# AN ITERATIVE TIME DOMAIN DENOISING METHOD

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**ABSTRACT**

This paper focuses on the classical additive noise signal restoration problem. The proposed time domain denoising method iteratively removes outliers. The proposed denoising filter incorporates a threshold operation to determine which sample values are outliers. This method is compared with wavelet soft/hard thresholding and empirical mode decomposition interval thresholding. The proposed method is shown to be a promising method to denoise signals where a frequency decomposition may not be a robust representation of the noise free signal. The paper provides a discussion on threshold selection and proposes future work to include applying noise estimation to automatically determine thresholds, possible stopping (convergence) criteria, and possible inclusion of an weighted mean computation to improve denoising performance.

***Index Terms*—** Denoising, nonlinear filtering, time domain methods

## I. INTRODUCTION

The classic signal restoration problem [1] is to determine the ideal signal $x(n)$ from the detected signal $y(n)$ given by the additive model

$$y(n) = p(n) * x(n) + \eta(n) \tag{1}$$

where $p(n)$ denotes the impulse response of the data acquisition system, the asterisk $(*)$ represents linear convolution, and $\eta(n)$ is a random process. The noise samples $\eta(n)$ are generally assumed to be white and Gaussian distributed. Moreover, the noise is typically assumed to be independent of $x(n)$. Generally, there is an unmentioned assumption that the noise is a wide sense stationary process. In the models given in equation (1), the distorted signal is modeled as a convolution by an impulse response, $p(n)$. This distortion is usually a blurring or smoothing of the desired signal. Thus, the impulse response of the data acquisition system is modeled as a low pass filter. This paper will not focus on the restoration of the adverse effects of $p(n)$, *i.e.* deconvolve. Rather the focus of the paper is to remove or reduce the additive noise, *i.e.* denoise. Thus, the problem addressed in this paper is to extract the noise free signal $x(n)$ from an acquired noisy signal $y(n)$ modeled as

$$y(n) = x(n) + \eta(n). \tag{2}$$

Previously proposed solutions to the additive noise signal restoration problem given in equation (2) include linear filtering such as a moving average and Weiner filtering [2], non-linear filters such as various adaptive median and regression filters [3], [4], wavelet thresholding [1], *etc.* Since white noise has a constant spectrum, the current state of the art denoising ideology is to decompose the noisy signal into various subbands. A thresholding of each subband decomposed signal provides beneficial denoising [5]–[7]. The subband decomposition that is widely used is the wavelet decomposition. A newer subband decomposition method is the Empirical Mode Decompositon (EMD). The current state of the art wavelet and EMD denoising methods will be evaluated in this paper along with the proposed novel approach.

The recently published Iterative Truncated Mean (ITM) and weighted ITM (WITM) filters [8]–[10] have sparked a reinvestigation on the usefulness of computing the arithmetic mean to replace noisy samples. The proposed method in this paper approaches the classical signal reconstruction problem modeled in equation (2) by combining an outlier removal method, which was originally suggested by Abreu *et al.* in [11], with thresholding into the arithmetic mean filtering process. Provided the noise is stationary, zero mean, Gaussian, independent, and a random process, the proposed reconstruction method will be shown as a promising method in removing noise and increasing the Signal to Noise Ratio (SNR).

## II. EMPIRICAL MODE DECOMPOSITION

The EMD with Interval Thresholding (IT) denoising method proposed in [13] applies a thresholding operation to the EMD intrinsic mode signals. The EMD for 1D signals provides a spectral analysis method for non-linear and non-stationary signals. [14] The EMD decomposes a signal into Intrinsic Mode Functions (IMFs). Each IMF is the result of the sifting process, which attempts to satisfy the following two conditions:

1) The number of zero crossings and the number of local extrema must be the same or off by at most one.
2) The mean defined by the average of the local maxima envelop and local minimum envelop must be zero.

After an IMF is found, the residue function is determined by subtracting the IMF from the previous residue. The EMD iterates by extracting another IMF from the current residue. Precisely, let $y(n)$ be some 1D signal and $y_i(t)$ be the $i$-th IMF where $i \in \mathbb{Z}^+$. The $i$-th IMF $y_i(t)$ is determined by a iterative sifting process of the residue function $r_{i-1}(n)$ where $r_0(n) = y(n)$. The sifting process of some generic function $y(n)$ is

1) determine the maximum envelop by a spline interpolation of the local maxima, $\lceil y(n) \rceil$;
2) determine the minimum envelop by a spline interpolation of the local minima, $\lfloor y(n) \rfloor$;
3) the mean signal is $\bar{y}(n) = \lceil y(n) \rceil - \lfloor y(n) \rfloor$;
4) subtract $\bar{y}(n)$ from $y(n)$, $\hat{y}(n) = y(n) - \bar{y}(n)$;

5) if $\hat{y}(n)$ satisfies the two IMF conditions, then the iteration stops and $\hat{y}(n)$ defines the IMF of $y(n)$. Otherwise, repeat the sifting process for $y(n) = \hat{y}(n)$.

The first residue is defined as

$$r_1(n) = y(n) - y_1(n).$$

For other $i \neq 1$ the residue $r_i(n)$ is

$$r_i(n) = r_{i-1}(n) - y_i(n).$$

The results in [13] shows that robust noise reduction can be achieved by applying IT to each EMD IMF. For each IMF $y_i(n)$ IT is defined as

$$\widehat{y}_i(n) = \begin{cases} \widetilde{y}(n) & \text{if } |y_i(n)| > T_i \text{ and } n = n_e \\ 0 & \text{if } |y_i(n)| \leq T_i \text{ and } n = n_e \\ y_i(n) & \text{otherwise} \end{cases} \quad (3)$$

where $y_i(n_e)$ is an extremum, $\widetilde{y}_i(n)$ is an interpolated version of $y_i(n)$ within an interval about $n$, and threshold values $T_i > 0$. The noise free signal is approximated by adding all the noise removed IMFs with the last residue

$$\widehat{y}(n) = \sum_i \widehat{y}_i(n) + r_i(n).$$

## III. THE PROPOSED DENOISING METHOD

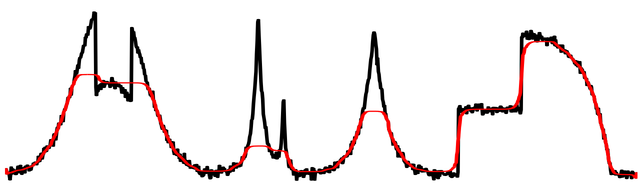


**Fig. 1**. The result of SBF denoising (red) from a noisy signal (black).

The publications of the ITM and WITM have provided evidence in the robustness of iteratively applying a threshold operation(s) to the mean or weighted mean filters on windowed samples to reduce noise. The proposed time domain method iteratively replaces samples that are determined as outliers with the local arithmetic mean. The outliers are identified as samples either greater than or less than some threshold values. The 2D version of the outlier denoising method, referred to as the Squeeze Box Filter (SBF), was proposed in [15]. It was shown to provide a reliable method to contrast enhance ultrasound images, while preserving major image edges. A 1D example of the SBF is shown in Fig. 1. The original noisy signal is shown in Fig. 1 as the black plot. The result of the SBF applied to the original noisy signal in Fig. 1 is shown as the red plot. The red plot indicates that almost all the noise has been removed. However, the peaks and valleys of the ideal signal are adversely diminished. Thus, the 1D SBF would not be considered as robust method in denoising by any metric. To improve the denoising performance of the SBF thresholding modifications were incorporated into the SBF algorithm to alleviate the problem of peak and valley removal.

The SBF with thresholds (SBFT) algorithm is iteratively applied as follows. Let $y(n)$ be a length $N$ noisy signal.

1) Set iteration indices $i, j = 0$ and $y_{i,j}(n) = y(n)$.
2) Set iteration limits $\lambda_1, \lambda_2 > 0$, thresholds $T_{i,1}, T_{i,2} \geq 0$ for $i = 0, 1, 2, \ldots, \lambda_2$, and convergence criteria $\epsilon > 0$.
3) Each iteration $j$ ($j$ starts at one) begins by determining the set of locations of local maxima (peaks) and local minima (valleys). The locations of these extrema are defined by the set

$$\mathcal{N}_E = \{n \mid y_{i,j-1}(n) \text{ meets condition 1 or 2 }\}$$

Condition 1: $y_{i,j-1}(n) > y_{i,j-1}(n-1)$ and $y_{i,j-1}(n) > y_{i,j-1}(n+1)$
Condition 2: $y_{i,j-1}(n) < y_{i,j-1}(n+l)$ and $y_{i,j-1}(n) < y_{i,j-1}(n+1)$

4) Without using the local extrema values, samples within an odd length $L$ window centered at $y_{i,j-1}(n)$ are used to determine the local mean. These extrema maybe replaced with the local mean values. That is for $n \in \mathcal{N}_E$ the local mean is computed as:

$$\bar{y}_{i,j-1}(n) = \frac{1}{L-1}\left(\left(\sum_{l=-\lfloor \frac{L}{2} \rfloor}^{\lfloor \frac{L}{2} \rfloor} y_{i,j-1}(n+l)\right) - y_{i,j-1}(n)\right)$$

where $\lfloor \cdot \rfloor$ is the greatest integer function.

5) The minimum and maximum values within the length $L$ window centered at $y_{i-1}(n)$ are determined

$$m = \min\left(\left\{ y_{i,j-1}(n+l) \;\middle|\; l = 0, \pm 1, \pm 2, \pm \left\lfloor \frac{L}{2} \right\rfloor \right\}\right)$$

and

$$M = \max\left(\left\{ y_{i,j-1}(n+l) \;\middle|\; l = 0, \pm 1, \pm 2, \pm \left\lfloor \frac{L}{2} \right\rfloor \right\}\right).$$

6) The outlier maybe replace according to

$$y_{i,j}(n) = \begin{cases} y_{i,j-1}(n) & \text{if } |M - m| \geq T_{i,1} \text{ or } \\ & |\bar{y}_{i,j-1}(n) - y_{i,j-1}(n)| \geq T_{i,2} \\ \bar{y}_{i,j-1}(n) & \text{otherwise.} \end{cases}$$

7a) If $j < \lambda_1$ and convergence in the Cauchy sense is not attained, that is

$$\sum_{n=0}^{N-1} |y_{i,j-1}(n) - y_{i,j}(n)| > \epsilon, \quad (4)$$

then $j$ is incremented by one and another iteration, starting from Step 3, is performed.

7b) If $j = \lambda_1$ or Cauchy convergence, contrary to equation (4), is attained, then when $i < \lambda_2$, $i$ is incremented by one, $j = 0$, and

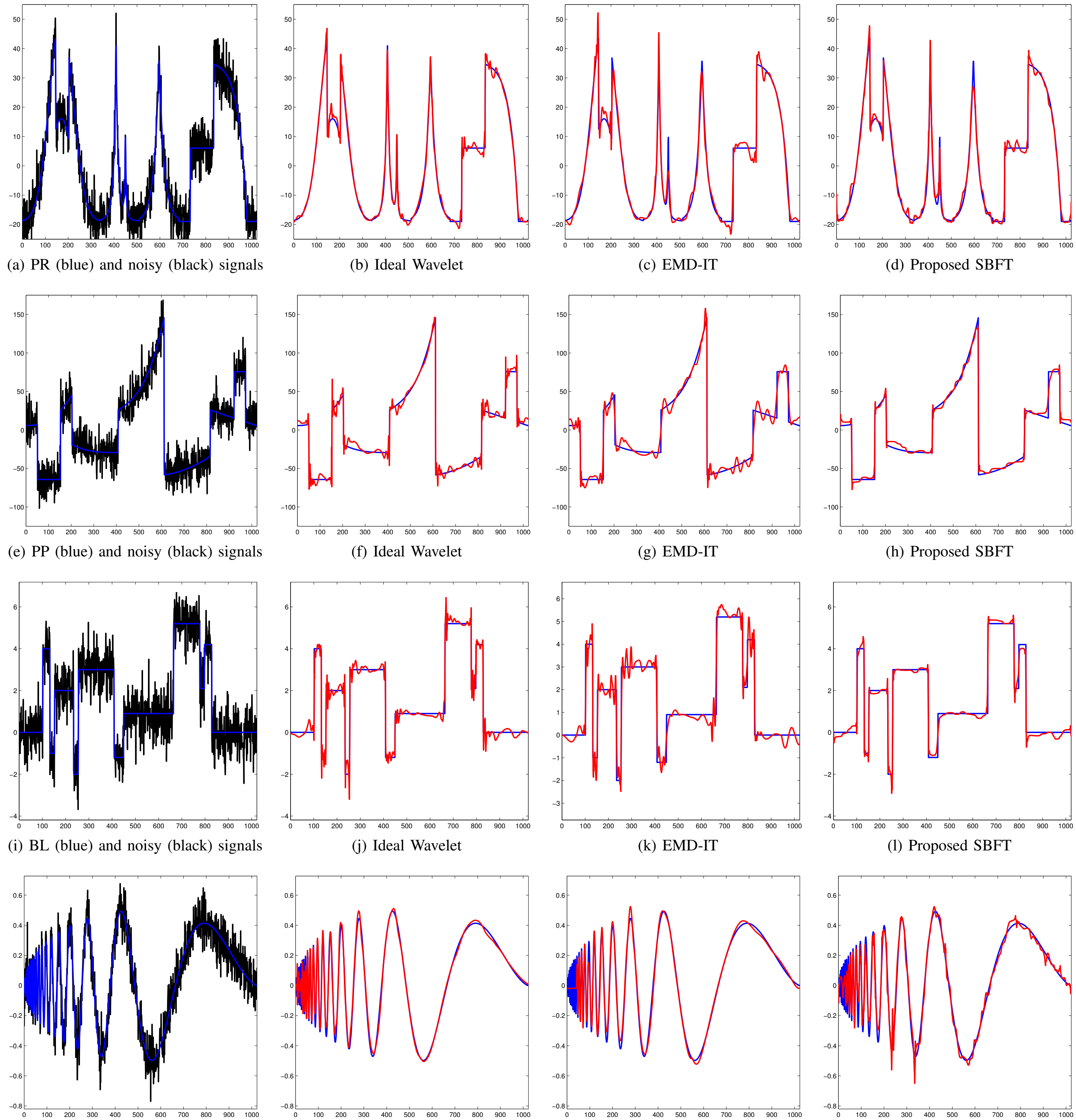
$$y_{i,j}(n) = y_{i-1,\lambda_1}(n) * h(n)$$

where $h(n)$ is a low pass filter. The process continues starting at Step 3.

Step 8: The algorithm stops when $i = \lambda_2$. An approximation of the noise free signal is produced as

$$\widehat{y}(n) = y_{\lambda_2,\lambda_1}(n).$$

Clearly like the ITM and WITM denoising method, the robustness of the proposed SBFT method is dependent on the choices of thresholds $T_{i,1}$, $T_{i,2}$, stopping criteria parameters ($\epsilon$, $\lambda_1$, and $\lambda_2$), and window length $L$. The threshold values $T_{i,1}$ and $T_{i,2}$ should be dynamically adjusted at each iteration. The threshold values should be based on the standard deviation of the noise.



(a) PR (blue) and noisy (black) signals (b) Ideal Wavelet (c) EMD-IT (d) Proposed SBFT

(e) PP (blue) and noisy (black) signals (f) Ideal Wavelet (g) EMD-IT (h) Proposed SBFT

(i) BL (blue) and noisy (black) signals (j) Ideal Wavelet (k) EMD-IT (l) Proposed SBFT

(m) DP (blue) and noisy (black) signals (n) Ideal Wavelet (o) EMD-IT (p) Proposed SBFT

**Fig. 2**. Noise free (blue) PR, PP, BL, DP, and noisy signals (black) and results (red) from various denoising techniques.

**Table I**. SBFT Parameters

| | PR | PP | BL | DP |
|---|---|---|---|---|
| $\lambda_1$ | 3 | 45 | 30 | 2 |
| $T_{0,1}$ | 21 | 70 | 4 | 0.34 |
| $T_{0,2}$ | 18 | 60 | 2 | 0.35 |
| $T_{1,1}$ | 12 | 40 | 2.4 | 0.29 |
| $T_{1,2}$ | 5 | 25 | 1.4 | 0.13 |

The noise standard deviation decrease as the iterations increase. Thus, an incorporation of a noise estimation method would be required for a fully automated denoising method. The choices for stopping criteria should be based on the benefits of further iterations and signal degradation. Allowing to many iterations would remove the peaks and valleys of the true noise free signal. Of course not allowing a sufficient number of iterations would not provide beneficial denoising results. Since motivation of this paper is to provide insight on the proposed time domain denoising technique versus subband decomposition, namely wavelet and EMD transform denoising methods, the noise estimation step at the beginning of each iteration and the possible use of a weighted mean are omitted and left for future research. The stopping criteria and window lengths parameters should allow be carefully investigated and also left for future research. All parameters of the proposed SBFT denoising method are empirical set in the following experiment.

## IV. EXPERIMENTS AND RESULTS

Experiments to compare the various wavelet, EMD-IT, and proposed SBFT methods were performed. The wavelet-ST, wavelet-HT, ideal wavelet methods were from the WaveLab [16] library of Matlab functions. Four WaveLab noise free signals used were Piece-Regular (PR), Piece-Polynomial (PP), Blocks (BL), and Doppler (DP). All four signals are length 1024. These signals were generated by the WaveLab MakeSignal.m function. Gaussian noise with standard deviations ($\sigma$) of 5, 5, 1, and 0.1 were added to PR, PP, BL, and DP, *resp.* to create the noisy version.

The quadrature mirror filter bank in the wavelet transform used a length 8 Daubechies wavelet. Both the wavelet-ST and wavelet-HT used a four level (five subbands) decomposition. The Donoho threshold was applied in the HT and ST operations. The ideal wavelet denoising method uses the noise free signal to adjust the threshold. The ideal wavelet denoising method is included in these experiments for the sake of providing an upper bound on an attainable SNR of the wavelet threholding denoising methods.

The EMD-IT employed a HT operations to the IMF decomposition. The threshold parameters of EMD-IT in equation (3) were manually optimized using a greedy search method to provide the largest possible SNR.

The proposed SBFT method used a length $L = 9$ window to perform the local averaging in Step 4. The convergence parameter in equation (4) is set to $\epsilon = 0.01$ and iteration parameter $\lambda_2 = 2$. The iteration parameter $\lambda_1$ and the threshold parameters used in the proposed SBFT to denoise each signal are given in Table I and were determined from a greedy search. The low pass filter in Step 7b was a simple three point averaging filter.

The SNR of the denoised signal is used to evaluate the performance of each method. The SNR of a restored or noisy signal $\widehat{y}(n)$ is defined as

$$\text{SNR}\left\{\widehat{\mathbf{y}}\right\} = 20\log_{10}\left(\frac{\|\mathbf{x}\|}{\|\mathbf{x}-\widehat{\mathbf{y}}\|}\right)\ \text{dB}$$

**Table II**. Quantitative SNR (dB) Improvements

| | WaveLab Signal | | | |
|---|---|---|---|---|
| Method | PR | PP | BL | DP |
| unprocessed | 11.30 | 10.49 | 10.05 | 9.65 |
| wavelet-ST | 12.24 | 16.13 | 9.61 | 11.42 |
| wavelet-HT | 18.05 | 12.61 | 14.89 | 18.31 |
| EMD-IT | 18.26 | 16.24 | 16.48 | 18.51 |
| SBFT | 20.50 | 18.53 | **19.64** | 16.37 |
| Ideal Wavelet | **22.83** | **18.71** | 19.59 | **20.58** |

where $\|\cdot\|$ denotes the $l_2$-norm and $x(n)$ is the noise free signal. The SNRs of the noisy signal and each denoising method tested in these experiments are given in Table II. The ideal wavelet method, which requires the noise free signal as an input, provided the best SNR in all but the BL signal. In general the wavelet-ST, wavelet-HT, and EMD-IT does provide significant denoising improvements with an increase in SNR over the unprocessed SNR in all cases except the wavelet-ST degraded the SNR in the BL signal. The proposed SBFT provides over 2 dB improvement over the wavelet-ST, wavelet-HT, and EMD-IT methods in restoring the PR and PP signals and over 3 dB improvements in restoring the BL signal. The SNR of the proposed SBFT method is about 2 dB less than the wavelet-HT method and the EMD-IT method in denoising the DP signal. Thus, the DP example provides evidence that the wavelet-HT and EMD-IT are more robust when the signal is strictly band limited. The other cases are evidence that SBFT could achieve better performance than the subband decomposition methods. The results of each restoration method applied to the four tested signals are shown in Fig. 2.

## V. CONCLUSION

An overview of the wavelet HT, wavelet-ST, and EMD-IT methods is provided. The wavelet and EMD-IT methods rely on a subband decomposition and subsequent thresholding operation (HT, ST, or IT) to restore a signal corrupt with additive noise. The SBFT incorporates a thresholding operation into the SBF algorithm to preserve a signal's peaks and valleys, while the additive noise is being reduced. Experiments using the length 1024 PP, PR, BL, and DP signals were performed. The results of these experiments show the proposed SBFT is capable of in excess of 2 dB SNR improvement over the wavelet-ST, wavelet-HT, and EMD-IT methods on the PR, PP, and BL signals. In the PP restoration the SBFT nearly attained the same SNR as the ideal wavelet method. In the BL example the SNR of SBFT exceeds the SNR of the ideal wavelet method and provided over 3 dB improvement over the other methods. When restoring the band limited DP signal, the SBFT did not perform on par with wavelet-HT and EMD-IT where the subband decomposition may have been advantageous. These experiments provide evidence that the SBFT may be a more robust denoising method than subband decomposition based methods with certain signals. In addition to more comprehensive testing, an automated threshold(s) selection method, which will incorporate noise estimation, and the possible use of a weighted mean are in the works.

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