HADAMARD PRODUCT PERSPECTIVE ON SOURCE RESOLVABILITY OF SPATIAL-SMOOTHING-BASED SUBSPACE METHODS

Zai Yang[*] *and Petre Stoica*[†]

[*]School of Automation, Nanjing University of Science and Technology, Nanjing 210094, China
[†]Department of Information Technology, Uppsala University, Uppsala, SE 75105, Sweden

ABSTRACT

Spatial smoothing is a common preprocessing scheme for subspace methods that resolves their sensitivity to coherent sources. The source resolvability problem of spatial-smoothing-based subspace methods has been extensively investigated using different analysis techniques. In this paper, a unified Hadamard product technique is provided to recover these results. This is done by answering a long-standing question in linear algebra as to under what conditions the Hadamard product of two singular positive-semidefinite matrices is positive definite.

Index Terms— DOA estimation, subspace methods, spatial smoothing, source resolvability, Hadamard product.

1. INTRODUCTION

Direction-of-arrival (DOA) estimation using antenna arrays is of major interest in array processing [1]. Since the 1970s, a prominent class of methods known as subspace methods has been developed and extensively studied for DOA estimation. Two representatives of such methods are the multiple signal classification (MUSIC) and the estimation of parameters by rotational invariant techniques (ESPRIT) [2, 3]. In subspace methods, the direction parameters are estimated from a subspace obtained from the eigenstructure of the array output covariance matrix, resulting in high resolution and good statistical properties at a modest computational cost.[1]

A major drawback of subspace methods is their poor performance in a multipath environment in which coherent (or completely correlated) sources are present and the source covariance matrix is singular. To resolve this problem, spatial smoothing, which was pioneered by Evans *et al.* [5, 6], has been a commonly adopted preprocessing scheme in the past decades (see, e.g., [7–15]). Concisely speaking, spatial smoothing aims at obtaining a full-rank smoothed source covariance matrix by dividing the whole antenna array into several overlapping subarrays so that coherent sources can be resolved by subspace methods as in the case of noncoherent sources.

Note that spatial-smoothing-based subspace methods resolve coherent sources at the cost of reduced effective aperture. In other words, more antennas are required to resolve the same number of coherent sources. A fundamental question therefore is this: how many antennas are sufficient to guarantee the source resolvability? Several answers to this question have been derived using different analysis techniques. In particular, $K+1$ antennas suffice to resolve *any* K noncoherent sources [16]. In the case when coherent sources are present, Shan *et al.* [7] showed that $2K$ antennas suffice to resolve *any* K sources. Moreover, if the source covariance matrix has rank r, then it was shown in [17, 18] that $2K-r+1$ antennas are sufficient. Additionally, a modified spatial smoothing technique was developed which ensures the generic resolvability of *almost any* sources with an array size equal to $\left\lceil \frac{3}{2}K \right\rceil$ [8,9] (here $\lceil\cdot\rceil$ denotes the smallest integer no less than the argument).

In this paper, we formulate, apparently for the first time, the smoothed source covariance matrix as a Hadamard product and show that the source resolvability problem of spatial-smoothing-based subspace methods can be investigated by studying positive definiteness of the Hadamard product of two positive semidefinite matrices. We show that the result in [7] is a simple consequence of a classical result on the Hadamard product dating back to 1968. We recover the result in [17, 18] by answering a question asked explicitly in 1973 as to under what conditions the Hadamard product of two singular positive-semidefinite matrices is positive definite. In future studies, not only new progresses on source resolvability are expected using the novel Hadamard product perspective of this paper, but also new applications of our technical result on the Hadamard product are anticipated.

The research of the project was supported by the National Natural Science Foundation of China, under grant 61603187, and the Natural Science Foundation of Jiangsu Province, China, under grant BK20160845.

[1]As compared to the subspace methods, the main advantages of recently developed sparse methods for DOA estimation lie in the scenarios in which the former have difficulties, e.g., in the case of missing and heterogeneous data and in the case when the number of sources is unknown, see the review in [4].

2. PRELIMINARIES

2.1. The DOA Estimation Problem

Consider K narrowband, far-field sources impinging on an N-element uniform linear array (ULA) from directions θ_k, $k = 1, \ldots, K$. Suppose that L snapshots of the array output are acquired that can be expressed as [1]:

$$\boldsymbol{y}(l) = \sum_{k=1}^{K} \boldsymbol{a}(\omega_k) s_k(l) + \boldsymbol{e}(l), \quad l = 1, \ldots, L, \tag{1}$$

where $\boldsymbol{y}(l)$ is a vector of size N and denotes the lth snapshot, $s_k(l)$ is the kth source, $\boldsymbol{e}(l)$ is the noise vector, and $\boldsymbol{a}(\omega_k)$ is the steering vector defined as

$$\boldsymbol{a}(\omega_k) = \left[1, e^{i\omega_k}, \ldots, e^{i(N-1)\omega_k}\right]^T, \tag{2}$$

$$\omega_k = 2\pi \frac{d \cos\theta_k}{\lambda}, \tag{3}$$

where $\cdot^T$ is the matrix transpose, d represents the distance between adjacent antennas, and λ is the wavelength. Let $\boldsymbol{A}$ be the $N \times K$ Vandermonde matrix defined as

$$\boldsymbol{A} = [\boldsymbol{a}(\omega_1), \ldots, \boldsymbol{a}(\omega_K)] \tag{4}$$

and $\boldsymbol{s}(l) = [s_1(l), \ldots, s_K(l)]^T$. The data model in (1) can then be written as:

$$\boldsymbol{y}(l) = \boldsymbol{A}\boldsymbol{s}(l) + \boldsymbol{e}(l), \quad l = 1, \ldots, L. \tag{5}$$

To properly define the DOA estimation problem, we assume that $N > K$ and $d \leq \frac{\lambda}{2}$. It follows that the poles $\left\{z_k = e^{i\omega_k}\right\}$ of the Vandermonde matrix $\boldsymbol{A}$ are distinct and that $\boldsymbol{A}$ has full column rank.

2.2. Subspace Methods and Spatial Smoothing

Subspace methods are derived based on the standard assumption that the sources and the noise are stationary and uncorrelated zero-mean random processes so that the array output covariance matrix $\boldsymbol{R}$ is given as:

$$\boldsymbol{R} = \mathbb{E}\boldsymbol{y}(l)\boldsymbol{y}^H(l) = \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}^H + \sigma^2\boldsymbol{I}, \tag{6}$$

where $\cdot^H$ is the Hermitian transpose, $\boldsymbol{\Sigma} = \mathbb{E}\boldsymbol{s}(l)\boldsymbol{s}^H(l)$ is the source covariance matrix, σ^2 is the noise power, and $\boldsymbol{I}$ is an identity matrix. They utilize the eigenstructure of $\boldsymbol{R}$ and resolve all K sources if and only if

$$N \geq K + 1, \tag{7}$$

$$\boldsymbol{\Sigma} > \boldsymbol{0}. \tag{8}$$

Here $\boldsymbol{\Sigma} > \boldsymbol{0}$ means that $\boldsymbol{\Sigma}$ is positive definite (and $\boldsymbol{\Sigma} \geq \boldsymbol{0}$ means that $\boldsymbol{\Sigma}$ is positive semidefinite). The necessity of the condition in (8) implies that subspace methods fail to resolve coherent sources for which $\boldsymbol{\Sigma}$ is singular.

Spatial smoothing is a technique that ensures the applicability of subspace methods in the presence of coherent sources. It starts with dividing the N-element ULA into P overlapping subarrays of M elements, with

$$N = P + M - 1, \tag{9}$$

where the pth subarray starts with the pth antenna. It follows that the pth subarray output covariance matrix is an order-M principal submatrix of $\boldsymbol{R}$ that is given by

$$\boldsymbol{R}_p = \boldsymbol{A}_M \boldsymbol{Z}^{p-1} \boldsymbol{\Sigma} \boldsymbol{Z}^{1-p} \boldsymbol{A}_M^H + \sigma^2 \boldsymbol{I}, \tag{10}$$

where $\boldsymbol{A}_M$ is as defined in (4) but with only M rows, and $\boldsymbol{Z}$ is a diagonal matrix with the poles $\{z_k\}$ on the diagonal. The smoothed covariance matrix $\widetilde{\boldsymbol{R}}$ is obtained as the mean of the subarray output covariance matrices that is given by

$$\widetilde{\boldsymbol{R}} = \frac{1}{P} \sum_{p=1}^{P} \boldsymbol{R}_p = \frac{1}{P} \boldsymbol{A}_M \widetilde{\boldsymbol{\Sigma}} \boldsymbol{A}_M^H + \sigma^2 \boldsymbol{I}, \tag{11}$$

where

$$\widetilde{\boldsymbol{\Sigma}} = \sum_{p=1}^{P} \boldsymbol{Z}^{p-1} \boldsymbol{\Sigma} \boldsymbol{Z}^{1-p} \tag{12}$$

denotes the smoothed source covariance matrix (up to the scaling factor $\frac{1}{P}$). Note that $\widetilde{\boldsymbol{R}}$ has the form of $\boldsymbol{R}$ in (6). According to the discussions above, therefore, all sources can be resolved by spatial-smoothing-based subspace methods from $\widetilde{\boldsymbol{R}}$ in (11) if and only if

$$M \geq K + 1, \tag{13}$$

$$\widetilde{\boldsymbol{\Sigma}} > \boldsymbol{0}. \tag{14}$$

3. A NOVEL HADAMARD PRODUCT PERSPECTIVE

The source resolvability problem of spatial-smoothing-based subspace methods concerns under what conditions on N the conditions in (13) and (14) are satisfied. Inserting (13) into (9), we have that

$$N \geq K + P. \tag{15}$$

Therefore, this problem can be investigated by studying how large P should be so that (14) is satisfied. To this end, we provide a new Hadamard product perspective for this problem which is crucial in our analysis.

We will use the following identity:

$$\operatorname{diag}(\boldsymbol{a})\, \boldsymbol{C} \operatorname{diag}(\boldsymbol{b}) = \boldsymbol{C} \odot \boldsymbol{a}\boldsymbol{b}^T, \tag{16}$$

which holds for vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and a matrix $\boldsymbol{C}$ of proper dimensions and can easily be shown. In (16), $\operatorname{diag}(\boldsymbol{a})$ is a diagonal matrix with the entries of $\boldsymbol{a}$ on the diagonal and $\odot$ is the Hadamard (or elementwise) product.

Let us define $\boldsymbol{z}^j = \left[z_1^j, \ldots, z_K^j\right]^T$ for integer j. Making use of (16) in (12), we obtain that

$$\begin{aligned}\widetilde{\boldsymbol{\Sigma}} &= \boldsymbol{\Sigma} \odot \sum_{p=1}^{P} \boldsymbol{z}^{p-1}\left(\boldsymbol{z}^{p-1}\right)^H \\ &= \boldsymbol{\Sigma} \odot \left[\boldsymbol{z}^0, \ldots, \boldsymbol{z}^{P-1}\right]\left[\boldsymbol{z}^0, \ldots, \boldsymbol{z}^{P-1}\right]^H \\ &= \boldsymbol{\Sigma} \odot \boldsymbol{A}_P^T \boldsymbol{A}_P^*,\end{aligned} \tag{17}$$

where $\boldsymbol{A}_P$ is defined as $\boldsymbol{A}_M$ and $\cdot^*$ denotes the complex conjugate operator. Therefore, the smoothed source covariance matrix $\widetilde{\boldsymbol{\Sigma}}$ is formulated as the Hadamard product of two positive semidefinite matrices $\boldsymbol{\Sigma}$ and $\boldsymbol{A}_P^T\boldsymbol{A}_P^*$, and we need to study under what conditions on P the Hadamard product is positive definite. Here it is worth noting that both $\boldsymbol{\Sigma}$ and $\boldsymbol{A}_P^T\boldsymbol{A}_P^*$ can be singular.

4. PREVIOUS RESULTS ON THE HADAMARD PRODUCT AND IMPLICATIONS

Positive definiteness of the Hadamard product is exactly what is concerned by Schur product theorem dating back to the early twentieth century [19]. It is formally stated below.

Theorem 1 *If $\boldsymbol{B} \geq \boldsymbol{0}$ and $\boldsymbol{C} \geq \boldsymbol{0}$, then $\boldsymbol{B} \odot \boldsymbol{C} \geq \boldsymbol{0}$; if $\boldsymbol{B} > \boldsymbol{0}$ and $\boldsymbol{C} > \boldsymbol{0}$, then $\boldsymbol{B} \odot \boldsymbol{C} > \boldsymbol{0}$.*

The Schur product theorem has been strengthened in different ways. When the positive definiteness is the main concern, to the best of our knowledge, the state-of-the-art result is due to Ballantine [20].

Theorem 2 *If either $\boldsymbol{B} \geq \boldsymbol{0}$ or $\boldsymbol{C} \geq \boldsymbol{0}$ is positive definite and the other matrix has a positive diagonal, then $\boldsymbol{B} \odot \boldsymbol{C} > \boldsymbol{0}$.*

Next, we study the positive definiteness of $\widetilde{\boldsymbol{\Sigma}}$ in (17) by making use of Theorem 2. Evidently, both $\boldsymbol{\Sigma}$ and $\boldsymbol{A}_P^T\boldsymbol{A}_P^*$ are positive semidefinite and have positive diagonals. According to Theorem 2, therefore, either $\boldsymbol{\Sigma} > \boldsymbol{0}$ or $\boldsymbol{A}_P^T\boldsymbol{A}_P^* > \boldsymbol{0}$ guarantees the positive definiteness of $\widetilde{\boldsymbol{\Sigma}}$.

First consider the case of $\boldsymbol{\Sigma} > \boldsymbol{0}$. This means that all sources are noncoherent and spatial smoothing is not required. Inserting the trivial identity $P = 1$ into (15), we obtain the well-known bound in this case that $N \geq K + 1$.

We next consider the other case in which $\boldsymbol{\Sigma}$ is singular and $\boldsymbol{A}_P^T\boldsymbol{A}_P^* > \boldsymbol{0}$. This means that coherent sources are present. Moreover, the Vandermonde matrix $\boldsymbol{A}_P$ has full column rank K. Making use of the following identity that is well-known for a Vandermonde matrix:

$$\operatorname{rank}\left(\boldsymbol{A}_P\right) = \min\left(P,\ K\right), \tag{18}$$

we have that $P \geq K$. Inserting this bound into (15), we obtain that

$$N \geq 2K. \tag{19}$$

Therefore, this recovers the result of [7] via simple arguments.

5. A NEW RESULT ON THE HADAMARD PRODUCT

To possibly improve the lower bound in (19) in the presence of coherent sources, we need to improve the lower bound on P. If this can be done, then both factors of the Hadamard product in (17) will be singular. Therefore, we need to study the problem as to under what conditions the Hadamard product of two singular positive-semidefinite matrices is positive definite. In fact, this question was asked explicitly by Styan [21] nearly half a century ago and was studied in [22, p. 214] as well. It was shown in [21] that the answer to this question is always negative when the matrix order is one (the scalar case) or two. When the order is at least three, the Hadamard product can be nonsingular; however, no sufficiently general answers to this question have been provided. This section is devoted to providing such an answer.

We start with the introduction of the Kruskal rank or k-rank that was implicitly defined by Kruskal [23] and whose name was coined in [24].

Definition 1 *The k-rank of matrix $\boldsymbol{B}$, denoted by $k_{\boldsymbol{B}}$, equals k if and only if* any *k columns of $\boldsymbol{B}$ are linearly independent, but either $\boldsymbol{B}$ has exactly k columns, or $\boldsymbol{B}$ has* at least one *collection of $k+1$ linearly dependent columns.*

By definition, it holds for any matrix $\boldsymbol{B}$ that $k_{\boldsymbol{B}} \leq \operatorname{rank}(\boldsymbol{B})$. Our main result of this section is given below.

Theorem 3 *If $\boldsymbol{B} \geq \boldsymbol{0}$ and $\boldsymbol{C} \geq \boldsymbol{0}$, both of order F, have positive diagonals and*

$$\max\left(\operatorname{rank}\left(\boldsymbol{B}\right) + k_{\boldsymbol{C}},\ \operatorname{rank}\left(\boldsymbol{C}\right) + k_{\boldsymbol{B}}\right) \geq F + 1, \tag{20}$$

then $\boldsymbol{B} \odot \boldsymbol{C} > \boldsymbol{0}$.

Ballantine's result in Theorem 2 is generalized by Theorem 3. The former corresponds to the special case of Theorem 3 in which $\operatorname{rank}(\boldsymbol{B}) = F$ and $k_{\boldsymbol{C}} \geq 1$, or $\operatorname{rank}(\boldsymbol{C}) = F$ and $k_{\boldsymbol{B}} \geq 1$. To satisfy the assumptions of Theorem 3, neither $\boldsymbol{B}$ nor $\boldsymbol{C}$ has to be positive definite; in particular, this provides an appropriate answer to the question asked by Styan.

Next, we give a proof of Theorem 3. Our proof uses the following connection between the Hadamard and the Khatri-Rao (or columnwise Kronecker) products [22, Proposition 6.4.2]:

$$\left(\boldsymbol{D} \star \boldsymbol{E}\right)^H \left(\boldsymbol{D} \star \boldsymbol{E}\right) = \boldsymbol{D}^H\boldsymbol{D} \odot \boldsymbol{E}^H\boldsymbol{E}. \tag{21}$$

The Khatri-Rao product $\boldsymbol{D} \star \boldsymbol{E}$ is defined as:

$$\boldsymbol{D} \star \boldsymbol{E} = \left[d_{ij}\boldsymbol{e}_j\right], \tag{22}$$

where d_{ij} is the (i,j)-th entry of $\boldsymbol{D}$ and $\boldsymbol{e}_j$ is the jth column of $\boldsymbol{E}$.

To proceed we present the following lemma whose proof is omitted due to the page limit.

Lemma 1 *For any matrix $\boldsymbol{D}$,*

$$k_{\boldsymbol{D}^H\boldsymbol{D}} = k_{\boldsymbol{D}}. \tag{23}$$

Instead of directly proceeding to a proof of Theorem 3, we first see what can be implied by Theorem 3 by making use of (21) and Lemma 1. For matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ that satisfy the assumptions of Theorem 3, there exist matrices $\boldsymbol{D}$ and $\boldsymbol{E}$, both with F columns, satisfying $\boldsymbol{B} = \boldsymbol{D}^H\boldsymbol{D}$ and $\boldsymbol{C} = \boldsymbol{E}^H\boldsymbol{E}$, and neither $\boldsymbol{D}$ nor $\boldsymbol{E}$ has zero columns. By Lemma 1 we have that $k_{\boldsymbol{B}} = k_{\boldsymbol{D}}$ and $k_{\boldsymbol{C}} = k_{\boldsymbol{E}}$. Substituting these identities and $\text{rank}(\boldsymbol{B}) = \text{rank}(\boldsymbol{D})$ and $\text{rank}(\boldsymbol{C}) = \text{rank}(\boldsymbol{E})$ into the condition in (20), we have that

$$\max\left(\text{rank}(\boldsymbol{D}) + k_{\boldsymbol{E}},\ \text{rank}(\boldsymbol{E}) + k_{\boldsymbol{D}}\right) \geq F + 1. \tag{24}$$

Moreover, using the identity in (21), $\boldsymbol{B} \odot \boldsymbol{C} > \boldsymbol{0}$ implies that $\boldsymbol{D} \star \boldsymbol{E}$ has full column rank. Consequently, Theorem 3 implies the following theorem.

Theorem 4 *If matrices $\boldsymbol{D}$ and $\boldsymbol{E}$, both with F columns, have no zero columns and satisfy* (24)*, then $\boldsymbol{D} \star \boldsymbol{E}$ has full column rank.*

By arguments similar to those above, it can easily be seen that Theorem 4 also implies Theorem 3. Therefore, to prove Theorem 3, it suffices to prove Theorem 4. Readers are referred to [25] for a proof of Theorem 4 where the result was apparently given for the first time.

6. IMPLICATION OF THE NEW RESULT

In this section we recover the source resolvability result in [17, 18] by applying Theorem 3. To this end, let us recall (17) and note that a matrix and its conjugate have the same rank and k-rank. Applying Lemma 1, we have that

$$k_{\boldsymbol{A}_P^T\boldsymbol{A}_P^*} = k_{\boldsymbol{A}_P^*} = k_{\boldsymbol{A}_P} = \min(P, K), \tag{25}$$

where the last equality follows from the fact that any square Vandermonde matrix with distinct poles is nonsingular. Applying Theorem 3 and using (25), it can readily be shown that the smoothed source covariance matrix $\widetilde{\boldsymbol{\Sigma}}$ in (17) is positive definite if

$$\text{rank}(\boldsymbol{\Sigma}) + k_{\boldsymbol{A}_P^T\boldsymbol{A}_P^*} = \text{rank}(\boldsymbol{\Sigma}) + \min(P, K) \geq K + 1, \tag{26}$$

or equivalently, if

$$\min(P, K) \geq K - \text{rank}(\boldsymbol{\Sigma}) + 1. \tag{27}$$

Because the inequality $K \geq K - \text{rank}(\boldsymbol{\Sigma}) + 1$ always holds, the condition in (27) can be simplified to

$$P \geq K - \text{rank}(\boldsymbol{\Sigma}) + 1. \tag{28}$$

Inserting (28) into (15), we obtain that

$$N \geq 2K - \text{rank}(\boldsymbol{\Sigma}) + 1, \tag{29}$$

which recovers the result in [17, 18]. Note that our arguments are much simpler than those in the cited papers.

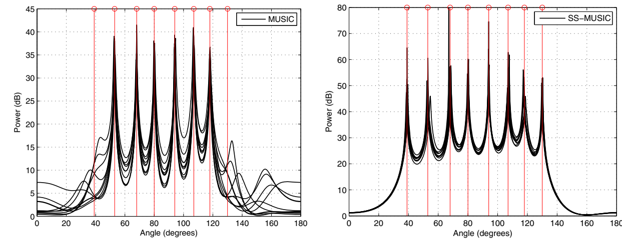

Fig. 1. Power spectra of MUSIC (left) and spatial-smoothing-based MUSIC (right) with a ten-element ULA, in ten Monte Carlo runs. The first and the last sources are coherent. The eight true DOAs are indicated by vertical lines.

7. NUMERICAL RESULTS

We provide numerical results to complement our theoretical analysis. In particular, we consider $K = 8$ sources with identical powers that equal the noise power. The first seven sources are independently generated from an identical Gaussian distribution. The last source is a replica of the first up to a global phase. This means that the source covariance matrix has rank $r = 7$. Consequently, according to our analysis, a ULA consisting of $N = 2K - r + 1 = 10$ antennas that are mutually separated by half a wavelength guarantees source resolvability. This increases the minimum array size $K+1 = 9$ just by one and is considerably smaller than the requirement $2K = 16$ of [7].

We consider $L = 500$ snapshots and the MUSIC algorithm for DOA estimation. The power spectra of MUSIC and spatial-smoothing-based MUSIC in ten Monte Carlo runs are presented in Fig. 1. It can be seen that MUSIC fails to resolve the coherent sources, as expected. All the sources are well resolved by the spatial-smoothing-based MUSIC algorithm, corroborating our analysis results.

8. CONCLUSION

In this paper, a novel Hadamard product perspective was provided on the source resolvability problem of spatial-smoothing-based subspace methods. By studying positive definiteness of the Hadamard product and answering an open question dating back to 1973, we recovered several previous results on the source resolvability problem by means of simplified arguments. We expect that the novel Hadamard product approach of this paper can be used to provide further insights into results about the source resolvability problem, and that our main technical result in Theorem 3 will be useful to other areas as well.

9. REFERENCES

[1] P. Stoica and R. L. Moses, *Spectral analysis of signals*. Pearson/Prentice Hall Upper Saddle River, NJ, 2005.

[2] R. Schmidt, “A signal subspace approach to multiple emitter location spectral estimation,” Ph.D. dissertation, Stanford University, 1981.

[3] A. Paulraj, R. Roy, and T. Kailath, “A subspace rotation approach to signal parameter estimation,” *Proceedings of the IEEE*, vol. 74, no. 7, pp. 1044–1046, 1986.

[4] Z. Yang, J. Li, P. Stoica, and L. Xie, “Sparse methods for direction-of-arrival estimation,” *Academic Press Library in Signal Processing Volume 7 (R. Chellappa and S. Theodoridis,* Eds.*)*, pp. 509–581, 2018.

[5] J. E. Evans, “High resolution angular spectrum estimation technique for terrain scattering analysis and angle of arrival estimation,” in *1st IEEE ASSP Workshop Spectral Estimation*, 1981, pp. 134–139.

[6] J. E. Evans, J. R. Johnson, and D. Sun, “Application of advanced signal processing techniques to angle of arrival estimation in ATC navigation and surveillance systems,” Lincoln Laboratory, Tech. Rep., 1982.

[7] T.-J. Shan, M. Wax, and T. Kailath, “On spatial smoothing for direction-of-arrival estimation of coherent signals,” *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 33, no. 4, pp. 806–811, 1985.

[8] R. T. Williams, S. Prasad, A. K. Mahalanabis, and L. H. Sibul, “An improved spatial smoothing technique for bearing estimation in a multipath environment,” *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 36, no. 4, pp. 425–432, 1988.

[9] S. U. Pillai and B. H. Kwon, “Forward/backward spatial smoothing techniques for coherent signal identification,” *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 37, no. 1, pp. 8–15, 1989.

[10] K.-C. Tan and G.-L. Oh, “Estimating directions-of-arrival of coherent signals in unknown correlated noise via spatial smoothing,” *IEEE Transactions on Signal Processing*, vol. 45, no. 4, pp. 1087–1091, 1997.

[11] A. N. Lemma, A.-J. Van Der Veen, and E. F. Deprettere, “Analysis of joint angle-frequency estimation using ESPRIT,” *IEEE Transactions on Signal Processing*, vol. 51, no. 5, pp. 1264–1283, 2003.

[12] P. Pal and P. Vaidyanathan, “Nested arrays: A novel approach to array processing with enhanced degrees of freedom,” *IEEE Transactions on Signal Processing*, vol. 58, no. 8, pp. 4167–4181, 2010.

[13] B. Liao and S.-C. Chan, “DOA estimation of coherent signals for uniform linear arrays with mutual coupling,” in *IEEE International Symposium on Circuits and Systems (ISCAS)*, 2011, pp. 377–380.

[14] F. Liu, J. Wang, C. Sun, and R. Du, “Spatial differencing method for DOA estimation under the coexistence of both uncorrelated and coherent signals,” *IEEE Transactions on Antennas and Propagation*, vol. 60, no. 4, pp. 2052–2062, 2012.

[15] C.-L. Liu and P. Vaidyanathan, “Remarks on the spatial smoothing step in coarray MUSIC,” *IEEE Signal Processing Letters*, vol. 22, no. 9, pp. 1438–1442, 2015.

[16] V. F. Pisarenko, “The retrieval of harmonics from a covariance function,” *Geophysical Journal International*, vol. 33, no. 3, pp. 347–366, 1973.

[17] Y. Bresler and A. Macovski, “On the number of signals resolvable by a uniform linear array,” *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 34, no. 6, pp. 1361–1375, 1986.

[18] A. Nehorai, D. Starer, and P. Stoica, “Direction-of-arrival estimation in applications with multipath and few snapshots,” *Circuits, Systems and Signal Processing*, vol. 10, no. 3, pp. 327–342, 1991.

[19] J. Schur, “Bemerkungen zur theorie der beschränkten bilinearformen mit unendlich vielen veränderlichen.” *Journal für die reine und Angewandte Mathematik*, vol. 140, pp. 1–28, 1911.

[20] C. S. Ballantine, “On the Hadamard product,” *Mathematische Zeitschrift*, vol. 105, no. 5, pp. 365–366, 1968.

[21] G. P. H. Styan, “Hadamard products and multivariate statistical analysis,” *Linear Algebra and Its Applications*, vol. 6, pp. 217–240, 1973.

[22] C. R. Rao and M. B. Rao, *Matrix algebra and its applications to statistics and econometrics*. World Scientific, 1998.

[23] J. B. Kruskal, “Three-way arrays: Rank and uniqueness of trilinear decompositions, with application to arithmetic complexity and statistics,” *Linear Algebra and Its Applications*, vol. 18, pp. 95–138, 1977.

[24] R. A. Harshman and M. E. Lundy, “The PARAFAC model for three-way factor analysis and multidimensional scaling,” *Research Methods for Multimode Data Analysis*, vol. 46, pp. 122–215, 1984.

[25] X. Guo, S. Miron, D. Brie, S. Zhu, and X. Liao, “A CANDECOMP/PARAFAC perspective on uniqueness of DOA estimation using a vector sensor array,” *IEEE Transactions on Signal Processing*, vol. 59, no. 7, pp. 3475–3481, 2011.