HYBRID SPARSE ARRAY DESIGN FOR UNDER-DETERMINED MODELS

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ABSTRACT

Sparse arrays are typically configured considering either the environmental dependent or independent design objectives. In this paper, we investigate hybrid sparse array design satisfying dual design objectives. We consider enhancing the source identifiability and maximizing the Signal-to-Interference-plus-noise-ratio (SINR) as our design criteria. We pose the problem as designing fully augmentable sparse arrays for receive beamforming achieving maximum SINR (MaxSINR) for desired point sources operating in an interference active environment. The problem is formulated as a re-weighted l_1-norm squared quadratically constraint quadratic program (QCQP). Simulation results are presented to show the effectiveness of the proposed algorithm for designing fully augmentable arrays in case of under-determined scenarios.

***Index Terms*—** SINR, MaxSINR, fully augmentable sparse arrays, QCQP, l_1-norm.

1. INTRODUCTION

Sparse arrays have gained considerable attention because of their affordable architecture, in addition to significant reduction in computational complexity. For environment-dependent design objectives, sensor selections assume sensor placements on a pre-defined grid points. Emerging sensor switching technologies enable activations of small number of optimally selected sensors in response to the rapidly changing environment [1–4]. This allows reduced cost and simplified hardware by limiting the number of transceivers chains employed by the system. Environment blind objectives, on the other hand, produce sparse arrays with fixed positions, independent of the sources or interferences in the array field of view (FOV). These objectives typically seek filled co-arrays which permit dealing with a large number of sources and interferences impinging on the array [5–7].

In many applications, it is desirable to optimize array configuration per environment-independent objective, such as maximizing the identifiability of the locations of sources in the FOV. It is also of interest to apply, at the same time, environment-dependent objective, e.g., achieving MaxSINR, tangible interference nulling, and accurate estimation of source power [8].

In this paper, we consider sparse array design meeting the dual design objectives, one is environment-independent, while the other is environment-dependent. The former seeks to obtain a filled co-array that maximizes the identifiability for DOA estimation, whereas the latter seeks maximizing SINR for desirable sources operating in active interference environment. Therefore, we restrict the sparse array optimization over a class of fully augmentable arrays. Full augmentability implies that the sparse array has an associated filled co-array that can potentially resolve more sources than available sensors. Given a limited array aperture, the full augmentability constraint is typically satisfied by prefixing few of the available sensors at specific grid locations [9]. In so doing, the remaining sensors can be utilized to achieve superior SINR performance. It is noted that the prefixed sensor positions in the proposed hybrid sparse array design simplifies sensor switching, as now only a limited number of sensors need to be switched according to the SINR objectives.

The problem is posed as optimally selecting K sensor locations out of N possible equally spaced grid points. Maximizing SINR amounts to maximizing the principal eigenvalue of the product of the inverse of data correlation matrix and the desired source correlation matrix [10]. Since it is an NP hard optimization problem, we pose this problem as QCQP with weighted l_1-norm squared to promote sparsity. The re-weighted l_1-norm convex relaxation has been exploited previously for sensor selection problem for beampattern synthesis [11, 12], whereas, the re-weighted l_1-norm squared relaxation effectively reduces sensors to minimize the transmit power for multicast beamforming [13]. We adopt an iterative partial re-weighting approach to control the sparsity of the optimum weight vector so that K sensor fully augmentable hybrid array is finally selected. This modified regularization weighting matrix is cognizant of the environment-independent criterion in the design, and works by minimizing the objective function around the presumed pre-fixed array. Unlike previous contributions, our approach operates directly on the received data correlation matrix and does not assume knowledge of the interfering environment, which is unavailable in many applications [14–16].

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The rest of the paper is organized as follows: In Section 2, we state the problem formulation for maximizing the output SINR. Section 3 deals with the optimum sparse array design by iterative semidefinite relaxation for designing K-sensor fully augmentable hybrid sparse array. In section 4, we demonstrate the usefulness of fully augmentable arrays achieving MaxSINR followed by concluding remarks.

2. PROBLEM FORMULATION

Consider a desired source in the presence of P narrowband interfering signals impinging on a linear array with N uniformly placed sensors. The received signal at the array at time instant n is then given by;

$$\mathbf{x}(n) = \alpha(n)\mathbf{s}(\theta) + \sum_{p=1}^{P} \beta_p(n)\mathbf{i}(\theta_p) + \mathbf{v}(n), \quad (1)$$

where, $\mathbf{s}(\theta)$ and $\mathbf{i}(\theta_p) \in \mathbb{C}^N$ are defined as the corresponding steering vectors with respect to directions of arrival, θ or θ_p as follows;

$$\mathbf{s}(\theta) = [1 \;\; e^{j(2\pi/\lambda)dcos(\theta)} \ldots e^{j(2\pi/\lambda)d(N-1)cos(\theta)}]^T. \quad (2)$$

The inter-element spacing is denoted by d, $(\alpha(n), \beta_p(n)) \in \mathbb{C}$ are the complex amplitudes of the incoming baseband signals. The additive Gaussian noise $\mathbf{v}(n) \in \mathbb{C}^N$ has a variance of σ_v^2 at the receiver output. The received signal vector $\mathbf{x}(n)$ is combined by the N-sensor beamformer, that strives to maximize the output SINR. The output signal $y(n)$ of the optimum beamformer for maximum SINR is given by [10],

$$y(n) = \mathbf{w}_0^H \mathbf{x}(n), \quad (3)$$

where, $\mathbf{w}_0$ is given by the following optimization problem;

$$\begin{aligned} \underset{\mathbf{w}}{\text{minimize}} \quad & \mathbf{w}^H \mathbf{R}_i \mathbf{w}, \\ \text{s.t.} \quad & \mathbf{w}^H \mathbf{R}_s \mathbf{w} = 1. \end{aligned} \quad (4)$$

For signals that are statistically independent, the desired source correlation matrix is approximated by, $\mathbf{R}_s = \sigma^2 \mathbf{s}(\theta)$ $\mathbf{s}^H(\theta)$, where, $\sigma^2 = E\{\alpha(n)\alpha^H(n)\}$. Likewise, we have the interference and noise correlation matrix $\mathbf{R}_i$= $\sum_{p=1}^{P}(\sigma_p^2 \mathbf{i}(\theta_p)$ $\mathbf{i}^H(\theta_p))$ + $\sigma_v^2 \mathbf{I}^{N*N}$, with $\sigma_p^2 = E\{\beta_p(n)\ \beta_p^H(n)\}$ representing the power of the pth interfering source. The problem in (4) is written equivalently, by replacing $\mathbf{R}_i$ with the received data covariance matrix, $\mathbf{R}_x = \mathbf{R}_s + \mathbf{R}_i$ as follows [10],

$$\begin{aligned} \underset{\mathbf{w}}{\text{minimize}} \quad & \mathbf{w}^H \mathbf{R}_x \mathbf{w}, \\ \text{s.t.} \quad & \mathbf{w}^H \mathbf{R}_s \mathbf{w} = 1. \end{aligned} \quad (5)$$

The analytical solution of the above optimization problem is given by $\mathbf{w}_0 = \mathscr{P}\{\mathbf{R}_i^{-1}\mathbf{R}_s\} = \mathscr{P}\{\mathbf{R}_x^{-1}\mathbf{R}_s\}$. The operator $\mathscr{P}\{.\}$ denotes the principal eigenvector of the input matrix. Substituting $\mathbf{w}_0$ into (3) yields the corresponding optimum output SINR;

$$\text{SINR}_o = \frac{\mathbf{w}_0^H \mathbf{R}_s \mathbf{w}_0}{\mathbf{w}_0^H \mathbf{R}_i \mathbf{w}_0} = \Lambda_{max}\{\mathbf{R}_i^{-1}\mathbf{R}_s\}. \quad (6)$$

Eq. (6) shows that the optimum output SINR is the maximum eigenvalue (Λ_{max}) of the product of the inverse of interference plus noise correlation matrix and the desired source correlation matrix. Therefore, the optimum beamformer for maximizing the output SINR is affected by the desired and interference plus noise correlation matrix that are intrinsically dependent on the array configuration.

3. OPTIMUM SPARSE ARRAY DESIGN

Sparse array design amounts to maximizing the principal eigenvector of the product of the two correlation matrices over all possible array configurations. Eigenvalue maximization for sensor placement design is combinatorial optimization and, therefore, cannot be solved in polynomial time [17]. To realize convex relaxation of sparse sensor selection, we assume that we can estimate the N dimensional correlation matrix corresponding to a full sensor array. This assumption has become possible in the proposed hybrid array approach that ensures committing degrees of freedom to construct sparse array with corresponding filled co-array. It is noted that in previous work on MaxSINR sparse array design, all degrees of freedom were designated towards satisfying the objective. In this case, the full augmented correlation matrix was only possible through prior knowledge of the source and interference angles of arrival [18, 19].

The problem expressed in Eq. (5) is penalized with l_1 norm to invoke the sparsity in the beamforming weight vector $\mathbf{w}$, as follows;

$$\begin{aligned} \underset{\mathbf{w} \in \mathbb{C}^N}{\text{minimize}} \quad & \mathbf{w}^H \mathbf{R}_x \mathbf{w} + \beta(||\mathbf{w}||_1), \\ \text{s.t.} \quad & \mathbf{w}^H \mathbf{R}_s \mathbf{w} = 1. \end{aligned} \quad (7)$$

Here, $||.||_1$ denotes the l_1-norm that is well known to encourage sparse solutions and β is a parameter to control the desired sparsity in the solution. To further promote sparse solutions, the problem in (7) is penalized instead by the weighted l_1-norm formulation [20],

$$\begin{aligned} \underset{\mathbf{w} \in \mathbb{C}^N}{\text{minimize}} \quad & \mathbf{w}^H \mathbf{R}_x \mathbf{w} + \beta(||(\mathbf{z}^i \circ |\mathbf{w}|)||_1), \\ \text{s.t.} \quad & \mathbf{w}^H \mathbf{R}_s \mathbf{w} = 1. \end{aligned} \quad (8)$$

where, $'\circ'$ denotes the element wise product, $|.|$ is the modulus operator and $\mathbf{z}^i \in \mathbb{R}^N$ is the regularization weighting vector at the ith iteration. It is shown in [13] that the re-weighted l_1-norm regularization is replaced by the l_1-norm squared function while preserving the sparsity promoting property of the weighted l_1-norm function, leading naturally to a semidefinite program (SDP),

$$\begin{aligned} \underset{\mathbf{w} \in \mathbb{C}^N}{\text{minimize}} \quad & \mathbf{w}^H \mathbf{R}_x \mathbf{w} + \beta(||(\mathbf{z}^i \circ |\mathbf{w}|)||_1^2), \\ \text{s.t.} \quad & \mathbf{w}^H \mathbf{R}_s \mathbf{w} \geq 1. \end{aligned} \quad (9)$$

Algorithm 1 Proposed algorithm for MaxSINR fully augmentable array design

Input: Data correlation matrix $\mathbf{R}_x$, N, K, θ, $\tilde{\mathbf{z}}$.
Output: K sensor beamforming weight vector $\mathbf{w}_0$,
Initialization:
Calculate the regularization weighting matrix $\mathbf{Z} = \tilde{\mathbf{z}}\tilde{\mathbf{z}}^T$.
Initialize β, ϵ.
while (Solution is not K sparse) **do**
Run the SDR of Eq. (10).
Update the regularization weighting matrix $\mathbf{Z}$ according to Eq. (12).
Increase or decrease β according to the binary search algorithm to converge to K sparse solution.
end while
After achieving the desired cardinality, run SDP for reduced size correlation matrix corresponding to nonzero values of $\tilde{\mathbf{W}}$ and $\beta = 0$, yielding, $\mathbf{w}_0 = \mathscr{P}\{\mathbf{W}\}$.
return $\mathbf{w}_0$

The SDP formulation can subsequently be realized by re-expressing the quadratic form, $\mathbf{w}^H\mathbf{R}_x\mathbf{w}$=Tr($\mathbf{w}^H\mathbf{R}_x\mathbf{w}$) = Tr($\mathbf{R}_x\mathbf{w}\mathbf{w}^H$) = Tr($\mathbf{R}_x\mathbf{W}$), where Tr(.) is the trace of the matrix. Similarly, $||(\mathbf{z}^i \circ |\mathbf{w}|)||_1^2 = (|\mathbf{w}|^T\mathbf{z}^i)((\mathbf{z}^i)^T|\mathbf{w}|)$ = $|\mathbf{w}|^T\mathbf{Z}^i|\mathbf{w}|$ =Tr($\mathbf{Z}^i|\mathbf{W}|$). Here, $\mathbf{W} = \mathbf{w}\mathbf{w}^H$ and $\mathbf{Z}^i = \mathbf{z}^i(\mathbf{z}^i)^T$ is the regularization weighting matrix at the ith iteration. Utilizing these quadratic expressions in (9) yields the following problem [13,21,22],

$$\begin{aligned} \underset{\mathbf{W}\in\mathbb{C}^{N*N},\tilde{\mathbf{W}}\in\mathbb{R}^{N*N}}{\text{minimize}} \quad & \text{Tr}(\mathbf{R}_x\mathbf{W}) + \beta\text{Tr}(\mathbf{Z}^i\tilde{\mathbf{W}}), \\ \text{s.t.} \quad & \text{Tr}(\mathbf{R}_s\mathbf{W}) \geq 1, \\ & \tilde{\mathbf{W}} \geq |\mathbf{W}|, \\ & \mathbf{W} \succeq 0. \end{aligned} \tag{10}$$

Here, $\geq$ is the element wise comparison and $\succeq$ denotes the generalized matrix inequality. The quadratic forms represented above only hold if Eq. (10) is accompanied by an additional rank one constraint on the solution matrix $\mathbf{W}$. However, constraining the rank of the solution matrix is a non convex constraint and is omitted resulting in the above rank relaxed semidefinite program (SDR). It is noted that the solution matrix is generally rank one for the underlying receive beamforming application and the SDR performs well with reasonable accuracy.

3.1. Partial re-weighting

The array designed freely without the full augmentability constraint calls for the regularization weighting matrix $\mathbf{Z}$ to be initialized by an all ones matrix. The m,nth element of $\mathbf{Z}$ is iteratively updated as follows,

$$\mathbf{Z}_{m,n}^{i+1} = \frac{1}{|\mathbf{W}_{m,n}^i| + \epsilon}. \tag{11}$$

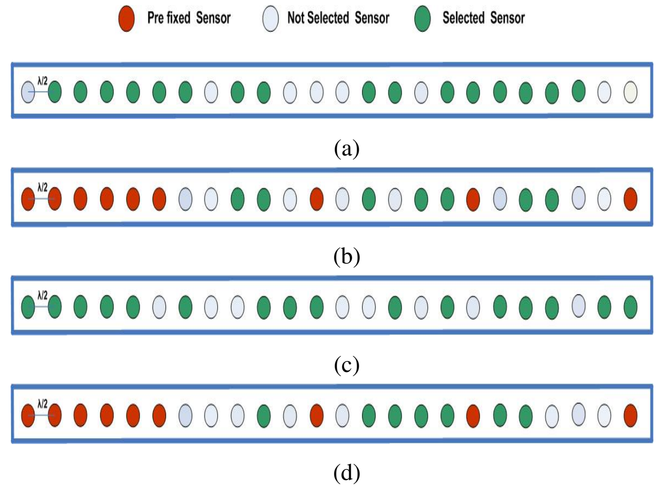

Fig. 1. (a) MaxSINR 16 element sensor array without augmentability constraint (desired source at 60^0) (b) Optimum Hybrid array for desired source at 60^0 (c) Array offering minimum SINR performance (d) Optimum Hybrid array for desired source at 90^0

The parameter ϵ prevents the unwanted case of division by zero and also avoids the solution to converge to local minima. It is noted that the above re-weighting treats all sensors fairly and does not incorporate the pre-fixed array configuration. To address this problem, we denote a selection vector $\tilde{z}$ containing the binary entries corresponding to each sensor location. The 0 entries are set against the pre-fixed sensor locations which signify that the corresponding sensors are not penalized relative to the 1 entries, thereby promoting pre-fixed sensor locations in the solution. Hence, the optimization is carried over the remaining degrees of freedom (locations corresponding to the 1 entries) to recover sparse solutions.

$$\mathbf{Z}^{i+1} = (\tilde{\mathbf{z}}\tilde{\mathbf{z}}^T) \oslash (|\mathbf{W}^i| + \epsilon). \tag{12}$$

The symbol $'\oslash'$ represents the element wise division. The pseudo-code for controlling the sparsity of the optimal weight vector $\mathbf{w}_0$ is summarized in **Algorithm 1**.

4. SIMULATIONS

We demonstrate the hybrid sparse array design for MaxSINR under-determined operating scenario having more sources than the selected sensors. The positions of $K = 16$ sensors are to be selected from $N = 24$ possible equally spaced locations with minimum sensors spacing of $\lambda/2$. There are 20 sources in the FOV, with respective DOAs ([25^0 27^0 29^0 31^0 33^0 39^0 41^0 43^0 55^0 60^0 65^0 85^0 90^0 95^0 104^0 106^0 108^0 110^0 130^0 150^0]). Consider beamforming for the desired source located at 60^0. The rest of the 19 sources are interferences. The SNR for the desired source is 0 dB, while the INR of each interference is 20 dB. Sensor location selection

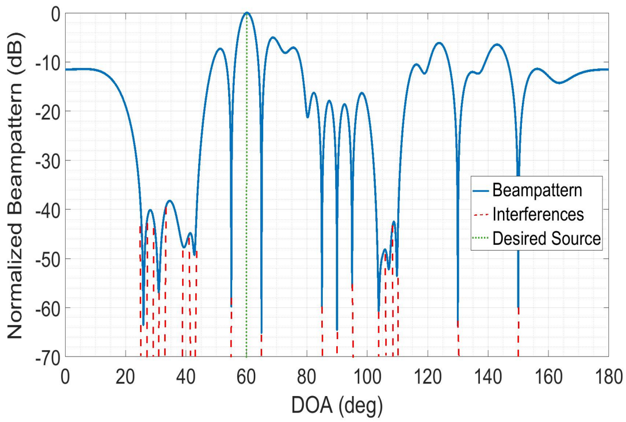

Fig. 2. Beampattern for desired source at 60^0

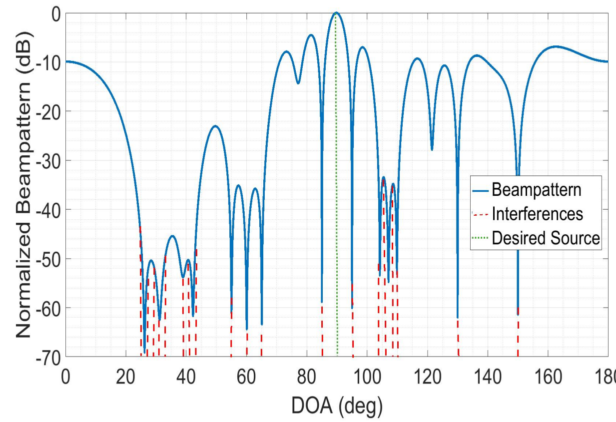

Fig. 3. Beampattern for desired source at 90^0

of $K = 16$ sensors from $N = 24$ total sensors, corresponds to 735471 different sparse array configurations that makes it computationally expensive to approach the problem by exhaustive search. Figure (1a) shows the array geometry found through exhaustive search that offers the maximum SINR of 10.5 dB. This array configuration is clearly missing quite a few correlation lags and is short of occupying the complete aperture.

To obtain a fully augmentable array configuration achieving MaxSINR, we fix nine sensor locations out of the sixteen sensors as a pre-fixed configuration. We choose a 9 element nested array configuration to satisfy the full augmentability constraint. Consequently, we optimize over the remaining 7 sensors to be placed in the 15 possible locations. Due to the few degrees of freedom, engaged by the pre-fixed sensors, the fully augmentable array can potentially deliver the maximum SINR of 9.7 dB (found through exhaustive search), that is around 0.8 dB lower than the maximum SINR without the augmentability constraint. The array configuration rendered by the SDR is shown in the Fig. (1b) (the red-filled circle is the pre-fixed nested array configuration, green-filled circle indicates sensor location selected and the gray-filled circle indicates sensor location not selected). This array configuration performs reasonably well having an output SINR of 8.6 dB, which is around 1 dB lower than the maximum SINR possible after the admittance of the augmentability constraint. It is to be noted that the 16 element compact uniform linear array with the optimal weights designed for the underlying scenario, can only manage an SINR of -3 dB. This is because the ULA can not resolve the source of interest effectively due to the lower array aperture. However, it is of interest to note that the bigger array aperture doest not warrant an improved SINR performance as the array configuration shown in Fig. (1c) offers an output SINR of -14 dB, although, it occupies the whole array aperture, thereby underscoring the importance of sparse array design.

Figure (2) depicts the beampattern of the optimal sparse array (Fig. (1b)) recovered through SDR. The considerably improved performance of the sparse array design can clearly be explained from the beampattern of the optimal sparse array configuration. The sparse array configuration uses its additional degrees of freedom to place the nulls at just the right locations to cancel the jammers efficiently, even if the jammers are more than the number of available sensors. We now consider the scenario where we are interested in the source at 90^0 and therefore would treat the source at 60^0 to be an unwanted interfering source. The optimum hybrid sparse array configuration for this scenario is shown in the Fig. (1d). Note that the pre-fixed configuration is kept the same, and the optimization carried over the remaining 7 sensors renders a different optimum sparse array than the previous scenario. The optimum sparse array configuration in this case shows comparable performance to the scenario for the desired source at 60^0, as is evident by the interference mitigation depicted from the beampattern in Fig. (3). We adopted a pre-fixed nested array for the above examples, however, other pre-fixed array topologies like minimum redundancy and coprime array can be considered to meet the full augmentability condition.

5. CONCLUSION

This paper considered fully augmentable sparse array configurations for maximizing the beamformer output SINR for under-determined design scenario. It proposed a hybrid sparse array design that simultaneously meets co-array and environment-dependent objectives. This design potentially offers reasonable SINR advantages as compared to sparse arrays that are freely designed without the augmentability constraint. The proposed methodology employs a subset of the available sensors to satisfy the fully augmentability condition while engaging the rest of sensors for achieving the highest SINR. We applied the partial re-weighting QCQP that effectively recovers the superior SINR performing hybrid sparse arrays in polynomial run times. The proposed algorithm and enumeration showed strong agreement in performance.

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