

ALTERNATING DIRECTION METHOD OF MULTIPLIERS FOR SEMI-BLIND ASTRONOMICAL IMAGE DECONVOLUTION

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ABSTRACT

High resolution astronomical imagery plays a critical role in multiple remote sensing applications. In this work, we introduce a novel post-acquisition computational technique aiming to recover the high-quality versions of blurry and degraded astronomical observations. Additionally, the proposed scheme is able to retrieve significant information regarding the characteristic properties of the blurring kernel, *i.e.* point spread function (PSF). In order to accomplish this goal, we exploit the mathematical frameworks of *Sparse Representations*, and the *Alternating Direction Method of Multipliers* (ADMM). Experimental results demonstrate the ability of the proposed approach to synthesize high-quality astronomical imagery.

1. INTRODUCTION

Astronomical image deconvolution is one of the most challenging problems in the remote sensing community due to the fact that reflected or emitted light from an astronomical object that is observed from the earth's surface and passes through the atmosphere, results into blurry and noisy observations. Due to the time and space varying fluctuations that are caused by atmospheric turbulence, characteristics of the blurring operator cannot be precisely defined [1]. Even for spaceborne telescopes, a spatially varying Point Spread Function (PSF) may characterize the acquisition process, leading to uneven blurring effects [2]. To overcome this issue, appropriate regularization terms should be imposed for both the PSF estimation and the sharp image reconstruction.

Formally, the problem of image deconvolution or de-blurring can be modeled as that of trying to estimate the unknown sharp image denoted by f , from the low quality blurred image g which is generated according to

$$\mathbf{g} = \mathbf{h} * \mathbf{f} + \eta, \quad (1)$$

where h corresponds to the imaging system's PSF and η is the additive noise. Image de-blurring techniques are divided into two main categories, the non-blind, where the PSF is known, and blind when the PSF is unknown. In cases where the true

PSF is unknown but prior knowledge related to its characteristics are known are classified as semi-blind.

In this work we investigate the core problem of semi-blind image deconvolution, where a remote sensing instrument acquires a low-quality blurred image and the objective is to estimate the high-quality version of the scene, by formulating the problem using the sparse representations framework [3]. Unlike established approaches, this work adheres to the following assumptions:(i) both the high quality image and the blurring kernel operator admit sparse representations when mapped to appropriate dictionaries; (ii) appropriate sparsifying dictionaries are available for both the image space and the blurring kernel space. Regarding the image space, we adhere to a data driven dictionary modelling approach exploiting the K-SVD algorithm [4], while for the blurring kernel, we assume a Gaussian mixture. We simultaneously estimate the clean signal and the blurring kernel via the proposed Alternating Direction Method of Multipliers (ADMM) [5] framework optimization framework. Figure 1 presents a visual illustration of the proposed algorithmic scheme.

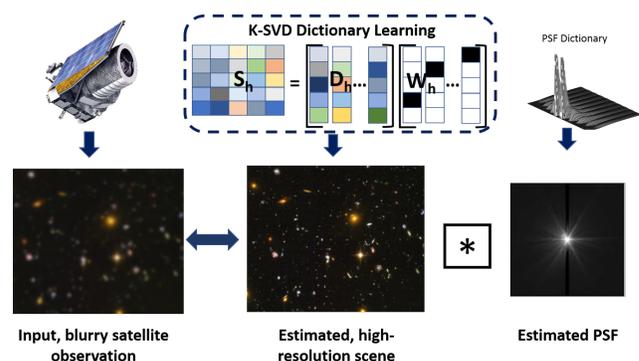


Fig. 1: Illustration of the proposed deconvolution approach. The low quality input image is expressed as the convolution of the image and the blurring signal, each of which is expressed as a sparse representation in the appropriate dictionary.

Consequently, our algorithm can be valuable in a wide range of remote sensing applications.

2. PREVIOUS WORK

The main goal of blind de-convolution approaches, is to estimate simultaneously the PSF and the de-blurred image. In fact, it was recently shown that the problem of blind signal deconvolution is, in general, non-identifiable, even for signals that are naturally sparse [6]. One class of methods for blind deconvolution for example assume the existence of multiple images of the same scene at different spectral bands [7], images with different spectral resolution as it is the case in pan-sharpening where both panchromatic and multispectral imagery is available [8], or side information from modules like inertia measurement units [9].

Regardless of the amount of available imagery, a critical parameter of deconvolution methods is related to the assumed signal prior, where Laplacian priors [10] or Total Variation (TV) priors [11] has been successfully employed. Sparsity has also been greatly investigated as a powerful signal prior for image deconvolution [12]. Specifically, in [7], the authors propose a blind de-blurring method, utilizing the sparsity as prior knowledge. Instead of solving the ill-posed deconvolution problem, the authors use the sparsity constraint subject to a dictionary matrix, created from patches extracted from the blurred image. Likewise, in [13] the authors propose a matching-pursuit optimization technique for semi-blind deconvolution. Another interesting frequency-domain deconvolution approach was proposed in [1], where the authors exploited a wavelet domain regularized filtering technique.

The problem of blind and semi-blind deconvolution of strictly sparse signals has been explored from multiple aspects including Bayesian [14] and sparse-based approaches [15]. Additionally, the authors in [16] proposed an interesting coupled sparsity prior technique for solving the problem of blind image deconvolution. Even more recently, it was shown that under minimal assumptions regarding the structure of subspaces that generate the observations, identification of the true signal is possible [17]. An example of such structure correspond to the model considered here where the convolved signals can be sparsely represented in appropriate dictionaries [18]. Extension of sparse coding, where the output signal corresponds to a linear mapping between a sparse vector and a dictionary matrix, have also be proposed in the case of *convolutional sparse coding* where the mapping is replaced by the convolution operator [19].

3. PROPOSED FORMULATION

Let $\mathbf{z} \in \mathbb{R}^m$ be the vector that corresponds to the patches of the observed blurred image \mathcal{Z} that is modulated by the spatially invariant convolution, produced as: $\mathbf{z} = \mathbf{s} * \mathbf{h}$, where $\mathbf{s} \in \mathbb{R}^m$ stand for the patches of the original image \mathcal{S} , \mathbf{h} is the Point Spread Function (PSF), and $*$ denotes a convolution operator. The task of blind image deconvolution is composed of two parts: estimation of the PSF \mathbf{h} , and deconvolution of

the original image, \mathcal{S} . In our proposed approach, we assume that the patches of the original, high-quality image, $\mathbf{s} \in \mathbb{R}^m$ can be represented as a sparse linear combination in a dictionary $\mathbf{D} \in \mathbb{R}^{m \times n}$, generated from multiple training images, as: $\mathbf{s} = \mathbf{D}\mathbf{w}$, $\mathbf{w} \in \mathbb{R}^n$, $\|\mathbf{w}\|_0 \ll m$.

Motivated by [20], we assume that a blurring dictionary composed of ℓ prototypical blurring functions, $\mathbf{B} \in \mathbb{R}^{p \times \ell}$ is available. This dictionary is constructed based on the assumption that an accurate approximation of the blurring function can be achieved by selecting a small number of elements from the blurring dictionary. Consequently, the blurring kernel can be expressed as a sparse linear combination between the blurring dictionary \mathbf{B} , and a sparse coding vector \mathbf{k} , $\mathbf{k} \in \mathbb{R}^\ell$, as: $\mathbf{h} = \mathbf{B}\mathbf{k}$, $\mathbf{k} \in \mathbb{R}^\ell$, $\|\mathbf{k}\|_0 \ll \ell$. In this work, we consider a dictionary consisting of mixtures of zero-mean Gaussian functions with different variances: $\mathbf{b}_i = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left\{-\frac{x^2}{2\sigma_i^2}\right\}$. The PSF is generated by linear combinations between the blurring dictionary elements and the sparse coefficients $\mathbf{k} \in \mathbb{R}^\ell$, as: $\mathbf{h} = \sum_{j=1}^{\ell} \mathbf{b}_j \mathbf{k}_j$. According to the aforementioned assumptions, the formulation of the recovery problem can be expressed as:

$$\min_{\mathbf{w}, \mathbf{k}} \frac{1}{2} \|\mathbf{z} - (\mathbf{D}\mathbf{w}) * (\mathbf{B}\mathbf{k})\|_F^2 + \lambda_1 \|\mathbf{w}\|_1 + \lambda_2 \|\mathbf{k}\|_1, \quad (2)$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$ denote the sparsity regularization parameters. In order to solve efficiently the problem expressed by Eq. (2), we exploit the scheme of the Alternating Direction Method of Multipliers (ADMM) [5]. Taking into consideration the very good performance of ADMM in several inverse image processing problems [21], it is a natural choice to utilize for the proposed blind image deconvolution scheme. Reformulating (2) in a suitable form for the ADMM by introducing auxiliary variables, along with their constraints, the minimization problem is given by:

$$\min_{\mathbf{w}, \mathbf{k}, \mathbf{p}, \mathbf{q}} \frac{1}{2} \|\mathbf{z} - (\mathbf{D}\mathbf{w}) * (\mathbf{B}\mathbf{k})\|_F^2 + \lambda_1 \|\mathbf{p}\|_1 + \lambda_2 \|\mathbf{q}\|_1 \quad (3)$$

subject to $\mathbf{p} = \mathbf{w}$, $\mathbf{q} = \mathbf{k}$

Following the general algorithmic strategy of the ADMM scheme, we seek for the stationary point, solving iteratively for one of the variables, while keeping the others fixed. The augmented Lagrangian function for our problem is formulated as:

$$\begin{aligned} \mathcal{L}(\mathbf{z}, \mathbf{w}, \mathbf{k}, \mathbf{p}, \mathbf{q}, \mathbf{y}_1, \mathbf{y}_2) = & \|\mathbf{z} - (\mathbf{D}\mathbf{w}) * (\mathbf{B}\mathbf{k})\|_F^2 + \\ & \lambda_1 \|\mathbf{p}\|_1 + \lambda_2 \|\mathbf{q}\|_1 + \mathbf{y}_1^T (\mathbf{p} - \mathbf{w}) + \mathbf{y}_2^T (\mathbf{q} - \mathbf{k}) + \\ & \frac{c_1}{2} \|\mathbf{p} - \mathbf{w}\|_F^2 + \frac{c_2}{2} \|\mathbf{q} - \mathbf{k}\|_F^2, \end{aligned} \quad (4)$$

where \mathbf{y}_1 and \mathbf{y}_2 denote the Lagrange multiplier vectors, while $c_1 > 0$ and $c_2 > 0$ stand for the step size parameters. Subsequently, the ADMM iterations for our problem are

- *Sparse Coding Sub-problems*: For minimizing the augmented Lagrangian functions subject to the sparse coding vectors \mathbf{w} and \mathbf{k} , we solve the individual sparse coding problems:

$$\begin{aligned} \mathbf{w}^{k+1} &= \underset{\mathbf{w}}{\operatorname{argmin}} \mathcal{L}(\mathbf{z}, \mathbf{w}^k, \mathbf{k}^k, \mathbf{p}^k, \mathbf{q}^k, \mathbf{y}_1^k, \mathbf{y}_2^k) \\ \mathbf{k}^{k+1} &= \underset{\mathbf{k}}{\operatorname{argmin}} \mathcal{L}(\mathbf{z}, \mathbf{w}^{k+1}, \mathbf{k}^k, \mathbf{p}^k, \mathbf{q}^k, \mathbf{y}_1^k, \mathbf{y}_2^k) \end{aligned} \quad (5)$$

Algorithm 1 Semi-Blind Signal Deblurring

Input: blurry signal \mathbf{z} , high resolution dictionary matrix \mathbf{D} , blurring dictionary matrix \mathbf{B} , number of iterations N and step size parameters c_1, c_2 .

Initialization: Initialize Lagrange multiplier vectors $\mathbf{y}_1 = \mathbf{y}_2 = \mathbf{0}$.

for $n = 1, \dots, N$ **do**

1. Update \mathbf{w} and \mathbf{k} via Eq. (7)
2. Update \mathbf{p} and \mathbf{q} via Eq. (9)
3. Update Lagrangian vectors \mathbf{y}_1 and \mathbf{y}_2 via Eq. (11)

end

By taking into consideration the periodic extensions of the acquired signals, the convolution operation can be expressed as the product of multiplication with an appropriately generated circular matrix, such that:

$$(\mathbf{D}\mathbf{w}) * (\mathbf{B}\mathbf{k}) = \mathbb{C}\{\mathbf{D}\mathbf{w}\} \cdot (\mathbf{B}\mathbf{k}) = (\mathbf{D}\mathbf{w}) \cdot \mathbb{C}\{\mathbf{B}\mathbf{k}\}, \quad (6)$$

where $\mathbb{C}\{\cdot\}$ denotes the operator that produces a Toeplitz matrix by performing the circular convolution of the signal with an appropriately sized identity matrix. Thus, setting $\nabla_{\mathbf{w}}\mathcal{L} = 0$ and $\nabla_{\mathbf{k}}\mathcal{L} = 0$, the sub-problems admit closed-form solutions:

$$\begin{aligned} \mathbf{w}^{k+1} &= (\mathbf{A}^T \mathbf{A} + c_1 \mathbf{I})^{-1} (\mathbf{A}^T \mathbf{z} + (\mathbf{y}_1^k)^T + c_1 \mathbf{p}^k) \\ \mathbf{k}^{k+1} &= (\mathbf{G}^T \mathbf{G} + c_2 \mathbf{I})^{-1} (\mathbf{G}^T \mathbf{z} + (\mathbf{y}_2^k)^T + c_2 \mathbf{q}^k) \end{aligned} \quad (7)$$

where $\mathbf{A} = \mathbf{D} \cdot \mathbb{C}\{\mathbf{B}\mathbf{k}\}$ and $\mathbf{G} = \mathbf{B} \cdot \mathbb{C}\{\mathbf{D}\mathbf{w}\}$.

- Sub-problems \mathbf{p} and \mathbf{q}

$$\begin{aligned} \nabla_{\mathbf{p}}\mathcal{L} &= \nabla_{\mathbf{p}} \left(\lambda_1 \|\mathbf{p}\|_1 + \mathbf{y}_1^T (\mathbf{p} - \mathbf{w}) + \frac{c_1}{2} \|\mathbf{p} - \mathbf{w}\|_2^2 \right) \\ \nabla_{\mathbf{q}}\mathcal{L} &= \nabla_{\mathbf{q}} \left(\lambda_2 \|\mathbf{q}\|_1 + \mathbf{y}_2^T (\mathbf{q} - \mathbf{k}) + \frac{c_2}{2} \|\mathbf{q} - \mathbf{k}\|_2^2 \right) \end{aligned} \quad (8)$$

Setting, $\nabla_{\mathbf{p}}\mathcal{L} = \nabla_{\mathbf{q}}\mathcal{L} = 0$, we have

$$\mathbf{p}^* = \mathcal{S}_{\lambda_1} \left(\left| \mathbf{w} - \frac{\mathbf{y}_1^T}{c_1} \right| \right), \quad \text{and} \quad \mathbf{q}^* = \mathcal{S}_{\lambda_2} \left(\left| \mathbf{k} - \frac{\mathbf{y}_2^T}{c_2} \right| \right), \quad (9)$$

where $\mathcal{S}_{\lambda_1}, \mathcal{S}_{\lambda_2}$ denote the soft-thresholding operators:

$$\mathcal{S}_{\lambda}(\mathbf{x}) = \operatorname{sign}(\mathbf{x}) \max(|\mathbf{x}| - \lambda, 0), \quad (10)$$

Finally, the Lagrangian vectors are updated as:

$$\begin{aligned} \mathbf{y}_1^{(n+1)} &= \mathbf{y}_1^{(n)} + c_1 (\mathbf{p} - \mathbf{w}) \\ \mathbf{y}_2^{(n+1)} &= \mathbf{y}_2^{(n)} + c_2 (\mathbf{q} - \mathbf{k}), \end{aligned} \quad (11)$$

where n denotes the number of iterations.

4. EXPERIMENTAL EVALUATION

In this section, we investigate the performance of the proposed semi-blind Deblurring technique when applied to the deconvolution of astronomical imagery. The reconstruction performance between the acquired ground truth and the reconstructed images, is quantified in terms of the Peak Signal to Noise Ratio (PSNR) metric defined as: $\operatorname{PSNR}(\mathbf{x}, \mathbf{y}) = 20 \log_{10} \frac{L_{max}}{\sqrt{\operatorname{MSE}(\mathbf{x}, \mathbf{y})}}$, where $\operatorname{MSE}(x, y) = \frac{1}{N} \sum_{i=1}^N (x_i - y_i)^2$, \mathbf{x} and \mathbf{y} , denote the pixel values in the reference and the recovered images, while L_{max} is the maximum possible pixel value of the image. Concerning the generation of the blurring dictionary, we utilized a mixture of Gaussian functions with 7×7 filter size, for variant values of standard deviation. Additionally, regarding the high-resolution dictionary, we exploited a K-SVD single dictionary learning approach [4] based on high-resolution astronomical image examples of 5×5 patch-sizes. For both dictionaries, the number of selected atoms was set to 512, balancing the computational cost with the robust reconstructions. In order to justify the quality of our reconstructions, we compare our results over competitive state-of-the-art deconvolution techniques. Specifically, we compare our algorithm against Zhang's *et al.* [16] blind deblurring technique, and against the frequency domain, regularized filtering approach that was proposed in [1].

Figures 2, 3 depict the proposed semi-blind deconvolution system's reconstruction compared to the state-of-the-art. In both experiments, we downgrade the quality of the original acquired images, by convolving them with a mixture of Gaussian point spread functions (PSF's) of 7×7 filter size and standard deviations varying between [0.5, 0.9]. As we may notice, in Figure 2, the proposed semi-blind deconvolution scheme outperforms both Zhang's [16] coupled adaptive sparse deblurring technique and the regularized filtering [1] approaches, both visually, but also in terms of the PSNR quantitative metric. For instance, the regularized filtering approach achieves a PSNR value of 23.18 dB, Zhang's reconstruction reaches the PSNR value of 26.34 dB, while the proposed algorithm achieves the largest of **32.87** dB. As we may observe, our semi-blind scheme provides a considerable improvement in the reconstruction quality of the blurred

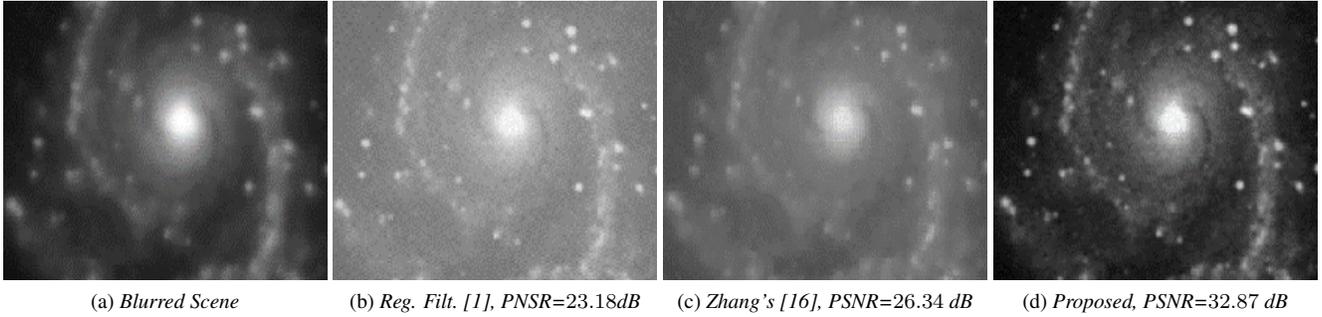


Fig. 2: Astronomical Image Recovery: In this simulation, we observe that the proposed scheme is able to extract significant information, and enhance the quality of this challenging scheme, when it is compared with the state-of-the-art, both visually and quantitatively in terms of the PSNR error metric.

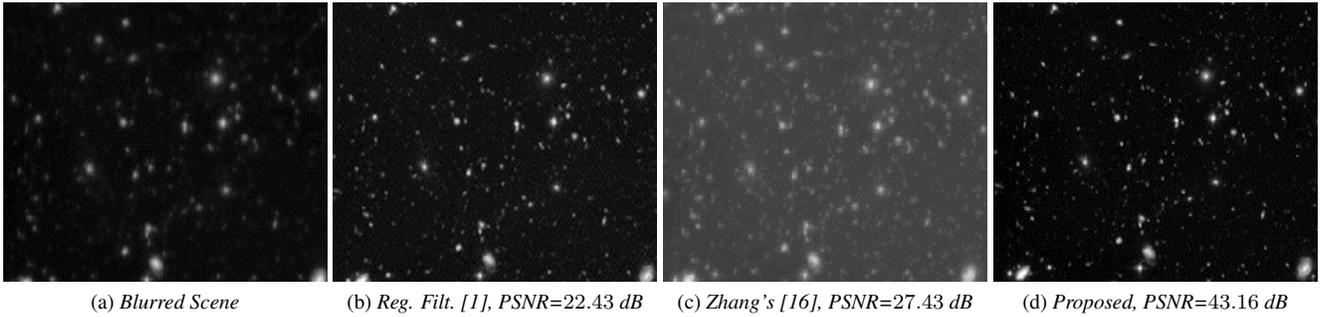


Fig. 3: Astronomical Scene Reconstruction: The quality of the proposed semi-blind system's reconstruction is investigated in this challenging scene. We observe, that under realistic conditions the proposed semi-blind deconvolution scheme provides high quality reconstruction compared with several competitive state-of-the-art techniques.

scene, in comparison with the blurry input imagery. Additionally, in Figure 3 we observe the high-performance of the proposed semi-blind deblurring scheme when is applied to another challenging astronomical scene. In this simulation, the PSNR reconstruction errors of the regularized filtering and Zhang's techniques are 22.43dB and 27.34 dB, respectively, while the proposed system achieves the highest value of **43.16** dB. As we may observe the proposed semi-blind deconvolution scheme is able to substantially improve the quality of this challenging scene.

Finally, Figure 4 illustrates the the empirical convergence of the proposed semi-blind deconvolution algorithm. Specifically, we depict the normalized reconstruction errors for the extraction of the blurry and high-resolution sparse coefficients. We note that both the blurry and the high-quality coefficients converge after approximately 5 iterations.

5. CONCLUSION

In this work, we proposed a novel semi-blind deconvolution approach for astronomical imagery, employing the mathematical framework of *Sparse Representations*, for encoding the relations between high and low quality observations. To achieve this goal, an efficient formulation that extracts both the high-quality and degraded sparse coefficients was

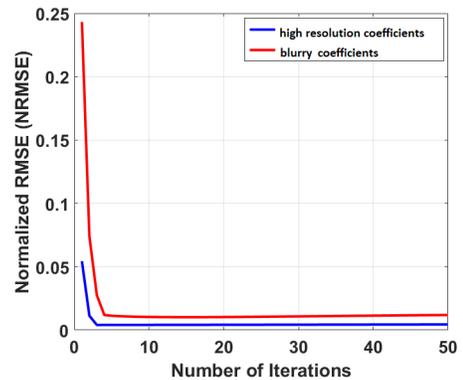


Fig. 4: Convergence Behaviour of the proposed ADMM semi-blind deconvolution scheme. For the extraction of both the blurry and the high-resolution coefficients, we observe that the proposed algorithm converges into a stationary point.

proposed based on the Alternating Direction Method of Multipliers. Experimental results support our claim that high quality reconstruction is obtained using our method. Our scheme can be extended to handle both arbitrary low-to-high resolution enhancements and additional sources of image degradation.

6. REFERENCES

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