleq |\mathbb{D}|$ is the translation factor and r is the array order. Note that since we assume $\mathbb{D}$ to be hole-free, the translation factor can be expressed as $M = 2\max(\mathbb{G}) + 1$. Notice that when $\mathbb{G} = [0\ 1]$, definition (5) reduces to the definition of Cantor arrays given in (4).

The array $\mathbb{F}_r$ defined in (5) is composed of copies of the generator $\mathbb{G}$ that are spatially arranged according to $\mathbb{G}$ and have a total of $N = L^r$ physical sensors. Notice that $\mathbb{F}_1$ is equal to the set $\mathbb{G}$ which is known as the *generator* in fractal terminology [27–30]. In contrast to previous related work [13, 23], we define the generator to be a sparse array with a hole-free difference array which directly determines the translation factor. We will later show that this enables the construction of arbitrarily large arrays which satisfy criteria 2 to 4 by designing appropriate generators.

Examples of the fractal arrays defined in (5) are shown in Fig. 1. A fractal array with $r = 2$ is demonstrated in Fig. 1(a) where the generator is a nested array $\mathbb{G} = [0\ 1\ 2\ 5]$. Figure 1(b) depicts a similar design where the generator is chosen to be a MRA $\mathbb{G} = [0\ 1\ 3]$. A standard Cantor array is shown in Fig. 1(c) with a generator defined by $\mathbb{G} = [0\ 1]$. We will prove in Corollary 1 of Theorem 2 that arrays (a) and (b) in Fig. 1 exhibit increased degrees of freedom compared with the Cantor array in Fig. 1(c).

We next study the properties of the proposed arrays. First, they are expressed in a closed-form, hence, they satisfy Criterion 1. The fulfillment of Criteria 2 to 4 depends on the generator, as shown below.

Theorem 1. *The array $\mathbb{F}_r$ is symmetric if $\mathbb{G}$ is symmetric.*

Proof. By induction:

- **Base** $(k = 0)$ - $\mathbb{F}_0 = \{0\}$ is symmetric.
- **Assumption** $(k = r)$ - $\mathbb{F}_r$ is symmetric.
- **Step** $(k = r + 1)$ - First, notice that

$$\min(\mathbb{G}) = 0,\ \min(\mathbb{F}_r) = 0.$$

In addition, we can rewrite $\mathbb{F}_{r+1}$ as

$$\mathbb{F}_{r+1} = \{m + nM^r\ :\ m \in \mathbb{F}_r,\ n \in \mathbb{G}\}.$$

Hence, we get

$$\begin{aligned} \max(\mathbb{F}_{r+1}) &= \max(\mathbb{F}_r) + \max(G)M^r, \\ \min(\mathbb{F}_{r+1}) &= \min(\mathbb{F}_r) + \min(G)M^r = 0. \end{aligned}$$

Let $\hat{\mathbb{G}}$, $\hat{\mathbb{F}}_r$ and $\hat{\mathbb{F}}_{r+1}$ denote the reversed versions of $\mathbb{G}$, $\mathbb{F}_r$ and $\mathbb{F}_{r+1}$ respectively. Then, we have

$$\begin{aligned} \hat{\mathbb{F}}_{r+1} &\triangleq \{\max(\mathbb{F}_{r+1}) + \min(\mathbb{F}_{r+1}) - l\ :\ l \in \mathbb{F}_{r+1}\} \\ &= \{\max(\mathbb{F}_{r+1}) - l\ :\ l \in \mathbb{F}_{r+1}\} \\ &= \{\max(\mathbb{F}_{r+1}) - (m + nM^r)\ :\ m \in \mathbb{F}_r,\ n \in \mathbb{G}\} \\ &= \{\max(\mathbb{F}_r) + \max(\mathbb{G})M^r - (m + nM^r)\ :\ m \in \mathbb{F}_r,\ n \in \mathbb{G}\} \\ &= \{\max(\mathbb{F}_r) - m + (\max(\mathbb{G}) - n)M^r\ :\ m \in \mathbb{F}_r,\ n \in \mathbb{G}\} \\ &= \{\hat{m} + \hat{n}M^r\ :\ \hat{m} \in \hat{\mathbb{F}}_r,\ \hat{n} \in \hat{\mathbb{G}}\} \\ &= \{\hat{m} + \hat{n}M^r\ :\ \hat{m} \in \mathbb{F}_r,\ \hat{n} \in \mathbb{G}\} \\ &= \mathbb{F}_{r+1}. \end{aligned}$$

Hence, $\mathbb{F}_{r+1}$ is symmetric. □

Theorem 1 above proves that if the generator $\mathbb{G}$ satisfies Criterion 2, then, $\mathbb{F}_r$ satisfies it as well. The hole-free property is given by the following theorem.

Theorem 2. *The difference coarray $\mathbb{D}_r$ of $\mathbb{F}_r$ is a hole-free array with $|\mathbb{D}_r| = M^r$ consecutive integers from $-\frac{M^r-1}{2}$ to $\frac{M^r-1}{2}$, i.e.,*

$$\mathbb{D}_r = \left[-\frac{M^r - 1}{2}, \frac{M^r - 1}{2}\right].$$

Proof. By induction:

- **Base** $(k = 0)$ - $\mathbb{D}_0 = \{0\}$ with size $|\mathbb{D}_0| = 1 = M^0$.
- **Assumption** $(k = r)$ - $\mathbb{D}_r = \left[-\frac{M^r-1}{2}, \frac{M^r-1}{2}\right]$.
- **Step** $(k = r + 1)$ - First, the difference coarray $\mathbb{D}$ of $\mathbb{G}$ is a hole-free array of size $|\mathbb{D}| = M$, hence,

$$\mathbb{D} = \left[-\frac{M - 1}{2}, \frac{M - 1}{2}\right].$$

Next, by the definition of the difference coarray, we have

$$\begin{aligned} \mathbb{D}_{r+1} &\triangleq \{n - m\ :\ n, m \in \mathbb{F}_{r+1}\} \\ &= \{(n + uM^r) - (m + vM^r)\ :\ n, m \in \mathbb{F}_r,\ u, v \in \mathbb{G}\} \\ &= \{(n - m) + (u - v)M^r\ :\ n, m \in \mathbb{F}_r,\ u, v \in \mathbb{G}\} \\ &= \{m + nM^r\ :\ m \in \mathbb{D}_r,\ n \in \mathbb{D}\} \\ &= \left[-\frac{M^{r+1} - 1}{2}, \frac{M^{r+1} - 1}{2}\right]. \end{aligned}$$

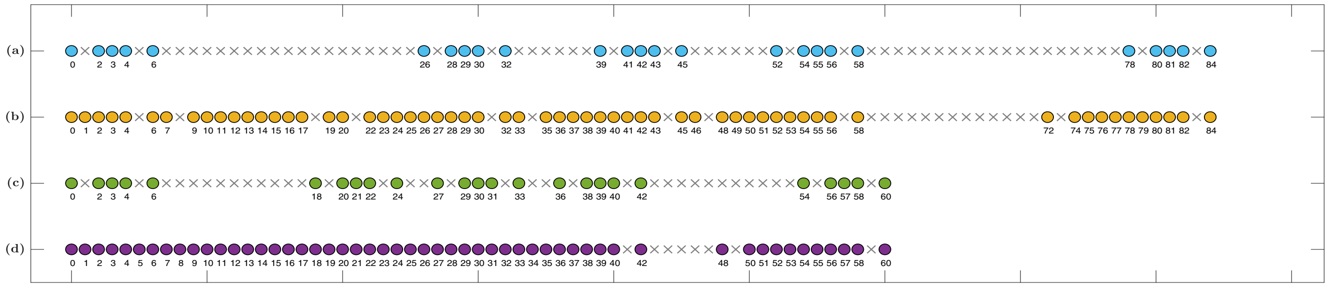

Fig. 2: **Coprime Array Generator.** The fractal arrays for a coprime array generator $\mathbb{G} = [0\ 2\ 3\ 4\ 6]$ with $r = 2$ where (a) $M = 2\max(\mathbb{G}) + 1 = 13$ and (c) $M = |\mathbb{U}| = 9$. (b) and (d) are the non-negative parts of the difference coarrays of (a) and (c) respectively.

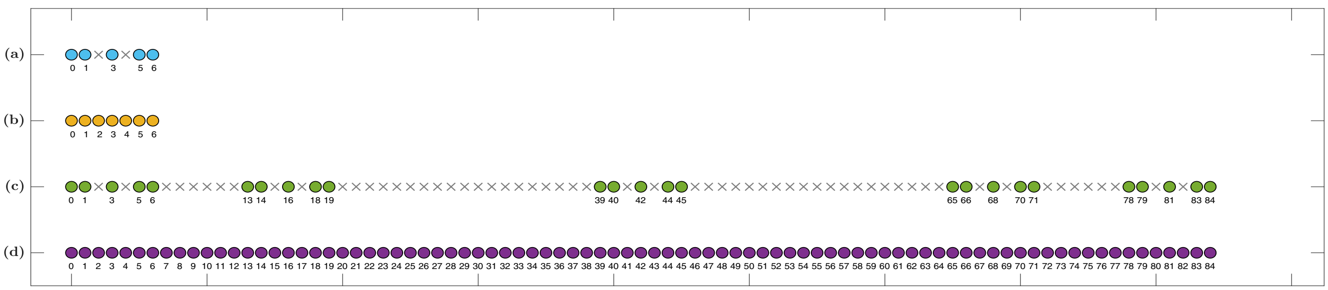

Fig. 3: **Optimal Generator.** (a) An optimal array $\mathbb{G} = [0\ 1\ 3\ 5\ 6]$, found by exhaustive search, which satisfies Criteria 2 to 4 and has a minimal number of sensors with respect to is aperture. (b) The difference coarray of $\mathbb{G}$. (c) The fractal array created by (5) using $\mathbb{G}$ as a generator with $r = 2$ and (d) the corresponding difference coarray.

Therefore, the size of difference coarray is

$$|\mathbb{D}_{r+1}| = 2\left(\frac{M^{r+1}-1}{2}\right) + 1 = M^{r+1}. \qquad \square$$

The result of Theorem 2 emphasizes the importance of the definition given in (5) where the generator is assumed to have a hole-free difference coarray and the translation factor is chosen according to its cardinality. This ensures that the proposed arrays have hole-free difference coarrays for any order r.

Note that sparse arrays whose difference coarrays have holes, such as coprime arrays, cannot be used as a generator according to definition (5). In the case where such an array is desired to be the generator, the resultant fractal array would have holes in its difference coarray. Moreover, its central ULA, denoted by $\mathbb{U}_r$, would be small. Hence, to maximize the size of $\mathbb{U}_r$, the translation factor should be modified to $M = |\mathbb{U}|$ where $\mathbb{U}$ is the central ULA of the generator. This is demonstrated in Fig. 2(a) where a coprime generator is used to construct a fractal array according to (5). The corresponding difference coarray, shown in Fig. 2(b), has holes and the central ULA is small. Using the same generator with a smaller translation factor leads to a different fractal array (Fig. 2(c)) whose difference coarray also has holes but its central ULA is much larger, as seen in Fig. 2(d).

Finally, the last result is the increased degrees of freedom property of the proposed design which follows from Theorem 2:

Corollary 1. *For a fixed order r, if the difference coarray of the generator satisfies $M \triangleq |\mathbb{D}| = \mathcal{O}(L^2)$, then the difference coarray of $\mathbb{F}_r$ satisfies*

$$|\mathbb{D}_r| = \mathcal{O}(N^2)$$

where $N = L^r$ is the number of physical sensors in $\mathbb{F}_r$.

Proof. According to Theorem 2, it holds that $|\mathbb{D}_r| = M^r$. Hence,

$$|\mathbb{D}_r| = M^r = \mathcal{O}(L^{2r}) = \mathcal{O}(N^2). \qquad \square$$

Thus, the proposed design allows to construct a small-scale generator array which satisfies Criteria 2 to 4, by exhaustive search for example, and then extend it in a fractal fashion to create an arbitrarily large array that exhibits the same properties, as displayed in Fig. 3. In contrast to standard Cantor arrays [13], the fractal arrays presented here have a number of sensors N that is a perfect power, not necessarily of two, and their difference coarrays can have $\mathcal{O}(N^2)$ degrees of freedom.

IV. CONCLUSION

This work considered sensor locations with closed-form expression, symmetry and a large hole-free difference coarray as leading characteristics for sparse arrays which most of existing sparse geometries do not exhibit simultaneously. In this paper, we introduced a fractal array design based on generalized Cantor where the generator is a sparse array with a hole-free difference coarray that dictates the translation factor. The proposed recursive scheme allows to extend any known sparse array in a fractal fashion and it was proven that the resulting array inherits the properties mentioned above from its generator. Thus, a generator array with a small number of sensors can be created and optimized and then extended to an arbitrarily large array which meets the same criteria. In particular, a symmetric fractal array can be constructed, which leads to $\mathcal{O}(N^2)$ degrees of freedom using N physical sensors in contrast to previously proposed standard Cantor arrays.

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