A HISTORY-BASED STOPPING CRITERION IN RECURSIVE BAYESIAN STATE ESTIMATION

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ABSTRACT

In dynamic state-space models, the state can be estimated through recursive computation of the posterior distribution of the state given all measurements. In scenarios where active sensing/querying is possible, a hard decision is made when the state posterior achieves a pre-set confidence threshold. This mandate to meet a hard threshold may sometimes unnecessarily require more queries. In application domains where sensing/querying cost is of concern, some potential accuracy may be sacrificed for greater gains in sensing cost. In this paper, we (a) propose a criterion based on a linear combination of state posterior and its changes, (b) show that for discrete-valued state estimation scenarios the proposed objective is more likely to sort correct and incorrect estimates appropriately compared to just looking at the posterior, and finally (c) demonstrate that the method can lead to significant human intent estimation speed increase without significant loss of accuracy in a brain-computer interface application.

Index Terms— Recursive Bayesian state estimation, Stopping criterion, Active sequential sensing or querying, Information transfer rate

1. INTRODUCTION

State estimation for discrete-time stochastic dynamic systems is still one of the major research topics, particularly in control and signal processing applications. Majority of proposed state estimation methods are based on a recursive framework [1]. In the recursive state estimation (RSE), the system queries the environment to acquire observations (evidence) about the state. Generally, for state estimation, obtained evidence is fused in a Bayesian manner with a prior information about the state, and the state estimation and the estimation confidence depend on the posterior distribution recursively calculated over the state. To achieve a high confidence and to decrease ambiguity in the state estimation, the system probes the environment through multiple iterations of queries [2, 3]. In general such evidence querying continues recursively until the maximum of the posterior distribution reaches up to a pre-set confidence threshold which is referred as maximum a-posteriori (MAP) estimation [2].

Recursive estimation procedure is not always precise and fast due to noisy evidence/observations. To compensate for the negative effects of the noise, the confidence threshold is set to a high value to decrease ambiguity in the state estimation. However, to reach to such high confidence levels, systems sacrifice budget/time for small increments in the maximum of the posterior of a particular estimate. In many applications, querying is expensive (e.g. brain-computer interfaces (BCI) [4, 5, 6], target-tracking [2, 7], communication network [8, 9], clinical studies [10]); hence the recursive Bayesian state estimation (RBSE) approaches need to balance between accuracy and speed. In RBSE, accuracy and speed could be balanced through optimum query selection and optimum stopping criterion.

There are many studies on query optimization. It has been shown in [11] that making decisions based on the current belief (exploitation), e.g. *N-best* selection [11, 12] will not always provide the best performance in the state estimation due to the misleading prior knowledge, transition noise, or changes in the environment distributions. Authors in [13] have shown that considering the weighted posterior changes of the state is informative and accelerates the estimation process. On the other hand, it is a major challenge to design a stopping criterion to terminate the recursion of the state posterior to make a final decision. There exists two approaches of stopping criteria; one approach is to stop whenever the algorithm objective exceeds a pre-set value that could be (i) the maximum number of recursions [14], (ii) the minimum value of the posterior probability (confidence level) [5, 6], (iii) maximum/overall uncertainty [15]. This fixed value is typically chosen based on the training dataset or computational resources which is not an optimal way to terminate the estimation procedure, particularly for noisy observations of a dy-

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namic system. Another approach of stopping criterion is estimating the convergence rate of the objective term [16, 17, 18]. The main problem of this approach is that in RBSE, there is no analytical way to estimate the convergence rate for the posterior when its trajectory is unknown. Unfortunately, both approaches have their own disadvantage; not taking the knowledge from history of posterior changes into account.

In this paper, we develop a novel stopping criterion design method that is based on the posterior distribution defined over the state and a history-based objective which we denote as *Momentum*. More specifically, our novel contribution through this method is using a linear combination of the state posterior and its average changes over multiple recursions. We also show that under certain assumptions, our method has theoretical guarantees to achieve the state with higher probability in compared to pre-defined threshold-based approaches. To assess the performance of the stopping criterion in a real-world RBSE problem, we utilize an actual human-in-the-loop typing scheme employing a language-model-assisted Electroencephalogram (EEG)-based BCI typing system called RSVP Keyboard.

2. METHOD

In this section, we explain the state estimation in the framework of a RBSE problem. We use a *sequence* as a term to denote a recursion during estimation process. We denote state with σ, query and evidence at sequence s with Φ_s and ε_s respectively. The state σ is an element of a dictionary denoted by $\mathcal{D}$. We also use $\mathcal{H}_s$ notation to indicate a set of evidence and queries up to sequence s, e.g. $\mathcal{H}_s = \{\varepsilon_{1:s}, \Phi_{1:s}, \mathcal{H}_0\}$, where $\mathcal{H}_0$ is the prior information. We also consider the case where each sequence consists multiple queries, where each query is previously determined before the querying starts. Calling each independent query and its corresponding evidence as *trial*, it is assumed each sequence consists of multiple independent trials. We represent $p(\varepsilon_s|\sigma, \Phi_s)$ with class conditional evidence distributions as presented in our previous work [11]. This allows us to assume all evidence is observed from noisy channels of two separate unimodal probability distributions conditioned on class definition including conditioned on target class (state) and conditioned on non-target class.

In the MAP estimation, system picks an estimate with highest posterior probability value conditioned on the evidence [2]. Accordingly, the number of sequences is determined by the confidence level called minimum posterior probability, τ. The constraint, conventionally, is cast on the posterior as a hyper-parameter to prevent (premature) incorrect decisions caused by the noisy observations and hence forces system to continue querying until finding a feasible solution. This procedure can be formulated as the following;

$$\hat{\sigma} = \arg\max_{\sigma \in \mathcal{D}} p(\sigma|\mathcal{H}_S) \quad \text{s.t. } p(\sigma|\mathcal{H}_S) \geq \tau \tag{1}$$

Using the Bayes' rule and assuming noisy observations are independent and identically distributed, we can rewrite the posterior as,

$$p(\sigma|\varepsilon_{1:S}, \Phi_{1:S}, \mathcal{H}_0) = p(\sigma|\mathcal{H}_0) \prod_{s=1}^{S} \frac{p(\varepsilon_s|\sigma, \Phi_s)}{p(\varepsilon_s|\Phi_s)} \tag{2}$$

where $p(\sigma|\mathcal{H}_0) = p(\sigma)$ is the prior knowledge. Hence, confidence is affected by two sources; prior information and the evidence information. Assuming by sufficient querying the environment, there exists a query sequence for an estimate that makes its posterior converge to 1 and consequently, for the other estimates the corresponding posteriors converge to 0. We can represent this, using the following representation;

$$\forall \sigma \; \exists \; \Phi_{1:s} \text{ s.t. }, \lim_{s \to \infty} p(\sigma|\mathcal{H}_s) \in \{0, 1\} \tag{3}$$

For an estimate, $p(\sigma|\mathcal{H}_s) \to 1$ for $p(\sigma) < 1$, indicates that the posterior increases over recursions. Whereas, for the other estimates posterior decreases. Which states that probability mass shifts towards one direction in expectation. Accordingly, the probability displacement of the an estimate at sequence s is informative. To take advantage of probability changes, we propose a stopping condition described as a linear combination of posterior and average of posterior displacement for each particular estimate. We introduce a new term called *Momentum* that is function of posterior changes across sequences, which is represented as the following;

$$m_s(a) = p(a|\mathcal{H}_{s-1}) \left(\log p(a|\mathcal{H}_s) - \log p(a|\mathcal{H}_{s-1})\right) \tag{4}$$

where $p(a|\mathcal{H}_s) = p(a|\varepsilon_s, \Phi_s, \mathcal{H}_{s-1})$. Since ε_s is already observed, posterior changes can be directly computed in the RBSE framework. Additionally, we define $m(\Phi|\mathcal{H}_0) = 0$, $\forall \Phi$, due to absence of history. The average momentum over sequences is calculated as the following;

$$M_S(a) = \frac{1}{S} \sum_{s=1}^{S} m_s(a) \tag{5}$$

It should be noted that the average momentum at sequence S, $M_S(a)$, is non-negative when $p(a|\mathcal{H}_s)/p(a|\mathcal{H}_{s-1}) \geq 1$, which means a is the desired state. Otherwise, $M_S(\cdot)$ is negative. By linearly combining (5) and (1), we can define a new stopping criterion for MAP estimation in the following;

$$\begin{aligned} \hat{\sigma} &= \arg\max_{\sigma \in \mathcal{D}} p(\sigma|\mathcal{H}_S) \\ &\text{s.t. } p(\sigma|\mathcal{H}_S) + \alpha M_S(\sigma) > \tau \end{aligned} \tag{6}$$

where α is a hyperparameter. This formulation allows us to capture the estimate better since, the state will increase its probability mass over time. At the beginning of the recursive process, if the state has a low prior probability in compared with the other estimates, the system benefits from momentum.

Increasing α forces the system to rely more on history of the posterior changes rather than the posterior itself.

The stopping criterion defined in (6) including the history-based objective allows the system to estimate state σ with higher probability compared to criterion defined in (1). In fact, using (6), the system is considering both posterior and the average speed of it's changes at each sequence. We proved our claim through the following proposition.

Proposition 1. *Given $a, b \in \mathcal{D}$ where $a \neq b$ and $\alpha \geq 0$, if $\exists\, \mathcal{H}_{s-1}$ s.t. $p(a|\mathcal{H}_{s-1}) < p(b|\mathcal{H}_{s-1})$, then;*

$$\begin{aligned} &p\left(p(a|\mathcal{H}_s) + \alpha M_s(a) > p(b|\mathcal{H}_s) + \alpha M_s(b)\right) \\ &\qquad \geq p\left(p(a|\mathcal{H}_s) > p(b|\mathcal{H}_s)\right) \end{aligned}$$

Proof. Without loss of generality, we picked $\alpha = 1$. By using the mathematical identity on sum of random variables, $p(A + B > C + D) \geq p(A > C|B > D)p(B > D)$, $\forall A, B, C, D$, we can state the following;

$$\begin{aligned} &p\left(p(a|\mathcal{H}_s) + M_s(a) > p(b|\mathcal{H}_s) + M_s(b)\right) \\ &\quad \geq p\left(M_s(a) > M_s(b)|p(a|\mathcal{H}_s) > p(b|\mathcal{H}_s)\right) \\ &\qquad p\left(p(a|\mathcal{H}_s) > p(b|\mathcal{H}_s)\right) \end{aligned}$$

Now assume $M_s(a) \leq M_s(b)$, where $p(a|\mathcal{H}_s) > p(b|\mathcal{H}_s)$.

$$\begin{aligned} &\log p(a|\mathcal{H}_s) > \log p(b|\mathcal{H}_s) \\ &\log \frac{p(a|\mathcal{H}_s)}{p(a|\mathcal{H}_{s-1})} > \log \frac{p(b|\mathcal{H}_s)}{p(b|\mathcal{H}_{s-1})} \rightarrow M_s(a) > M_s(b) \end{aligned}$$

which is a contradiction. Hence, if $p(a|\mathcal{H}_s) > p(b|\mathcal{H}_s)$ then, $M_s(a) > M_s(b)$. This allows us to use the following relation;

$$p\left(M_s(a) > M_s(b)|p(a|\mathcal{H}_s) > p(b|\mathcal{H}_s)\right) = 1$$

and hence;

$$\begin{aligned} &p\left(p(a|\mathcal{H}_s) + M_s(a) > p(b|\mathcal{H}_s) + M_s(b)\right) \\ &\qquad \geq p\left(p(a|\mathcal{H}_s) > p(b|\mathcal{H}_s)\right) \end{aligned}$$

Thus, the condition holds. □

According to the above proposition, the constraint defined in (6) has a higher probability of capturing the state compared to the constraint in (1), independent of α. Therefore, we can estimate the state with a higher probability given the same set of observations (same history). Observe that, in RBSE, the constraint also determines the number of sequences. The analysis of the number of sequences versus accuracy is not trivial. Therefore, we empirically decide on the α value that decreases number of querying sequences without decreasing accuracy significantly. We will discuss our decision on α in the following section.

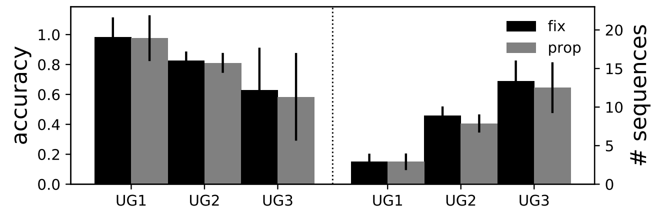

Fig. 1. The average accuracy and average number of sequences spent for estimation. UG1, UG2, UG3 represent user groups with AUC belongs to [%90, %100], [%78, %90], and [%50, %78] interval, respectively. It shows that number of sequences increases and accuracy decreases with respect to the user performance decrements. *'fix'* and *'prop'* represent stopping criteria obtained from (1) and (6), respectively.

3. RESULTS

To assess the performance of the proposed stopping criterion, we used a language-model-assisted EEG based BCI typing system called RSVP Keyboard [4]. In such typing systems, users are instructed to type one symbol at a time. We refer to the user-intended symbol as a state in the RBSE framework, which is an element of a finite set of alphabet (dictionary). The system queries the user with letters from the alphabet and obtains EEG evidence as noisy observations. To examine the significance of our approach in practice, we used 35 calibration session data obtained from 15 users across different sessions. Subjects were recruited under IRB-130107 approved by Northeastern University. EEG signals were acquired from 16 sensors according to International 10-20 System locations: Fp1, Fp2, F3, F4, Fz, Fc1, Fc2, Cz, P1, P2, C1, C2, Cp3, Cp4, P5 and P6. We also used an FIR linear-phase bandpass filter passing [1.5,42] Hz with zero DC gain and a notch filter at 60Hz. Each subject attended multiple calibration sessions in different days. During calibration, the users were asked to type a set of pre-defined target symbols within randomly ordered sequences to enable the system to learn the class conditional EEG evidence distributions. Here, each calibration session includes 100 sequences (queries), each containing randomly ordered 14 letters. Labelled calibration data is later used to learn optimal parameters for class conditioned evidence distributions. To evaluate the performance of the proposed stopping criterion, a *Copy Phrase* task was used. In the Copy Phrase task simulation, under a RBSE framework, a simulated user (using learned evidence distributions from calibration data) interacts with the system to type letters in the pre-defined phrase. A Monte-Carlo sampling method has been used to draw samples from class distributions. More details about the simulation framework can be found in [19].

To present the changes of the information gain trough the RBSE process, here we reported two measures: changes of entropy and information transfer rate (ITR) [20, 5] which is a common performance measure for BCI applications.

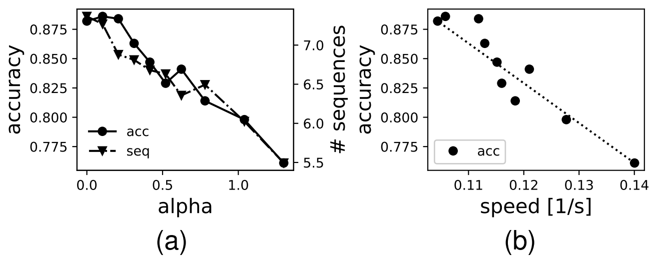

Fig. 2. Average results for a user with AUC= $\%82.5$. (a) Changes in accuracy and number of sequences spent with varying alpha. (b)Speed accuracy plot without alpha values. Speed is determined by 1 over number sequences, where each sequence takes 1.3s in our application. Given the figures it is desired to pick the point which has the largest distance to the line and on the right hand side to achieve better accuracy.

Figure1 shows the average accuracy and the average number of sequences for all users categorized into three groups according to their performance in the calibration session using the area under receiver operation characteristics (AUC) values. Users in UG1, UG2, and UG3 have AUCs values in $[\%90, \%100]$, $[\%78, \%90]$, and $[\%50, \%78]$ intervals, respectively. It can be observed that for users with high performance (high AUCs), the proposed stopping criterion does not save significant number of sequences on average; because it takes a few number of sequences (around 3 sequences) for this group of users to type the phrase with the high confidence level. However, for the other groups of users, we observed statistically significant changes in the average number of sequences with $p < 0.002$ using Wilcoxon signed-rank test; with small changes in the average accuracy values (at most $\%2$). Across all users, on average the proposed stopping criterion method in compared with the fix method, shortens the number of sequences by $\%14.10$, with only $\%1.96$ drop in accuracy.

Figure2.(a) illustrates the accuracy and sequences spent for decision as a function of α, averaged over 1000 Monte Carlo simulations for a user in UG2. Figure2.(b) illustrates the relation between accuracy and speed (1/sequences) using the same α values in figure2.(a). This figure shows that we can select α such that it gives us highest accuracy/speed ratio without dramatics decreases in accuracy.

Figure3.(a) illustrates the average changes of the posterior probability and accuracy as a function of number of sequences. It shows that by using different stopping criteria, i.e. fix and proposed history-based approaches, the system achieves similar accuracy/confidence. Figure3.(b) also illustrates the average changes of entropy and ITR across sequences. As expected, both entropy and ITR increased at the beginning of the RBSE process. However, after a certain point, they both started to decrease and converged to 0. For an average performing user, we visualized the mean posterior for the target letter (state) and the best competing letter (the letter with highest prior value) and accuracy. Figure 3 allows

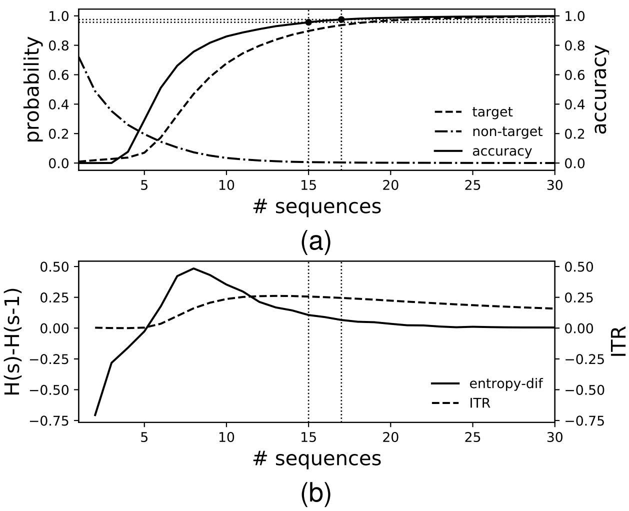

Fig. 3. (a) Probability of the state (target letter) and a competing candidate which has the highest prior probability (non-target) over all possible 28 choices in the dictionary. Accuracy is computed over 1000 Monte-Carlo simulations. Dots indicate stopping accuracy and average posterior for the proposed early stopping (computed according to (6)) and conventional one (computed according to (1)). The system saves 2 sequences while sacrificing $\%1.1$ average accuracy. (b) ITR and entropy changes during RBSE. When posterior probability is saturated, the changes (in terms of entropy and ITR) are negligible.

us to conclude that the history-based proposed method terminated the estimation earlier in compared with the fix method, with marginal accuracy loss after entropy changes converged to 0.

4. CONCLUSION

Being motivated by previous findings for the query selection problem, we proposed a new stopping criterion based on the history-based objective. We formulated the stopping objective as a linear weighted combination of the posterior and the average changes of log of the posterior displacement, which is called momentum. We illustrated that using the proposed momentum-based objective gives us better accuracy/speed ratio. We also analytically showed that using performing RBSE under the proposed stopping criterion gives us higher probability than the conventional fix method. To examine the performance of the momentum-based stopping criterion, we used a BCI typing system simulations employing the actual human-in-the-loop calibration data. As a next step of this study, we aim to improve accuracy/time-spent ratio of state estimation problem by jointly optimizing querying and stopping.

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